Week 11-12: Special Counting Sequences

April 17, 2019

We have considered several special counting sequences. For instance, the sequence $n!$ counts the number of permutations of an n-set; the sequence D_n counts the number of derangements of an n-set; and the Fibonacci sequence f_n counts the pairs of rabbits.

1 Catalan Numbers

Definition 1.1. The **Catalan sequence** is the sequence

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

The number C_n is called the n**th Catalan number**. The first few Catalan numbers are

$$C_0 = 1, \quad C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14, \quad C_5 = 42.$$

Theorem 1.2. *The number of words $a_1a_2 \ldots a_{2n}$ of length $2n$ having exactly n positive ones $+1$'s and exactly n negative ones -1's and satisfying*

$$a_1 + a_2 + \cdots + a_i \geq 0 \quad \text{for all} \quad 1 \leq i \leq 2n, \tag{1}$$

equals the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

Proof. We call a word of length $2n$ with exactly n positive ones $+1$'s and n negative ones -1's **acceptable** if it satisfies (1) and **unacceptable** otherwise. Let A_n denote the set of acceptable words, and U_n the set of unacceptable words of length $2n$. Then $A_n \cup U_n$ is the set of words of length $2n$ with exactly n positive ones and exactly n negative ones, and

$$|A_n| + |U_n| = \binom{2n}{n} = \frac{(2n)!}{n!n!}.$$

Let S_n denote the set of words of length $2n$ with exactly $n+1$ ones and $n-1$ negative ones.

We define a map $f : U_n \to S_n$ as follows: For each word $a_1 a_2 \ldots a_{2n}$ in U_n, since the word is unacceptable there is a smallest integer k such that

$$a_1 + a_2 + \cdots + a_k < 0.$$

Since the number k is smallest, we have $k \geq 1$, $a_1 + a_2 + \cdots + a_{k-1} = 0$, and $a_k = -1$ (we assume $a_0 = 0$). Note that the integer k must be an odd number. Now switch the signs of the first k terms in the word $a_1 a_2 \ldots a_{2n}$ to obtain a new word $a'_1 a'_2 \ldots a'_k a_{k+1} \ldots a_{2n}$, where

$$a'_1 = -a_1, \quad a'_2 = -a_2, \quad \ldots, \quad a'_k = -a_k.$$

The new word $a'_1 a'_2 \ldots a'_k a_{k+1} \ldots a_{2n}$ has $n+1$ positive ones and $n-1$ negative ones. We then define

$$f(a_1 a_2 \ldots a_{2n}) = a'_1 a'_2 \ldots a'_k a_{k+1} \ldots a_{2n}.$$

We define another map $g : S_n \to U_n$ as follows: For each word $a'_1 a'_2 \ldots a'_{2n}$ in S_n, the word has exactly $n+1$ positive ones and exactly $n-1$ negatives ones. There is a smallest integer k such that

$$a'_1 + a'_2 + \cdots + a'_k > 0.$$

Then $k \geq 1$, $a'_1 + a'_2 + \cdots + a'_{k-1} = 0$, and $a'_k = 1$ (we assume $a_0 = 1$). Switch the signs of the first k terms in $a'_1 a'_2 \ldots a'_{2n}$ to obtain a new word $a_1 a_2 \ldots a_k a'_{k+1} \ldots a'_{2n}$, where

$$a_1 = -a'_1, \quad a_2 = -a'_2, \quad \ldots, \quad a_k = -a'_k.$$

The word $a_1 a_2 \ldots a_k a'_{k+1} \ldots a'_{2n}$ has exactly n ones and exactly n negative ones, and is unacceptable because $a_1 + a_2 + \cdots + a_k < 0$. We set

$$g(a'_1 a'_2 \ldots a'_{2n}) = a_1 a_2 \ldots a_k a'_{k+1} \ldots a'_{2n}.$$

Now it is easy to see that the maps f and g are inverses each other. Hence

$$|U_n| = |S_n| = \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!(n-1)!}.$$

It follows from $|A_n| + |U_n| = (2n)!/(n!n!)$ that

$$\begin{aligned} |A_n| &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\ &= \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{(2n)!}{n!(n-1)!} \cdot \frac{1}{n(n+1)} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

□

Corollary 1.3. *The number of nondecreasing lattice paths from* $(0,0)$ *to* (n,n) *and above the straight line* $x = y$ *is equal to the* n*th Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

Proof. Viewing the $+1$ as a unit move upward and -1 as a unit move to the right, then each word of length $2n$ with exactly n positive ones ($+1$'s) and n negative ones (-1's) can be interpreted as a nondecreasing lattice path from $(0,0)$ to (n,n) and above the straight line $x = y$. □

Example 1.1. There are $2n$ people line to get into theater. Admission is 50 cents. Of the 2n people, n have a 50 cent piece and n have a 1 dollar bill. Assume the box office at the theater begin with empty cash register. In how many ways can the people line up so that whenever a person with a dollar bill buys a ticket and the box office has a 50 cent piece in order to make change?

