Week 8-9: The Inclusion-Exclusion Principle

March 31, 2005

1 The Inclusion-Exclusion Principle

Let S be a finite set, and let A, B, C be subsets of S. Then

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Let $P_1, P_2, \ldots, P_n$ be properties referring to the objects in S. Let A_i be the set of all elements of S that have the property P_i, i.e.,

$$A_i = \{x \in S : x \text{ has the property } P_i\}, \quad 1 \le i \le n.$$

The elements of A_i may possibly have properties other than P_i. In many occasions we need to find the number of objects having none of the properties $P_1, P_2, \ldots, P_n$.

Theorem 1.1. *The number of objects of S which have none of the properties $P_1, P_2, \ldots, P_n$ is given by*

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| &= |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots \\ &\quad \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|. \end{aligned} \tag{1}$$

Proof. The left side of (1) counts the number of objects of S with none of the properties. We establish the identity (1) by showing that an object with none the properties makes a net contribution of 1 to the right side of (1), and for an object with at least one of the properties makes a net contribution of 0.

Let x be an object having none of the properties. Then the net contribution of x to the right side of (1) is

$$1 - 0 + 0 - 0 + \cdots + (-1)^n 0 = 1.$$

Let x be an object of S having exactly r properties of $P_1, P_2, \ldots, P_n$. The net contribution of x to the right side of (1) is

$$\binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = (1-1)^r = 0.$$

□

Corollary 1.2. *The number of objects of S which have at least one of the properties $P_1, P_2, \ldots, P_n$ is given by*

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots \\ &\quad \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|. \end{aligned} \tag{2}$$

Proof. Note that the set $A_1 \cup A_2 \cup \cdots \cup A_n$ consists of all those objects in S which possess at least one of the properties, and

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |S| - |\overline{A_1 \cup A_2 \cup \cdots \cup A_n}|.$$

Then by the DeMorgan law we have

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n.$$

Thus

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |S| - |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n|.$$

Putting this into the identity (1), the identity (2) follows immediately. □

2 Combinations with Repetition

Let M be a multiset. Let x be an object of M and its repetition number is larger than r. Let M' be the multiset whose objects have the same repetition numbers as those objects in M, except that the repetition number of x in M' is exactly r. Then

$$\#\{r\text{-combinations of } M\} = \#\{r\text{-combinations of } M'\}.$$

Example 2.1. Determine the number of 10-combinations of the multiset $M = \{3a, 4b, 5c\}$.

Let S be the set of 10-combinations of the multiset $M' = \{\infty a, \infty b, \infty c\}$. Let P_1 be the property that a 10-combination of M' has more than 3 a's, let P_2 be the property that a 10-combination of M' has more than 4 b's, and let P_3 be the property that a 10-combination of M' has more than 5 c's. Then the number of 10-combinations of M is the number of 10-combinations of M' which have none of the properties P_1, P_2, and P_3. Let A_i be the sets consisting of the 10-combinations of M' which have the property P_i, $1 \le i \le 3$. Then by inclusion-exclusion principle the number to be determined in the problem is given by

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - \Big(|A_1| + |A_2| + |A_3|\Big) + \Big(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\Big) - |A_1 \cap A_2 \cap A_3|.$$

Note that

$$\begin{aligned}
|S| &= \left\langle {3 \atop 10} \right\rangle = \binom{3+10-1}{10} = \binom{12}{10} = 66,\\
|A_1| &= \left\langle {3 \atop 6} \right\rangle = \binom{3+6-1}{6} = \binom{8}{6} = 28,\\
|A_2| &= \left\langle {3 \atop 5} \right\rangle = \binom{3+5-1}{5} = \binom{7}{5} = 21,\\
|A_3| &= \left\langle {3 \atop 4} \right\rangle = \binom{3+4-1}{4} = \binom{6}{4} = 15,\\
|A_1 \cap A_2| &= \left\langle {3 \atop 1} \right\rangle = \binom{3+1-1}{1} = \binom{3}{1} = 3,\\
|A_1 \cap A_3| &= \left\langle {3 \atop 0} \right\rangle = \binom{3+0-1}{0} = \binom{2}{0} = 1,\\
|A_2 \cap A_3| &= 0,\\
|A_1 \cap A_2 \cap A_3| &= 0.
\end{aligned}$$

Putting all these results into the inclusion-exclusion formula, we have

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = 66 - (28 + 21 + 15) + (3 + 1 + 0) - 0 = 6.$$

The six 10-combinations are listed as

$$\{3a, 4b, 3c\}, \quad \{3a, 3b, 4c\}, \quad \{3a, 2b, 5c\}, \quad \{2a, 4b, 4c\}, \quad \{2a, 3b, 5c\}, \quad \{a, 4b, 5c\}.$$

