

Week 6-7: The Binomial Coefficients

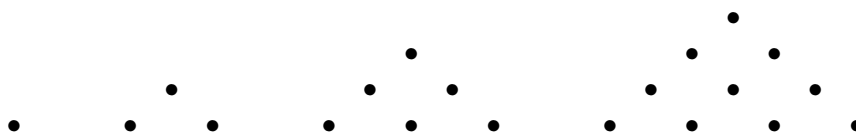
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1 The Pascal Formula

Theorem 1.1 (Pascal's Formula). For integers n and k such that $1 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The numbers $\binom{n}{2} = \frac{n(n-1)}{2}$ ($n \geq 2$) are *triangle numbers*, i.e.,



The numbers $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$ ($n \geq 3$) are *tetrahedral numbers*, i.e., $\binom{n}{3}$ is the number of lattice points of the polytope $\Delta_3(n)$ defined by

$$\Delta_3(n) = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq n - 3\}.$$

Theorem 1.2. The number of nondecreasing coordinate paths from $(0, 0)$ to (m, n) with $m, n \geq 0$ equals

$$\binom{m+n}{m}.$$

2 The Binomial Theorem

Theorem 2.1 (Binomial Expansion). For integer $n \geq 1$ and variables x and y ,

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k},$$

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$$

3 Binomial Identities

Definition 3.1. For any real number α and integer k , define the *binomial coefficients*

$$\binom{\alpha}{k} = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k > 0 \end{cases}$$

Proposition 3.2. 1. For real number α and integer k ,

$$\binom{\alpha}{k} = \binom{\alpha-1}{k} + \binom{\alpha-1}{k-1}.$$

2. For real number α and integer k ,

$$k \binom{\alpha}{k} = \alpha \binom{\alpha - 1}{k - 1}.$$

3. For nonnegative integers m, n , and k such that $m + n \geq k$,

$$\binom{m + n}{k} = \sum_{i=0}^k \binom{m}{i} \binom{n}{k - i}.$$

Proposition 3.3. For integers $n, k \geq 0$,

$$\binom{n + 1}{k + 1} = \sum_{m=0}^n \binom{m}{k}$$

Proof. Applying the Pascal formula again and again, we have

$$\begin{aligned} \binom{n + 1}{k + 1} &= \binom{n}{k + 1} + \binom{n}{k} \\ &= \binom{n - 1}{k + 1} + \binom{n - 1}{k} + \binom{n}{k} \\ &= \dots \\ &= \binom{0}{k + 1} + \binom{0}{k} + \binom{1}{k} + \dots + \binom{n}{k}. \end{aligned}$$

Note that $\binom{0}{k + 1} = 0$. □

4 The Multinomial Theorem

Theorem 4.1 (Multinomial Expansion). For any positive integer n ,

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{\substack{n_1 + n_2 + \dots + n_k = n \\ n_1, n_2, \dots, n_k \geq 0}} \binom{n}{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k},$$

where the coefficients

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$$

are called multinomial coefficients.

Proof.

$$\begin{aligned} (x_1 + x_2 + \dots + x_k)^n &= \underbrace{(x_1 + x_2 + \dots + x_k) \cdots (x_1 + x_2 + \dots + x_k)}_n \\ &= \sum u_1 u_2 \cdots u_n \quad (u_i = x_1, x_2, \dots, x_k, 1 \leq i \leq n) \\ &= \sum_{\substack{n_1 + n_2 + \dots + n_k = n \\ n_1, n_2, \dots, n_k \geq 0}} \left\{ \begin{array}{l} \text{Number of permutations of the} \\ \text{multiset } \{n_1 x_1, n_2 x_2, \dots, n_k x_k\} \end{array} \right\} \\ &= \sum_{\substack{n_1 + n_2 + \dots + n_k = n \\ n_1, n_2, \dots, n_k \geq 0}} \binom{n}{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}. \end{aligned}$$