If the $2n$ people are considered indistinguishable, then the answer is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

If the $2n$ people are consider distinguishable, then the answer is

$$\frac{1}{n+1}\binom{2n}{n}\cdot n!n! = \frac{(2n)!}{n+1}.$$

2 Difference Sequences and Stirling Numbers

Definition 2.1. The **first order difference sequence** (or just **difference sequence**) of a sequence $a = (a_n; n \geq 0)$ is the sequence $\Delta a = (\Delta a_n; n \geq 0)$ defined by

$$\Delta a_n := (\Delta a)_n = a_{n+1} - a_n, \quad n \geq 0.$$

Example 2.1. The difference sequence of the sequence 3^n $(n \geq 0)$ is the sequence

$$\Delta 3^n = 3^{n+1} - 3^n = 2 \times 3^n, \quad n \geq 0.$$

The difference sequence of 2×3^n is

$$\Delta(2 \times 3^n) = 2 \times 3^{n+1} - 2 \times 3^n = 2^2 \times 3^n, \quad n \geq 0.$$

The difference sequence $\Delta(\Delta a_n; n \geq 0)$ of the sequence $(\Delta a_n; n \geq 0)$ is called the **second order difference sequence** of $(a_n; n \geq 0)$, and is denoted by $(\Delta^2 a_n; n \geq 0)$. More specifically,

$$\begin{aligned} \Delta^2 a_n = (\Delta^2 a)_n : &= \Delta(\Delta a_n) = \Delta a_{n+1} - \Delta a_n \\ &= (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) \\ &= a_{n+2} - 2a_{n+1} + a_n. \end{aligned}$$

Similarly, the p**th order difference sequence** $(\Delta^p a_n; n \geq 0)$ of $(a_n; n \geq 0)$ is the difference sequence $\Delta(\Delta^{p-1} a_n; n \geq 0)$ of the sequence $(\Delta^{p-1} a_n; n \geq 0)$, namely,

$$\Delta^p a_n = (\Delta^p a)_n := \Delta(\Delta^{p-1} a_n) = \Delta^{p-1} a_{n+1} - \Delta^{p-1} a_n, \quad n \geq 0.$$

We define the 0**th order difference sequence** $(\Delta^0 a_n; n \geq 0)$ to be the sequence itself, namely,

$$\Delta^0 a_n = (\Delta^0 a)_n := a_n, \quad n \geq 0.$$

To avoid the cumbersome on notations of the higher order difference sequences, we view each sequence $(a_n; n \geq 0)$ as a function

$$f : \{0, 1, 2, \ldots\} \to \mathbb{C}, \quad f(n) = a_n, \quad n \geq 0.$$

Let S_∞ denote the vector space of functions defined on the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of nonnegative integers. Then S_∞ is a vector space under the ordinary addition and scalar multiplication of functions. Now the difference operator Δ is a linear function from S_∞ to itself. For each $f \in S_\infty$, Δf is the sequence defined by

$$(\Delta f)(n) = f(n+1) - f(n), \quad n \geq 0.$$

Lemma 2.2. *The operator $\Delta : S_\infty \to S_\infty$ is a linear map.*

Proof. Given sequences $f, g \in S_\infty$ and numbers α, β. We have

$$\begin{aligned} \Delta(\alpha f + \beta g)(n) &= (\alpha f + \beta g)(n+1) - (\alpha f + \beta g)(n) \\ &= \alpha[f(n+1) - f(n)] + \beta[g(n+1) - g(n)] \\ &= \alpha(\Delta f)(n) + \beta(\Delta g)(n) \\ &= (\alpha \Delta f + \beta \Delta g)(n), \quad n \geq 0. \end{aligned}$$

This means that $\Delta(\alpha f + \beta g) = \alpha \Delta f + \beta \Delta g$. □

Theorem 2.3. *For each sequence $f \in S_\infty$, the pth order difference sequence $\Delta^p f$ has the form*

$$\Delta^p f(n) = (\Delta^p f)(n) := \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} f(n+k), \quad n \geq 0.$$

Proof. For $p = 0$, it is clear that $\Delta^0 f(n) = (\Delta^0 f)(n) = f(n)$. For $p = 1$,

$$\sum_{k=0}^{1} (-1)^{1-k} \binom{1}{k} f(n+k) = f(n+1) - f(n) = (\Delta f)(n).$$

Let $p \geq 2$. We assume that it is true for $p - 1$, that is,

$$(\Delta^{p-1} f)(n) = \sum_{k=0}^{p-1} (-1)^{p-1-k} \binom{p-1}{k} f(n+k).$$

By definition of difference, $\Delta^p f = \Delta(\Delta^{p-1} f)$, we have

$$
\begin{aligned}
(\Delta^p f)(n) &= \left(\Delta(\Delta^{p-1} f)\right)(n) = (\Delta^{p-1} f)(n+1) - (\Delta^{p-1} f)(n) \\
&= \sum_{k=0}^{p-1} (-1)^{p-1-k} \binom{p-1}{k} f(n+1+k) - \sum_{k=0}^{p-1} (-1)^{p-1-k} \binom{p-1}{k} f(n+k) \\
&= \sum_{k=1}^{p} (-1)^{p-k} \binom{p-1}{k-1} f(n+k) + \sum_{k=0}^{p-1} (-1)^{p-k} \binom{p-1}{k} f(n+k).
\end{aligned}
$$

Applying the Pascal formula $\binom{p}{k} = \binom{p-1}{k-1} + \binom{p-1}{k}$ for $1 \le k \le p-1$, we obtain

$$
(\Delta^p f)(n) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} f(n+k), \quad n \ge 0.
$$

□

Definition 2.4. The **difference table** of a sequence $f(n)$ $(n \ge 0)$ is the array

$$
\begin{array}{ccccccccc}
f(0) & & f(1) & & f(2) & & f(3) & & \cdots \\
& (\Delta f)(0) & & (\Delta f)(1) & & (\Delta f)(2) & & \cdots & \\
& & (\Delta^2 f)(0) & & (\Delta^2 f)(1) & & \cdots & & \\
& & & (\Delta^3 f)(0) & & \ddots & & & \\
& & & & \ddots & & & &
\end{array}
$$

where the pth row is the pth order difference sequence $(\Delta^p f)(n)$, $n \ge 0$.