Example 2.2. Find the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

which satisfy the conditions

$$2 \le x_1 \le 6, \quad -2 \le x_2 \le 1, \quad 0 \le x_3 \le 6, \quad 3 \le x_4 \le 8.$$

Let $y_1 = x_1 - 2$, $y_2 = x_2 + 2$, $y_3 = x_3$, and $y_4 = x_4 - 3$. Then the problem becomes to find the number of nonnegative integral solutions of the equation

$$y_1 + y_2 + y_3 + y_4 = 12$$

subject to

$$0 \le y_1 \le 4, \quad 0 \le y_2 \le 3, \quad 0 \le y_3 \le 6, \quad 0 \le y_4 \le 5.$$

Let S be the set of all nonnegative integral solutions of the equation $y_1 + y_2 + y_3 + y_4 = 12$. Let P_1 be the property that $y_1 \ge 5$, P_2 the property that $y_2 \ge 4$, P_3 the property that $y_3 \ge 7$, and P_4 the property that $y_4 \ge 6$.

Let A_i denote the subset of S consisting of the solutions satisfying the property P_i, $1 \le i \le 4$. Then the problem is to find the cardinality $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4|$ by the inclusion-exclusion principle. In fact,

$$|S| = \left\langle \begin{matrix} 4 \\ 12 \end{matrix} \right\rangle = \binom{4+12-1}{12} = \binom{15}{12} = 455.$$

Similarly,

$$\begin{aligned} |A_1| &= \left\langle \begin{matrix} 4 \\ 7 \end{matrix} \right\rangle = \binom{4+7-1}{7} = \binom{10}{7} = 120, \\ |A_2| &= \left\langle \begin{matrix} 4 \\ 8 \end{matrix} \right\rangle = \binom{4+8-1}{8} = \binom{11}{8} = 165, \\ |A_3| &= \left\langle \begin{matrix} 4 \\ 5 \end{matrix} \right\rangle = \binom{4+5-1}{5} = \binom{8}{5} = 56, \\ |A_4| &= \left\langle \begin{matrix} 4 \\ 6 \end{matrix} \right\rangle = \binom{4+6-1}{6} = \binom{9}{6} = 84. \end{aligned}$$

For the intersections of two sets, we have

$$|A_1 \cap A_2| = \left\langle \begin{matrix} 4 \\ 3 \end{matrix} \right\rangle = \binom{4+3-1}{3} = \binom{6}{3} = 20,$$

$$|A_1 \cap A_3| = 1, \quad |A_1 \cap A_4| = 4, \quad |A_2 \cap A_3| = 4, \quad |A_2 \cap A_4| = 10, \quad |A_3 \cap A_4| = 0.$$

For the intersections of more sets,

$$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = |A_1 \cap A_2 \cap A_3 \cap A_4| = 0.$$

Thus the number required is given by

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = 455 - (120 + 165 + 56 + 84) + (20 + 1 + 4 + 4 + 10) = 69.$$

3 Derangements

A permutation of $\{1, 2, \ldots, n\}$ is called a *derangement* if every integer i $(1 \le i \le n)$ is not placed at the ith position. We denote by D_n the number of derangements of $\{1, 2, \ldots, n\}$.

Let S be the set of all permutations of $\{1, 2, \ldots, n\}$. Then $|S| = n!$. Let P_i be the property that a permutation of $\{1, 2, \ldots, n\}$ has the integer i in its ith position, and let A_i be the set of all permutations satisfying the property P_i, where $1 \le i \le n$. Then

$$D_n = |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n|.$$

For each $(i_1, i_2, \ldots, i_k)$ such that $1 \le i_1 < i_2 < \cdots < i_k \le n$, a permutation of $\{1, 2, \ldots, n\}$ with $i_1, i_2, \ldots, i_k$ fixed at the i_1th, i_2th, …, i_kth position respectively can be identified as a permutation of the set $\{1, 2, \ldots, n\} - \{i_1, i_2, \ldots, i_k\}$ of $n - k$ objects. Thus

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!.$$

By the inclusion-exclusion principle, we have

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| &= |S| + \sum_{k=1}^{n} (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \\ &= n! + \sum_{k=1}^{n} (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} (n-k)! \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \simeq \frac{n!}{e}. \end{aligned}$$

Theorem 3.1. *For $n \geq 1$, the number D_n of derangements of $\{1, 2, \ldots, n\}$ is given by*