□

5 The Newton Binomial Theorem

Theorem 5.1 (Newton's Binomial Expansion). Let α be a real number. If $0 \leq |x| < |y|$, then

$$(x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k},$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

Proof. Apply the Taylor expansion formula for the function $(x + y)^\alpha$ of two variables. □

Corollary 5.2. If $|z| < 1$, then

$$\begin{aligned} (1+z)^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k, \\ \frac{1}{(1-z)^\alpha} &= \sum_{k=0}^{\infty} \binom{-\alpha}{k} z^k = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} z^k. \end{aligned}$$

The identity

$$\binom{-\alpha}{k} = (-1)^k \binom{\alpha+k-1}{k}.$$

is called the reciprocity law of binomial coefficients.

Proof. Apply the Taylor expansion formula. □

In particular, since $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$, we have

$$\begin{aligned} \frac{1}{(1-z)^n} &= \left(\sum_{k=0}^{\infty} z^k \right) \cdots \left(\sum_{k=0}^{\infty} z^k \right) \\ &= \sum_{k=0}^{\infty} z^k \sum_{k_1+\cdots+k_n=k} 1 \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k. \end{aligned}$$

This shows again that the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = k$$

equals the binomial coefficient

$$\binom{n+k-1}{k}.$$

6 The Unimodality of Binomial Coefficients

Definition 6.1. A sequence $s_0, s_1, s_2, \dots, s_n$ is called *unimodal* if there is an integer k ($0 \leq k \leq n$) such that

$$s_0 \leq s_1 \leq \cdots \leq s_k \geq s_{k+1} \geq \cdots \geq s_n.$$

Theorem 6.2. Let n be a positive integer. The sequence of binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$$

is an unimodal sequence. More precisely, if n is even,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{n/2} > \cdots > \binom{n}{n-1} > \binom{n}{n};$$

and if n is odd,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2} > \cdots > \binom{n}{n-1} > \binom{n}{n}.$$

Proof. Note that the quotient

$$\binom{n}{k} / \binom{n}{k-1} = \frac{n-k+1}{k} = \begin{cases} \geq 1 & \text{if } k \leq (n+1)/2 \\ \leq 1 & \text{if } k \geq (n+1)/2 \end{cases}$$

The unimodality follows immediately. □

A *cluster* of a set S is a collection \mathcal{A} of subsets of S such that no one is contained in another. A *chain* is a collection \mathcal{C} of subsets of S such that for any two subsets, one subset is always contained in another subset. For example, for $S = \{a, b, c, d\}$, the collection

$$\mathcal{A} = \{\{a, b\}, \{b, c, d\}, \{a, c\}, \{a, d\}\}$$

is a cluster; while the collection

$$\mathcal{C} = \{\emptyset, \{b, d\}, \{a, b, d\}, \{a, b, c, d\}\}$$

is a chain. In more general context, a cluster is an antichain of a partially ordered set.

Theorem 6.3 (Sperner). *Any cluster of an n -set S contains at most $\binom{n}{\lfloor n/2 \rfloor}$ subsets of S .*

Proof. Let $S = \{1, 2, \dots, n\}$. We actually prove the following stronger result by induction on n :

The power set $P(S)$ can be partitioned into m disjoint chains $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$, where $m = \binom{n}{\lfloor n/2 \rfloor}$.