Example 2.2. Let $f(n)$ be a sequence defined by

$$
f(n) = 2n^2 + 3n + 1, \quad n \ge 0.
$$

Then its difference table is

$$
\begin{array}{ccccccccccccc}
1 & & 6 & & 15 & & 28 & & 45 & & 66 & & \cdots \\
& 5 & & 9 & & 13 & & 17 & & 21 & & \cdots & \\
& & 4 & & 4 & & 4 & & 4 & & \cdots & & \\
& & & 0 & & 0 & & 0 & & \cdots & & & \\
& & & & 0 & & 0 & & \ddots & & & & \\
& & & & & 0 & & \ddots & & & & & \\
& & & & & & \ddots & & & & & &
\end{array}
$$

A sequence $f(n)$ $(n \geq 0)$ of the form

$$f(n) = \alpha_p n^p + a_{p-1} n^{p-1} + \cdots + \alpha_1 n + \alpha_0, \quad n \geq 0,$$

where $\alpha_1, \ldots, \alpha_p$ are constants and $\alpha_p \neq 0$, is called a **polynomial sequence of degree** p.

Theorem 2.5. *For each polynomial sequence $f(n)$ $(n \geq 0)$ of degree p, the $(p+1)$th order difference sequence $\Delta^{p+1} f$ is identically zero, that is,*

$$(\Delta^{p+1} f)(n) = 0, \quad n \geq 0.$$

Proof. We proceed by induction on p. For $p = 0$, the sequence $f(n) = \alpha_0$ is a constant sequence, and

$$(\Delta f)(n) = \alpha_0 - \alpha_0 = 0$$

is the zero sequence. Consider $p \geq 1$ and assume $\Delta^p g \equiv 0$ for all polynomial sequences g of degree at most $p - 1$. Compute the difference

$$
\begin{aligned}
(\Delta f)(n) &= \left[\alpha_p (n+1)^p + \alpha_{p-1}(n+1)^{p-1} + \cdots + \alpha_1 (n+1) + \alpha_0\right] \\
&\quad -[\alpha_p n^p + \alpha_{p-1} n^{p-1} + \cdots + \alpha_1 n + \alpha_0] \\
&= \alpha_p \binom{p}{1} n^{p-1} + \text{lower degree terms.}
\end{aligned}
$$

The sequence $g = \Delta f$ is a polynomial sequence of degree at most $p - 1$. Thus by the induction hypothesis,

$$\Delta^{p+1} f = \Delta^p (\Delta f) = \Delta^p g \equiv 0.$$

□

Theorem 2.6. *The difference table of a sequence $f(n)$ $(n \geq 0)$ is determined by its* 0**th diagonal sequence**

$$(\Delta^0 f)(0), \quad (\Delta^1 f)(0), \quad (\Delta^2 f)(0), \quad \ldots, \quad (\Delta^n f)(0), \quad \ldots$$

Moreover, the sequence $f(n)$ itself is determined as

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} (\Delta^k f)(0) = \sum_{k=0}^{\infty} \binom{n}{k} (\Delta^k f)(0), \quad n \geq 0. \tag{2}$$

Proof. We proceed by induction on $n \geq 0$. For $n = 0$, for any sequence $g \in S_\infty$,

$$g(0) = \sum_{k=0}^{0} \binom{0}{k} (\Delta^k g)(0) = (\Delta^0 g)(0) = g(0).$$

Let $n \geq 1$ and assume that it is true for the case $n-1$, that is, for each sequence $h \in S_\infty$,

$$h(n-1) = \sum_{k=0}^{n-1} \binom{n-1}{k} (\Delta^k h)(0).$$

Now for the sequence $f \in S_\infty$, by definition of Δf at $n-1$, we have

$$f(n) = f(n-1) + (\Delta f)(n-1) = (f + \Delta f)(n-1).$$

Applying the induction hypothesis to the sequences f and Δf, we have

$$\begin{aligned} f(n) &= \sum_{k=0}^{n-1} \binom{n-1}{k} (\Delta^k f)(0) + \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\Delta^k(\Delta f)\right)(0) \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} (\Delta^k f)(0) + \sum_{k=0}^{n-1} \binom{n-1}{k} (\Delta^{k+1} f)(0) \\ &= (\Delta^0 f)(0) + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] (\Delta^k f)(0) + (\Delta^n f)(0) \\ &= \sum_{k=0}^{n} \binom{n}{k} (\Delta^k f)(0), \quad \because \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \end{aligned}$$

□

Corollary 2.7. *If the 0th diagonal of the difference table for a sequence $f(n), n \geq 0$ is*

$$c_0, \quad c_1, \quad \ldots, \quad c_p \ (\neq 0), \quad 0, \quad \ldots,$$

then $f(n)$ is a polynomial sequence of degree p, and is explicitly given by

$$f(n) = c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots + c_p\binom{n}{p}, \quad n \geq 0.$$

In other words,

$$f(n) = \sum_{k=0}^{p} \binom{n}{k} (\Delta^k f)(0), \quad n \geq 0. \tag{3}$$

Proof. Note that $f(n) = \sum_{k=0}^{n} \binom{n}{k} (\Delta^k f)(0)$ for $n \geq 0$. For $n \leq p$, we have

$$f(n) = \sum_{k=0}^{n} c_k \binom{n}{k} + \sum_{k=n+1}^{p} c_k \binom{n}{k} = \sum_{k=0}^{p} c_k \binom{n}{k}.$$

For $n > p$, we have

$$f(n) = \sum_{k=0}^{p} c_k \binom{n}{k} + \sum_{k=p+1}^{n} (\Delta^k f)(0) \binom{n}{k} = \sum_{k=0}^{p} c_k \binom{n}{k}.$$

□

For sequence f_p such that $\Delta^n f_p(0) = \delta_{np}$ with $p = 3$, we have its difference table

$$\begin{array}{ccccccccccccc}
0 & & 0 & & 0 & & 1 & & 4 & & 10 & & 20 \\
& 0 & & 0 & & 1 & & 3 & & 6 & & 10 & \\
& & 0 & & 1 & & 2 & & 3 & & 4 & & \\
& & & 1 & & 1 & & 1 & & 1 & & & \\
& & & & 0 & & 0 & & 0 & & & & \\
& & & & & 0 & & 0 & & & & & \\
& & & & & & 0 & & & & & &
\end{array}$$