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right). \tag{3}$$

Corollary 3.2. *The number of permutations of $\{1, 2, \ldots, n\}$ with exactly k numbers displaced equals*

$$\binom{n}{k} D_k.$$

Here are a few derangement numbers:

$$D_1 = 0, \quad D_2 = 1, \quad D_3 = 2, \quad D_4 = 9, \quad D_5 = 44.$$

Proposition 3.3. *The derangement sequence D_n satisfies the recurrence relation*

$$D_n = (n-1)(D_{n-1} + D_{n-2}), \quad n \geq 3$$

with the initial condition $D_1 = 0, D_2 = 1$. The sequence D_n satisfies the recurrence relation

$$D_n = nD_{n-1} + (-1)^n, \quad n \geq 2.$$

Proof. The recurrence relations can be proved without using the formula (3). Let S_k be the set of derangements $ka_2a_3 \cdots a_n$ of $\{1, 2, \ldots, n\}$ that have k at the beginning, $k = 2, 3, \ldots, n$. The derangements in each S_k can be partitioned into two types:

$$ka_2a_3 \cdots a_k \cdots a_n \ (a_k \neq 1) \quad \text{and} \quad ka_2a_3 \cdots a_{k-1}1a_{k+1} \cdots a_n$$

There are D_{n-1} derangements of the first type and D_{n-2} derangements of the second type. We thus obtain the recurrence relation

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Let us rewrite the recurrence relations as

$$D_n - nD_{n-1} = -\Big(D_{n-1} - (n-1)D_{n-2}\Big), \quad n \geq 3.$$

Applying this recurrence formula continuously, we have

$$D_n - nD_{n-1} = (-1)^{n-2}(D_2 - D_1) = (-1)^n.$$

Hence $D_n = nD_{n-1} + (-1)^n$. □

4 Surjective Functions

Let X be a set with m objects and let Y be a set with n objects. Then the number of functions from X to Y is

$$n^m.$$

The number of injective functions from X to Y is

$$\binom{n}{m} m! = P(n, m).$$

Let $C(m, n)$ denote the number of surjective functions from X to Y. What is $C(m, n)$?

Theorem 4.1. *The number $C(m, n)$ of surjective functions from a set of m objects to a set of n objects is given by*

$$C(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Proof. Let S be the set of all functions from X to Y, and write $Y = \{b_1, b_2, \ldots, b_n\}$. Let A_i be the set of all functions f such that b_i is not assigned to any element of X by f, i.e., $b_i \notin f(X)$, where $1 \le i \le n$. Then

$$C(m,n) = |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n|.$$

For each $(i_1, i_2, \ldots, i_k)$ such that $1 \le i_1 < i_2 < \cdots < i_k \le n$, the set $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ can be identified to the set of all functions f from X to the complement $Y - \{i_1, i_2, \ldots, i_k\}$. Thus

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)^m.$$

By the inclusion-exclusion principle, we have

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| &= |S| + \sum_{k=1}^{n} (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \\ &= n^m + \sum_{k=1}^{n} (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} (n-k)^m \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m. \end{aligned}$$

□

Note that $C(m,n) = 0$ for $m < n$; we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = 0 \quad \text{if} \quad m < n.$$

Corollary 4.2. *For integers $m, n \ge 1$,*

$$\sum_{\substack{i_1 + \cdots + in = m \\ i_1, \ldots, in \ge 1}} \binom{m}{i_1, \ldots, i_n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Proof. The integer $C(m,n)$ can be interpreted as the number of ways to place the objects of X into n distinct boxes such that no one is empty. We then have

$$C(m,n) = \sum_{\substack{i_1 + \cdots + in = m \\ i_1, \ldots, in \ge 1}} \binom{m}{i_1, \ldots, i_n}.$$

□

5 The Euler Phi Function

Let n be a positive integer. We denote by $\phi(n)$ the number of integers of $[1, n]$ which are coprime to n. For example,

$$\phi(1) = 1, \quad \phi(2) = 1, \quad \phi(3) = 2, \quad \phi(4) = 2, \quad \phi(5) = 5, \quad \phi(6) = 2.$$

The integer-valued function ϕ is defined on the set of positive integers, and is called the *Euler phi function.*

Theorem 5.1. *Let n be a positive integer and be factorized into the form $n = p_1^{e_1} p_2^{e_r} \cdots p_r^{e_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes and $e_1, e_2, \ldots, e_r \ge 1$. Then the Euler function $\phi(n)$ is given by*

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right).$$

Proof. Let $S = \{1, 2, \ldots, n\}$. Let P_i be the property that an integer of S has p_i as a factor, and let A_i be the set of all integers in S that have the property P_i, where $1 \le i \le r$. Then $\phi(n)$ is the number of integers that have none of the properties $P_1, P_2, \ldots, P_r$, i.e.,

$$\phi(n) = |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_r|.$$

Note that

$$A_i = \left\{ p_i, 2p_i, 3p_i, \ldots, \left(\frac{n}{p_i}\right) p_i \right\}, \quad 1 \le i \le r.$$

More generally, if $1 \le i_1 < i_2 < \cdots < i_k \le r$, then

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} = \left\{ q, 2q, 3q \ldots, \left(\frac{n}{q}\right) q \right\}, \quad \text{where} \quad q = p_{i_1} p_{i_2} \cdots p_{i_k}.$$