If so, then for any cluster \mathcal{A} of S ,

$$|\mathcal{A} \cap \mathcal{C}_i| \leq 1 \quad \text{for all } 1 \leq i \leq m;$$

and consequently,

$$|\mathcal{A}| \leq m.$$

$$\text{For } n = 1, \binom{n}{\lfloor n/2 \rfloor} = \binom{1}{0} = 1,$$

$$\emptyset \subset \{1\}.$$

$$\text{For } n = 2, \binom{n}{\lfloor n/2 \rfloor} = \binom{2}{1} = 2,$$

$$\emptyset \subset \{1\} \subset \{1, 2\},$$

$$\{2\}.$$

$$\text{For } n = 3, \binom{n}{\lfloor n/2 \rfloor} = \binom{3}{1} = 3,$$

$$\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\},$$

$$\{2\} \subset \{2, 3\},$$

$$\{3\} \subset \{1, 3\}.$$

For $n = 4$, $\binom{n}{\lfloor n/2 \rfloor} = \binom{4}{2} = 6$; the 6 chains can be obtained in two ways: (i) Attach a new subset at the end to each chain of the chain partition for $n = 3$ (this new subset is obtained by appending 4 to the last subset of the chain); (ii) delete the last subsets in all chains of the partition for $n = 3$ and append 4 to all the remaining subsets.

$$\begin{aligned} \emptyset &\subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \{1, 2, 3, 4\}, \\ &\{2\} \subset \{2, 3\} \subset \{2, 3, 4\}, \\ &\{3\} \subset \{1, 3\} \subset \{1, 3, 4\}, \\ &\{4\} \subset \{1, 4\} \subset \{1, 2, 4\}, \\ &\{2, 4\}, \\ &\{3, 4\}. \end{aligned}$$

Note that the chain partition satisfies the properties: (i) Each chain is saturated in the sense that no subset can be added in between any two consecutive subsets; (ii) in each chain the size of the beginning subset plus the size of the ending subset equals n . A chain partition satisfying the two properties is called a *symmetric chain partition*. The above chain partitions for $n = 1, 2, 3, 4$ are symmetric chain partitions.

Given a symmetric chain partition for the case $n - 1$; we construct a symmetric chain partition for the case n : For each chain $A_1 \subset A_2 \subset \dots \subset A_k$ in the chain partition for the case $n - 1$,

$$\begin{aligned} \text{if } k \geq 2, \quad \text{do } & A_1 \subset A_2 \subset \dots \subset A_k \subset A_k \cup \{n\}, \quad \text{and} \\ & A_1 \cup \{n\} \subset A_2 \cup \{n\} \subset \dots \subset A_{k-1} \cup \{n\}; \\ \text{if } k = 1, \quad \text{do } & A_1 \subset A_1 \cup \{n\}. \end{aligned}$$

It is clear that the chains constructed form a symmetric chain partition. In fact, the chains constructed are obviously saturated. Since $|A_1| + |A_k| = n - 1$, then $|A_1| + |A_k \cup \{n\}| = |A_1| + |A_k| + 1 = n$, and when $k \geq 2$,

$$|A_1 \cup \{n\}| + |A_{k-1} \cup \{n\}| = |A_1| + |A_{k-1}| + 2 = |A_1| + |A_k| + 1 = n.$$

Now for any chain $B_1 \subset B_2 \subset \dots \subset B_l$ of the chain partition for the case n , since $B_1 \leq |B_l|$, we have $|B_1| \leq n/2 \leq |B_l|$; since $\lfloor n/2 \rfloor \leq n/2 \leq \lceil n/2 \rceil$, it follows that

$$|B_1| \leq \lfloor n/2 \rfloor \leq \lceil n/2 \rceil \leq |B_l|.$$

This means that the chain $B_1 \subset B_2 \subset \dots \subset B_l$ contains exactly one $\lfloor n/2 \rfloor$ -subset and exactly one $\lceil n/2 \rceil$ -subset. Note that the number of $\lfloor n/2 \rfloor$ -subsets of S is $\binom{n}{\lfloor n/2 \rfloor}$ and the number of $\lceil n/2 \rceil$ -subsets of S is $\binom{n}{\lceil n/2 \rceil}$. Thus the number of chains in the constructed symmetric chain partition is

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}.$$

Therefore the largest size of clusters must be $\binom{n}{\lfloor n/2 \rfloor}$. □