Example 2.3. Consider the sequence

$$f(n) = n^3 + 2n^2 - 3n + 2, \quad n \geq 0.$$

Computing the difference we obtain

$$\begin{array}{ccccccc} 2 & & 2 & & 12 & & 38 \\ & 0 & & 10 & & 26 & \\ & & 10 & & 16 & & \\ & & & 6 & & & \end{array}$$

Thus the sequence $f(n)$ can be written as

$$f(n) = 2\binom{n}{0} + 10\binom{n}{2} + 6\binom{n}{3}, \quad n \geq 0.$$

Corollary 2.8. *For any sequence $f(n)$, $n \geq 0$, its partial sum can be written as*

$$\sum_{k=0}^{n} f(k) = \sum_{k=0}^{n} \binom{n+1}{k+1} (\Delta^k f)(0), \quad n \geq 0. \tag{4}$$

Proof. Recall the identity $\sum_{k=i}^{n} \binom{k}{i} = \binom{n+1}{i+1}$. Then

$$\begin{aligned} \sum_{k=0}^{n} f(k) &= \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} (\Delta^i f)(0) \\ &= \sum_{i=0}^{n} \left[\sum_{k=i}^{n} \binom{k}{i} \right] (\Delta^i f)(0) \\ &= \sum_{i=0}^{n} \binom{n+1}{i+1} (\Delta^i f)(0). \end{aligned}$$

□

Example 2.4. For the sequence $f(n) = n^2$ $(n \geq 0)$, computing the difference we have

$$\begin{array}{ccccc} 0 & & 1 & & 4 \\ & 1 & & 3 & \\ & & 2 & & \end{array}$$

Thus

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \binom{n+1}{2} + 2\binom{n+1}{3} = \frac{(n+1)n(2n+1)}{6}.$$

For the sequence $f(n) = n^3$, computing the difference we obtain

$$\begin{array}{ccccccc} 0 & & 1 & & 8 & & 27 \\ & 1 & & 7 & & 19 & \\ & & 6 & & 12 & & \\ & & & 6 & & & \end{array}$$

Thus

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + n^3 &= \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4} \\ &= \left[\frac{(n+1)n}{2}\right]^2. \end{aligned}$$

For $f(n) = n^4$, computing the difference we have

$$\begin{array}{ccccccccc} 0 & & 1 & & 16 & & 81 & & 256 \\ & 1 & & 15 & & 65 & & 175 & \\ & & 14 & & 50 & & 110 & & \\ & & & 36 & & 60 & & & \\ & & & & 24 & & & & \end{array}$$

Hence

$$\begin{aligned} \sum_{k=1}^{n} k^4 &= \binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5} \\ &= \frac{(n+1)n(6n^3 + 9n^2 + n - 1)}{30}. \end{aligned}$$

Example 2.5. Consider the sequence $f(n) = n^p$, $n \geq 0$, with $p \in \mathbb{N}$. Write its 0th diagonal sequence as

$$C(p,0), \quad C(p,1), \quad \ldots, \quad C(p,p), \quad 0, \quad \ldots$$

Then by Corollary 2.7,

$$n^p = \sum_{k=0}^{p} C(p,k)\binom{n}{k}.$$

Definition 2.9. The **falling factorial** of n with length k is the number

$$\begin{aligned} [n]_0 &= 1, \\ [n]_k &= n(n-1)\cdots(n-k+1), \quad k \geq 1. \end{aligned}$$

We call the numbers

$$S(p,k) = \frac{C(p,k)}{k!}, \quad 0 \leq k \leq p.$$

the **Stirling numbers of the second kind**.

It is easy to see that the falling factorial $[n]_k$ with $n \geq k \geq 0$ satisfies the recurrence relation

$$[n]_{k+1} = (n-k)[n]_k;$$

$$[n]_k = \binom{n}{k} k!, \quad n, k \geq 0.$$

Then $[n]_0 = [n]_n = 1$ for $n \geq 0$ and $[n]_k = 0$ for $k > n$.

Corollary 2.10. *For any integer $p \geq 0$,*

$$n^p = \sum_{k=0}^{p} S(p,k)[n]_k. \tag{5}$$

Proof. $C(p,k)\binom{n}{k} = \big(C(p,k)/k!\big)[n]_k = S(p,k)[n]_k$. □

Theorem 2.11. *The Stirling numbers $S(p,k)$ of the second kind are integers, satisfying the recurrence relation:*

$$\begin{cases} S(0,0) = S(p,p) = 1 & \text{if} \quad p \geq 0 \\ S(p,0) = 0 & \text{if} \quad p \geq 1 \\ S(p,1) = 1 & \text{if} \quad p \geq 1 \\ S(p,k) = S(p-1,k-1) + kS(p-1,k) & \text{if} \quad p > k \geq 1 \end{cases} \tag{6}$$

Proof. For $p = 0$, since $n^0 = 1$ and $[n]_0 = 1$, (5) implies $S(0,0) = 1$. Since $[n]_k$ is a polynomial of degree k in n with leading coefficient 1, then (5) implies $S(p,p) = 1$.

Let $p \geq 1$. The constant term of the polynomial n^p is zero. Since $[n]_k$ is a polynomial of degree k, the constant term of $[n]_k$ is zero if $k \geq 1$. Then $[n]_0 = 1$

and (5) force that $S(p, 0) = 0$. Since $[n]_1 = n$, let $n = 1$ in (5), since $1^p = 1$, $S(p, 0) = 0$, $[1]_1 = 1$, and $[1]_k = 0$ for $k \geq 2$, we see that $S(p, 1) = 1$.