The proof of the Spencer theorem actually gives the construction of clusters of maximal size. When $n = \text{even}$, there is only one such cluster,

$$P_{\frac{n}{2}}(S) : \quad \text{the collection of all } \frac{n}{2}\text{-subsets of } S;$$

and when $n = \text{odd}$, there are exactly two such clusters,

$$P_{\frac{n-1}{2}}(S) : \quad \text{the collection of all } \frac{n-1}{2}\text{-subsets of } S, \text{ and}$$

$$P_{\frac{n+1}{2}}(S) : \quad \text{the collection of all } \frac{n+1}{2}\text{-subsets of } S.$$

Example 6.1. (a) Let $S = \{1\}$. Then $n = 1$ and $\binom{1}{0} = \binom{1}{1} = 1$. There are two clusters: \emptyset and $\{1\}$.

(b) Let $S = \{1, 2\}$. Then $n = 2$ and $\binom{2}{1} = 2$. There is only one cluster of maximal size: $\{\{1\}, \{2\}\}$.

(c) Let $S = \{1, 2, 3\}$. Then $n = 3$ and $\binom{3}{1} = \binom{3}{2} = 3$. There are two clusters of maximal size:

$$\{\{1\}, \{2\}, \{3\}\} \quad \text{and} \quad \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

(d) Let $S = \{1, 2, 3, 4\}$. Then $n = 4$ and $\binom{4}{2} = 6$. There is only one cluster of maximal size:

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$$

(e) Let $S = \{1, 2, 3, 4, 5\}$. Then $n = 5$ and $\binom{5}{2} = \binom{5}{3} = 10$. There are two clusters of maximal size:

$$\begin{aligned} & \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}, \\ & \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}. \end{aligned}$$

7 The Dilworth Theorem

Let (X, \leq) be a finite partially ordered set. A subset A of X is called an *antichain* if any two elements of A are incomparable. In contrast, a *chain* is a subset C of X whose any two elements are comparable. Thus a chain is a linearly ordered subset of X . It is clear that any subset of a chain is also a chain, and any subset of an antichain is also an antichain. The important connection between chains and antichains is:

$$|A \cap C| \leq 1 \quad \text{for any antichain } A \text{ and chain } C.$$

Example 7.1. Let $X = \{1, 2, \dots, 10\}$. The divisibility $|$ makes X into a partially ordered set. The subsets

$$\{2, 3, 5, 7\}, \{2, 3, 7, 10\}, \{3, 4, 7, 10\}, \{3, 5, 7, 8\}, \{4, 5, 6, 7, 9\}, \{4, 6, 7, 9, 10\}, \{5, 6, 7, 8, 9\}, \{6, 7, 8, 9, 10\}$$

are antichains, they are actually maximal antichains; while the subsets

$$\{1, 2, 4, 8\}, \{1, 3, 6\}, \{1, 3, 9\}, \{1, 5, 10\}, \{1, 7\}$$

are chains and they are actually maximal chains.

Let (X, \leq) be a finite poset. We are interested in partitioning X into disjoint union of antichains and partitioning X into disjoint union of chains. Let \mathcal{A} be an antichain partition of X and let C be a chain of X . Since no two elements of C can be contained in any antichain in \mathcal{A} , then

$$|\mathcal{A}| \geq |C|.$$

Similarly, for any chain partition \mathcal{C} and an antichain A of X , there are no two elements of A belonging to a chain of \mathcal{C} , we then have

$$|\mathcal{C}| \geq |A|.$$

Theorem 7.1. *Let (X, \leq) be a finite poset, and let r be the largest size of a chain. Then X can be partitioned into r but no fewer antichains. In other words,*

$$\min\{|\mathcal{A}| : \mathcal{A} \text{ is an antichain partition}\} = \max\{|C| : C \text{ is a chain}\}.$$