Now for $p > k \geq 1$, notice that

$$n^p = \sum_{k=0}^{p} S(p,k)[n]_k, \quad n^{p-1} = \sum_{k=0}^{p-1} S(p-1,k)[n]_k.$$

It follows that

$$\begin{aligned} n^p = n \times n^{p-1} &= n \sum_{k=0}^{p-1} S(p-1,k)[n]_k \\ &= \sum_{k=0}^{p-1} S(p-1,k)(n-k+k)[n]_k. \end{aligned}$$

Splitting $n-k+k$ into $(n-k)+k$, we have

$$\begin{aligned} n^p &= \sum_{k=0}^{p-1} S(p-1,k)(n-k)[n]_k + \sum_{k=0}^{p-1} S(p-1,k)k[n]_k \\ &= \sum_{k=0}^{p-1} S(p-1,k)[n]_{k+1} + \sum_{k=0}^{p-1} kS(p-1,k)[n]_k \\ &= \sum_{j=1}^{p} S(p-1,j-1)[n]_j + \sum_{k=1}^{p-1} kS(p-1,k)[n]_k. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k=0}^{p} S(p,k)[n]_k = S(p-1,p-1)[n]_p + \\ \sum_{k=1}^{p-1} [S(p-1,k-1) + k(S(p-1,k)][n]_k. \end{aligned}$$

Therefore $S(p,p) = S(p-1,p-1)$ and

$$S(p,k) = S(p-1,k-1) + kS(p-1,k), \quad 1 \leq k < p.$$

In particular, for $p \geq 2$ and $k = 1$, since $S(p-1,0) = 0$, we obtain

$$S(p,1) = S(p-1,0) + S(p-1,1) = S(p-1,1).$$

Applying the recurrence, we have

$$S(p,1) = S(p-1,1) = \cdots = S(2,1) = S(1,1) = 1.$$

The recurrence relation implies that $S(p,k)$ are integers for all $p \geq k \geq 0$. $\square$

(p,k)	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	1	1						
3	0	1	3	1					
4	0	1	7	6	1				
5	0	1	15	25	10	1			
6	0	1	31	90	65	15	1		
7	0	1	63	301	350	140	21	1	
8	0	1	127	966	1701	1050	266	28	1

Theorem 2.12.

$$\begin{aligned} \sum_{k=1}^{n} k^p &= \sum_{i=0}^{p} \binom{n+1}{i+1} C(p,i) \\ &= \sum_{i=0}^{p} \frac{S(p,i)}{i+1} [n+1]_{i+1} \\ &= \sum_{k=1}^{p+1} \frac{S(p,k-1)}{k} [n+1]_k. \end{aligned}$$

Definition 2.13. A partition of a set S is a collection $\mathcal{P}$ of disjoint nonempty subsets of S such that

$$S = \bigcup_{A \in \mathcal{P}} A.$$

The cardinality $|\mathcal{P}|$ is called the number of parts (or blocks) of the partition $\mathcal{P}$. We define

$$S_{n,k} = \text{number of partitions of an } n\text{-set into } k \text{ parts.}$$

We have $S_{n,k} = 0$ for $k > n$. We assume $S_{0,0} = 1$.

A partition of a set S into k parts can be viewed as a placement of S into k indistinguishable boxes so that each box is nonempty.

Example 2.6. (a) An n-set S with $n \geq 1$ cannot be partitioned into zero parts, can be partitioned into one part in only one way, and can be partitioned into n parts in only one way. So we have

$$S_{n,0} = 0, \quad S_{n,1} = S_{n,n} = 1, \quad n \geq 1.$$

(b) For $S = \{1, 2\}$, we have partitions

$$\{1, 2\}; \quad \big\{\{1\}, \{2\}\big\}.$$

(c) For $S = \{1, 2, 3\}$, we have partitions:

$$\{1, 2, 3\};$$

$$\big\{\{1\}, \{2, 3\}\big\},\ \big\{\{1, 2\}, \{3\}\big\},\ \big\{\{1, 3\}, \{2\}\big\};$$
$$\big\{\{1\}, \{2\}, \{3\}\big\}.$$

(d) For $S = \{1, 2, 3, 4\}$, there are 7 partitions of S into two parts:

$$\big\{\{1\}, \{2, 3, 4\}\big\},\ \big\{\{1, 3, 4\}, \{2\}\big\},\ \big\{\{1, 2, 4\}, \{3\}\big\},\ \big\{\{1, 2, 3\}, \{4\}\big\},$$

$$\big\{\{1, 2\}, \{3, 4\}\big\},\ \big\{\{1, 3\}, \{2, 4\}\big\},\ \big\{\{1, 4\}, \{2, 3\}\big\}.$$

There are 6 partitions of S into three parts:

$$\big\{\{1\}, \{2\}, \{3, 4\}\big\},\ \big\{\{1\}, \{3\}, \{2, 4\}\big\},\ \big\{\{1\}, \{4\}, \{2, 3\}\big\},$$

$$\big\{\{2\}, \{3\}, \{1, 4\}\big\},\ \big\{\{2\}, \{4\}, \{1, 3\}\big\},\ \big\{\{3\}, \{4\}, \{1, 2\}\big\}.$$

Theorem 2.14. *The numbers $S_{n,k}$ satisfy the recurrence relation:*

$$\begin{cases} S_{0,0} = S_{n,n} = 1 & \text{if} \quad n \geq 0 \\ S_{n,0} = 0 & \text{if} \quad n \geq 1 \\ S_{n,1} = 1 & \text{if} \quad n \geq 1 \\ S_{n,k} = S_{n-1,k-1} + kS_{n-1,k} & \text{if} \quad n-1 \geq k \geq 1 \end{cases} \tag{7}$$

Proof. Obviously, $S_{0,0} = S_{n,n} = 1$. For $n \geq 1$, it is also clear that $S_{n,0} = 0$ and $S_{n,1} = 1$.