Proof. It is enough to show that X can be partitioned into r antichains. Let $X_1 = X$ and let A_1 be the set of all minimal elements of X_1 . Let $X_2 = X_1 - A_1$ and let A_2 be the set of all minimal elements of X_2 . Let $X_3 = X_2 - A_2$ and let A_3 be the set of all minimal elements of X_3 . Continuing this procedure we obtain a decomposition of X into antichains A_1, A_2, \dots, A_p . By the previous argument we always have $p \geq r$. On the other hand, for any $a_p \in A_p$, there is an element $a_{p-1} \in A_{p-1}$ such that $a_{p-1} < a_p$. Similarly, there is an element $a_{p-2} \in A_{p-2}$ such that $a_{p-2} < a_{p-1}$. Continuing this process we obtain a chain $a_1 < a_2 < \dots < a_p$. Since C is a chain of largest size, we then have $r \geq p$. Thus $p = r$. \square

The following dual version of the theorem is known as the Dilworth Theorem.

Theorem 7.2 (Dilworth). *Let (X, \leq) be a finite poset, and let s be the largest size of an antichain. Then X can be partitioned into s but no fewer chains. In other words,*

$$\min\{|\mathcal{C}| : \mathcal{C} \text{ is a chain partition}\} = \max\{|A| : A \text{ is an antichain}\}.$$

Proof. It suffices to show that X can be partitioned into s chains. We proceed by induction on the number n of elements in X . For $n = 1$, it is trivially true. Assume that $n \geq 2$. We consider two cases:

Case 1. There is no chain $a < b < c$ of size 3, i.e., every element of X is either a minimal element or a maximal element. Then X can be partitioned into one or two maximal antichains: the antichain of all minimal elements, and the antichain of all maximal elements. If X is partitioned into only one maximal antichain, then X itself is an antichain of s elements and the theorem is obviously true. If X is partitioned into two maximal antichains, take a minimal element x and a maximal element y such that $x < y$. Then the maximal size of antichains in $X - \{x, y\}$ is $s - 1$. By induction hypothesis, $X - \{x, y\}$ can be partitioned into $s - 1$ chains. These chains together with the chain $x < y$ gives a partition of X into s chains.

Case 2. There is a chain of size 3. Then there is an antichain $A = \{a_1, a_2, \dots, a_s\}$ which is neither the set of all minimal elements nor the set of all maximal elements of X . Let

$$\begin{aligned} A^- &= \{x \in X : x \leq a_i \text{ for some } a_i \in A\}, \\ A^+ &= \{x \in X : x \geq a_i \text{ for some } a_i \in A\}. \end{aligned}$$

The sets A^+ and A^- satisfy the following properties:

1. $A \subsetneq A^-$ since there is a minimal element not in A ;
2. $A \subsetneq A^+$ since there is a maximal element not in A ;
3. $A^- \cap A^+ = A$, $A^- \cup A^+ = X$.

Since A^- and A^+ are smaller posets having antichains of size s , then by induction hypothesis, A^- can be partitioned into s chains $C_1^-, C_2^-, \dots, C_s^-$ with the maximal elements a_1, a_2, \dots, a_s respectively, and A^+ can be partitioned into s chains $C_1^+, C_2^+, \dots, C_s^+$ with the minimal elements a_1, a_2, \dots, a_s respectively. Thus we obtain a partition of X into s chains

$$C_1^- \cup C_1^+, \quad C_2^- \cup C_2^+, \quad \dots, \quad C_s^- \cup C_s^+.$$

□

Example 7.2. Let $X = \{1, 2, \dots, 20\}$ be the poset with the partial order of divisibility. Then the subset $\{1, 2, 4, 8, 16\}$ of 5 elements is a chain of maximal size. The set X can be partitioned into 5 antichains

$$\{1\}, \quad \{2, 3, 5, 7, 11, 13, 17, 19\}, \quad \{4, 6, 9, 10, 14, 15\}, \quad \{8, 12, 18, 20\}, \quad \{16\}.$$