Let S be a set of n elements, $n > k \geq 1$. Fix an element $a \in S$. The partitions of S into k parts can be divided into two categories: partitions in which $\{a\}$ is a single part, and the partitions that $\{a\}$ is not a single part. The formal partitions can be viewed as partitions of $S \setminus \{a\}$ into $k-1$ parts; there are $S_{n-1,k-1}$ such partitions. The latter partitions can be obtained by partitions of $S \setminus \{a\}$ into k parts and joining the element a in one of the k parts; there are $kS_{n-1,k}$ such partitions. Thus

$$S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}.$$

□

Corollary 2.15.

$$S(p,k) = S_{p,k}, \quad 0 \leq k \leq p.$$

Theorem 2.16.

$$C(p,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^p,$$

$$S(p,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^p.$$

Definition 2.17. The n**th Bell number** B_n is the number of partitions of an n-set into nonempty indistinguishable boxes, i.e.,

$$B_n = \sum_{k=0}^{n} S_{n,k}.$$

The first few Bell numbers are

$$\begin{array}{ll} B_0 = 1 & B_4 = 15 \\ B_1 = 1 & B_5 = 52 \\ B_2 = 2 & B_6 = 203 \\ B_3 = 5 & B_7 = 877 \end{array}$$

Theorem 2.18. *For $n \geq 1$,*

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

Proof. Let S be a set of n elements and fix an element $a \in S$. For each partition $\mathcal{P}$ of S, there is a part (or block) A which contains a. Then $A' = A - \{a\}$ is a subset of $S - \{a\}$. The other blocks of $\mathcal{P}$ except the block A form a partition $\mathcal{P}'$ of $S - A$. Let $k = |A'|$. Then $0 \leq k \leq p - 1$.

Conversely, for any subset $A' \subseteq S - \{a\}$ and any partition $\mathcal{P}'$ of $S - A' \cup \{a\}$, the collection $\mathcal{P}' \cup \{A \cup \{a\}\}$ forms a partition of S. If $|A'| = k$, then there are $\binom{p-1}{k}$ ways to select A'; there are B_{n-1-k} $(= B_k)$ partitions for the set $S - A' \cup \{a\}$. Thus

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

□

The falling factorial $[n]_p$ is a polynomial of degree k in n, and so can be written as a linear combination of the monomials $1, n, n^2, \ldots, n^p$. Let

$$\begin{aligned}
[n]_p &= n(n-1)\cdots(n-p+1) \\
&= \sum_{k=0}^{p} s(p,k) n^k \\
&= \sum_{k=0}^{p} (-1)^{p-k} c(p,k) n^k. \qquad (8)
\end{aligned}$$

The integers $s(p, k)$ are called the **(signed) Stirling numbers of the first kind**. For variables $x_1, x_2, \ldots, x_p$, the elementary symmetric polynomials

$s_0, s_1, s_2, \ldots, s_p$ are defined as follows:

$$\begin{aligned} s_0(x_1, x_2 \ldots, x_p) &= 1, \\ s_1(x_1, x_2 \ldots, x_p) &= \sum_{i=1}^{p} x_i, \\ s_2(x_1, x_2 \ldots, x_p) &= \sum_{i<j} x_i x_j, \\ &\vdots \\ s_p(x_1, x_2 \ldots, x_p) &= x_1 x_2 \cdots x_p, \end{aligned}$$

Since

$$[n]_p = n(n-1)\cdots(n-p+1) = \sum_{k=0}^{p} (-1)^{p-k} s_{p-k}(0, 1, \ldots, p-1) n^k,$$

we have

$$s(p,k) = (-1)^{p-k} s_{p-k}(0, 1, \ldots, p-1).$$

Theorem 2.19. *The integers $c(p,k)$ satisfy the recurrence relation*

$$\begin{cases} c(0,0) = c(p,p) = 1 & \text{if} \quad p \geq 0 \\ c(p,0) = 0 & \text{if} \quad p \geq 1 \\ c(p,k) = c(p-1,k-1) + (p-1)c(p-1,k) & \text{if} \quad p-1 \geq k \geq 1 \end{cases} \tag{9}$$

Proof. It follows from definition (8) that $c(0,0) = c(p,p) = 1$ and $c(p,0) = 0$ for $p \geq 1$.

Let $1 \leq k \leq p-1$. Note that

$$[n]_p = \sum_{k=0}^{p} (-1)^{p-k} c(p,k) n^k,$$

$$[n]_{p-1} = \sum_{k=0}^{p-1} (-1)^{p-1-k} c(p-1,k) n^k,$$

$$[n]_p = \big(n - (p-1)\big)[n]_{p-1}.$$

Then

$$
\begin{aligned}
[n]_p &= \sum_{k=0}^{p-1} (-1)^{p-1-k} \big(n-(p-1)\big) c(p-1,k) n^k \\
&= \sum_{k=0}^{p-1} (-1)^{p-1-k} c(p-1,k) n^{k+1} - \sum_{k=0}^{p-1} (-1)^{p-1-k} (p-1) c(p-1,k) n^k \\
&= \sum_{k=1}^{p} (-1)^{p-k} c(p-1,k-1) n^k + (p-1) \sum_{k=0}^{p-1} (-1)^{p-k} c(p-1,k) n^k.
\end{aligned}
$$

Comparing the coefficients of n^k, we obtain

$$
c(p,k) = c(p-1,k-1) + (p-1)c(p-1,k), \quad 1 \le k \le p-1.
$$

□

Recall that each permutation of n letters can be written as disjoint cycles. Let $c_{n,k}$ denote the number of permutations of an n-set S with exactly k cycles. We assume that $c_{0,0} = 1$.

Proposition 2.20. *The numbers $c_{n,k}$ satisfy the recurrence relation:*

$$
\begin{cases}
c_{0,0} = c_{n,n} = 1 & \text{if} \quad n \ge 0 \\
c_{n,0} = 0 & \text{if} \quad n \ge 1 \\
c_{n,k} = c_{n-1,k-1} + (n-1)c_{n-1,k} & \text{if} \quad n-1 \ge k \ge 1
\end{cases}
\tag{10}
$$

Proof. Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of n objects. If n is positive, then the number of cycles of any permutation must be positive. So $c_{n,0} = 0$ for $n \ge 1$. Note that only the identity permutation has exactly n cycles, so $c_{n,n} = 1$ for $n \ge 0$.

Now fix the object a_n of S. Then permutations of S with k cycles can be divided into two kinds: the permutations that the singleton $\{a_n\}$ is a cycle, and the permutations that the singleton $\{a_n\}$ is not a cycle. In the former case, deleting the cycle $\{a_n\}$, the permutations become the permutations of $n-1$ objects with $k-1$ cycles; there are $c_{n-1,k-1}$ such permutations. In the latter case, deleting the element a_n from the cycle that a_n is contained, the permutations of S become permutations of $S \smallsetminus \{a_n\}$ with k cycles; since each such permutation of S with k cycles can be obtained by putting a_n into the left of elements $a_1, a_2, \dots, a_{n-1}$, and since there are $n-1$ such ways, there are $(n-1)c_{n-1,k}$ permutations of the second type. Thus we obtain the recurrence relation:

$$c_{n,k} = c_{n-1,k-1} + (n-1)c_{n-1,k}, \quad 1 \le k \le n-1.$$

□

Corollary 2.21.

$$c(p,k) = c_{p,k}.$$

3 Partition Numbers

A **partition of a positive integer** n is a representation of n as an unordered sum of one or more positive integers (called **parts**). The number of partitions of n is denoted by p_n. For instance,

$$\begin{aligned} 2 &= 1+1, \\ 3 &= 2+1 = 1+1+1, \\ 4 &= 3+1 = 2+2 = 2+1+1 = 1+1+1+1. \end{aligned}$$

Thus $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $p_4 = 5$. The **partition sequence** is the sequence of numbers

$$p_0 = 1, \quad p_1, \quad p_2, \quad \dots, \quad p_n, \quad \dots.$$

A partition of n is sometimes symbolically written as

$$\lambda = 1^{a_1} 2^{a_2} \cdots n^{a_n}$$

where a_k is the number of parts equal to k. If k is not a part of the partition λ then $a_k = 0$, and in this case the term k^{a_k} is usually omitted. For instance, the partitions

$$\begin{aligned} 5 &= 4+1 = 3+2 = 3+1+1 = 2+2+1 \\ &= 2+1+1+1 = 1+1+1+1+1 \end{aligned}$$

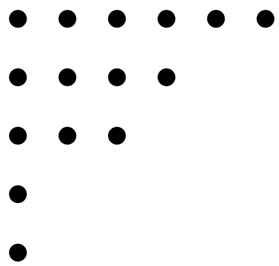
can be written as

$$5^1, \quad 1^1 4^1, \quad 2^1 3^1, \quad 1^2 3^1, \quad 1^1 2^2, \quad 1^3 2^1, \quad 1^5.$$

Let λ be the partition

$$n = n_1 + n_2 + \cdots + n_k$$

of n with $n_1 \geq n_2 \geq \cdots \geq n_k$. The **Ferrers diagram** of λ is a left-justified array of dots which has k rows with n_i dots in the ith row. For instance, the Ferrers diagram of the partition $15 = 6 + 4 + 3 + 1 + 1$ is



Theorem 3.1.

$$\sum_{n=0}^{\infty} p_n x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}. \tag{11}$$

Proof. Note that the right side of (11) is the product of the series

$$\frac{1}{1 - x^k} = 1 + x^k + x^{2k} + x^{3k} + \cdots$$

for $1 \leq k < \infty$. The term x^n arises in the product by choosing a term $x^{a_1 1}$ from the first factor, a term $x^{a_2 2}$ from the second factor, a term $x^{a_3 3}$ from the third factor, and so on, with

$$a_1 1 + a_2 2 + a_3 3 + \cdots + a_k k + \cdots = n.$$

Such choices are in one-to-one correspondent with the partitions

$$\lambda = 1^{a_1} 2^{a_2} 3^{a_3} \cdots k^{a_k} \cdots$$

of the integer n. □

Definition 3.2. Let λ and μ be partitions of a positive integer n, and

$$\begin{aligned} \lambda: \quad & n = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k, \\ \mu: \quad & n = \mu_1 + \mu_2 + \cdots + \mu_k, \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k. \end{aligned}$$

The partition λ is called **majorized by** the partition μ (or μ **majorizes** λ), denoted by $\lambda \leq \mu$, if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for} \quad 1 \leq i \leq k.$$

Example 3.1. Consider the three partitions of 9:

$$\begin{aligned} \lambda: &\quad 9 = 5+1+1+1+1, \\ \mu: &\quad 9 = 4+2+2+1, \\ \nu: &\quad 9 = 4+4+1. \end{aligned}$$

The partition μ is majorized by the partition ν because

$$\begin{aligned} &4 \le 4, \\ &4+2 \le 4+4, \\ &4+2+2 \le 4+4+1, \\ &4+2+2+1 \le 4+4+1. \end{aligned}$$

However, the partitions λ and μ are incomparable because

$$\begin{aligned} &5 > 4, \\ &4+2+2 > 5+1+1. \end{aligned}$$

Similarly, λ and ν are incomparable.