However, the size of the antichain $\{2, 3, 5, 7, 11, 13, 17, 19\}$ is not maximal. In fact, $\{4, 6, 7, 9, 10, 11, 13, 15, 17, 19\}$ is an antichain of 10 elements. The set X can be partitioned into 10 chains

$$\{1, 2, 4, 8, 16\}, \quad \{3, 6, 12\}, \quad \{5, 10, 20\}, \quad \{7, 14\}, \quad \{9, 18\}, \quad \{11\}, \quad \{13\}, \quad \{15\}, \quad \{17\}, \quad \{19\}.$$

This means that $\{4, 6, 7, 9, 10, 11, 13, 15, 17, 19\}$ is an antichain of maximal size.

Example 7.3. Let $X = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i, j \leq 3\}$ be a poset whose partial order \leq is defined by $(i, j) \leq (k, l)$ if and only if $i \leq k$ and $j \leq l$. The size of the longest chain is 7. For instance,

$$(0, 0) < (1, 0) < (1, 1) < (1, 2) < (2, 2) < (2, 3) < (3, 3)$$

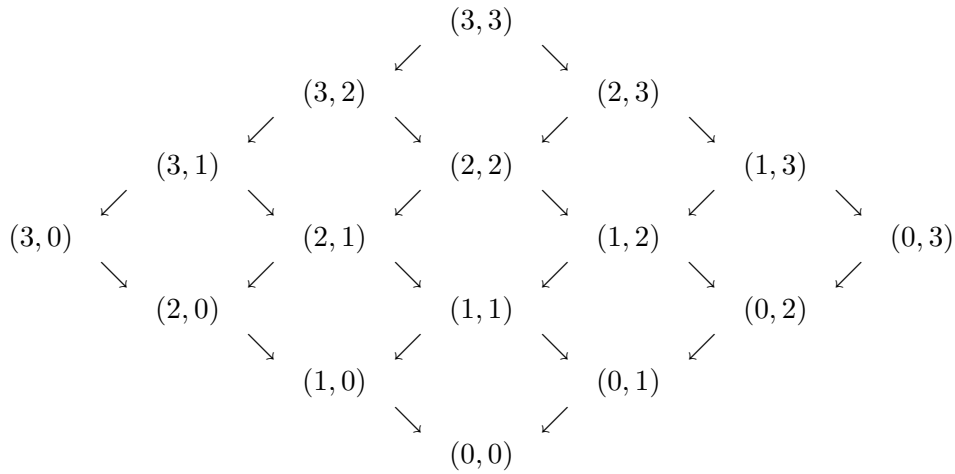
is chain of length 7. The the following collection of subsets

$$\begin{aligned} \{(0, 0)\}, \quad \{(1, 0), (0, 1)\}, \quad \{(2, 0), (1, 1), (0, 2)\}, \quad \{(3, 0), (2, 1), (1, 2), (0, 3)\}, \\ \{(3, 1), (2, 2), (1, 3)\}, \quad \{(3, 2), (2, 3)\}, \quad \{(3, 3)\} \end{aligned}$$

is an antichain partition of X . The maximal size of antichain is 4 and the poset X can be partitioned into 4 disjoint chains:

$$\begin{array}{ll}
 (0, 0) < (0, 1) < (0, 2) < (0, 3) < (1, 3) < (2, 3) < (3, 3), & \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 3), (2, 3), (3, 3)\}, \\
 (1, 0) < (1, 1) < (1, 2) < (2, 2) < (3, 2), & \{(1, 0), (1, 1), (1, 2), (2, 2), (3, 2)\}, \\
 (2, 0) < (2, 1) < (3, 1), & \{(2, 0), (2, 1), (3, 1)\}, \\
 (3, 0). & \{(3, 0)\}.
 \end{array}$$

The Hasse diagram of the poset X is



Finding an antichain of maximal size for a poset is a difficult problem. So far there is no canonical way to do this job.