Theorem 3.3. *The lexicographic order is a linear extension of the partial order of majorization on the set P_n of partitions of a positive integer n.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be distinct partitions of n. We need to show that if λ is majorized by μ then there exists an i such that

$$\lambda_1 = \mu_1, \quad \lambda_2 = \mu_2, \quad \ldots, \quad \lambda_{i-1} = \mu_{i-1}, \quad \text{and} \quad \lambda_i < \mu_i.$$

In fact we choose the smallest integer i such that $\lambda_j = \mu_j$ for all $j < i$ but $\lambda_i \ne \mu_i$. Since

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \le \mu_1 + \mu_2 + \cdots + \mu_i,$$

we conclude that $\lambda_i < \mu_i$, and hence λ precedes μ in the lexicographic order. □

4 A Geometric Problem

This section is to give a geometric and combinatorial interpretation for the numbers

$$h_n^{(m)} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}, \quad m \ge 0,\ n \ge 0.$$

For each fixed $m \geq 0$, we obtain a sequence

$$h_0^{(m)}, \quad h_1^{(m)}, \quad \ldots, \quad h_n^{(m)}, \quad \ldots$$

For each fixed n,

$$2^n = h_n^{(n)} = h_n^{(n+1)} = h_n^{(n+2)} = \cdots .$$

For $m = 0$, we have

$$h_n^{(0)} = \binom{n}{0} = 1, \quad n \geq 0.$$

For $m = 1$, we obtain

$$h_n^{(1)} = \binom{n}{0} + \binom{n}{1} = n + 1, \quad n \geq 0.$$

For $m = 2$, we have

$$h_n^{(2)} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} = \frac{n^2 + n + 2}{2}, \quad n \geq 0.$$

Using Pascal's formula $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ for $i \geq 1$, the difference of the sequence $h_n^{(m)}$ can be computed as

$$\begin{aligned}
\Delta h_n^{(m)} &= h_{n+1}^{(m)} - h_n^{(m)} \\
&= \sum_{i=0}^{m} \binom{n+1}{i} - \sum_{i=0}^{m} \binom{n}{i} \\
&= \sum_{i=0}^{m} \left[\binom{n+1}{i} - \binom{n}{i} \right] \\
&= \sum_{i=1}^{m} \binom{n}{i-1} \\
&= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m-1} = h_n^{(m-1)}.
\end{aligned}$$

Theorem 4.1. *The number $h_n^{(m)}$ counts the number of regions in an m-dimensional space divided by n hyperplanes in general position.*

Proof. Hyperplanes in general position in an m-dimensional vector space: Every k hyperplanes, where $1 \leq k \leq m$, meet in an $(m-k)$-plane; no $k+1$ hyperplanes meet in an $(m-k)$-plane.

Let $g_n^{(m)}$ denote the number of regions divided by n hyperplanes in an m-dimensional vector space in general position. Assume $g_n^{(0)} = 1$ for $n \geq 0$. We need to prove that $g_n^{(m)} = h_n^{(m)}$. We show it by induction on $m \geq 1$.

For $m = 1$, any 1-dimensional space is a straight line; a hyperplane of a straight line is just a point; any finite number of points on a line are in general position. If n distinct points are inserted on a straight line, the line is divided into $n+1$ parts (called regions). Thus the number of regions of a line divided by n distinct points is

$$g_n^{(1)} = n + 1 = \binom{n}{0} + \binom{n}{1} = h_n^{(1)}.$$

For $m = 2$, consider n lines in a plane in general position. Being in general position in this case means that any two lines meet at a common point, not three lines meet at a point (the intersection of any three lines is empty).

Given n lines in general position in a plane, we add a new line so that the total $n+1$ lines are in general position. The first n lines intersect the $(n+1)$th line at n distinct points, and the $(n+1)$th line is divided into

$$g_n^{(1)} = n + 1$$

open segments, including two unbounded open segments. Each of these $g_n^{(1)}$ segments divides a region formed by the first n lines into two regions. Thus the number of regions formed by $n+1$ lines is increased by $g_n^{(1)}$ (from the number of regions formed by the first n lines), i.e.,

$$\Delta g_n^{(2)} = g_{n+1}^{(2)} - g_n^{(2)} = g_n^{(1)} = h_n^{(1)} = \Delta h_n^{(2)}.$$

Note that $h_0^{(2)} = g_0^{(2)} = 1$ (the number of regions of a plane divided by zero lines is 1). The two sequences $g_n^{(2)}, h_n^{(2)}$ have the same difference and satisfy the same initial condition. We conclude that

$$g_n^{(2)} = h_n^{(2)} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}.$$

For $m = 3$, consider n planes in 3-space in general position: Every two planes meet at a line, every three planes meet at a point, and no four planes meet at point (the intersection of any four planes is empty).

Now for $m \geq 2$, consider n hyperplanes in m-dimensional vector space in general position. The number of regions of an m-dimensional space divided by 0 hyperplanes is 1, i.e., $g_0^{(m)} = 1$. Consider $n + 1$ hyperplanes in general position. The first n hyperplanes intersect the $(n+1)$th hyperplane in n distinct $(m - 2)$-planes in general position. These n planes of dimension $m - 2$ divide the $(n + 1)$th hyperplane into $g_n^{(m-1)}$ regions of dimension $m - 1$; each of these $(m - 1)$-dimensional regions divides an m-dimensional region (formed by the first n hyperplanes) into two m-dimensional regions. Then the number of m-dimensional regions formed by $n + 1$ hyperplanes is increased by $g_n^{(m-1)}$, i.e.,

$$\Delta g_n^{(m)} = g_{n+1}^{(m)} - g_n^{(m)} = g_n^{(m-1)} = h_n^{(m-1)} = \Delta h_n^{(m)}.$$

Note that $h_0^{(m)} = g_0^{(m)} = 1$ (having the same initial condition). The sequences $g_n^{(m)}, h_n^{(m)}$ for the fixed m have the same difference and satisfy the same initial condition. We conclude that

$$g_n^{(m)} = h_n^{(m)} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}, \quad n \geq 0.$$

□