Thus

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_k}}.$$

By the inclusion-exclusion principle, we have

$$\begin{aligned}
|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_r| &= |S| + \sum_{k=1}^{r} (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \\
&= n + \sum_{k=1}^{r} (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_k}} \\
&= n \left[1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_r} \right) \right. \\
&\quad + \left(\frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \cdots + \frac{1}{p_{r-1} p_r} \right) \\
&\quad - \left(\frac{1}{p_1 p_2 p_3} + \frac{1}{p_1 p_2 p_4} + \cdots + \frac{1}{p_{r-2} p_{r-1} p_r} \right) \\
&\quad \left. + \cdots + (-1)^r \frac{1}{p_1 p_2 \cdots p_r} \right] \\
&= n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i} \right).
\end{aligned}$$

□

Example 5.1. For the integer 36, since $2^2 3^2$ we have

$$\phi(36) = 36 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) = 12.$$

The following are the twelve specific integers of $[1, 36]$ that are coprime to 36:

$$1,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 25,\ 29,\ 31,\ 35.$$

Corollary 5.2. *For any prime number* p,

$$\phi(p^k) = p^k - p^{k-1}.$$

Proof. The result can be directly proved without Theorem 5.1. An integer a of $[1, p^k]$ that is not coprime to p^k must be of the form $a = ip$, where $1 \le i \le p^{k-1}$. Thus the number of integers of $[1, p^k]$ that is coprime to p^k equals $p^k - p^{k-1}$. Therefore $\phi(p^k) = p^k - p^{k-1}$. □

Lemma 5.3. *Let* $m = m_1 m_2$. *If* $\gcd(m_1, m_2) = 1$, *then the function*

$$f : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, m_1\} \times \{1, 2, \ldots, m_2\}, \quad a \mapsto f(a) = (r_1, r_2),$$

is a bijection, where $a = q_1m_1 + r_1 = q_2m_2 + r_2$, $1 \le r_1 \le m_1$, $1 \le r_2 \le m_2$. *Moreover, the restriction*

$$f : \Big\{a \in [1,m] : \gcd(a,m) = 1\Big\} \longrightarrow \Big\{a \in [1,m_1] : \gcd(a,m_1) = 1\Big\} \times \Big\{a \in [1,m_2] : \gcd(a,m_2) = 1\Big\}$$

is also a bijection.

Proof. It suffices to show that f is surjective. Since $\gcd(m_1, m_2) = 1$, there are integers x and y such that $xm_1 + ym_2 = 1$. For any $(r_1, r_2) \in [1, m_1] \times [1, m_2]$, let $r = r_2xm_1 + r_1ym_2$. Then

$$r = (r_2 - r_1)xm_1 + r_1(xm_1 + ym_2) = (r_1 - r_2)ym_2 + r_2(xm_1 + ym_2).$$

Putting $xm_1 + ym_2 = 1$ into the above expression, we have

$$r = (r_2 - r_1)xm_1 + r_1 = (r_1 - r_2)ym_2 + r_2.$$

Modify r by adding an appropriate multiple qm of m to obtain a number $a = qm + r$ so that $1 \le r \le m$. We thus have $f(a) = (r_1, r_2)$. Hence f is surjective. Since $[1, m]$ and $[1, m_1] \times [1, m_2]$ have the same cardinality, it follows that f must be a bijection.

The second part follows from the fact that an integer a is coprime to m_1m_2 if and only if a is coprime to both m_1 and m_2. □

Theorem 5.4. *If* $\gcd(m, n) = 1$, *then*

$$\phi(mn) = \phi(m)\phi(n).$$

Moreover, if $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ *with* $e_1, e_2, \ldots, e_r \ge 1$, *then*

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right).$$

Proof. The first part follows from Lemma 5.3. The second part follows from the first part, i.e.,

$$\phi(n) = \prod_{i=1}^{r} \phi(p_i^{e_i}) = \prod_{i=1}^{r} \left(p_i^{e_i} - p_i^{e_i - 1}\right) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right).$$

□

6 Permutations with Forbidden Positions

Let $S = \{1, 2, \ldots, n\}$. Let $X_1, X_2, \ldots, X_n$ be subsets (possibly empty) of S. We denote by $P(X_1, X_2, \ldots, X_n)$ the set of all permutations $a_1a_2 \cdots a_n$ of S such that

$$a_1 \notin X_1, \quad a_2 \notin X_2, \quad \ldots, \quad a_n \notin X_n.$$

In other words, a permutation of S belongs to $P(X_1, X_2, \ldots, X_n)$ provided that no elements of X_1 occupy the first place, no elements of X_2 occupy the second place, ..., and no elements of X_n occupy the nth place. We denote by $p(X_1, X_2, \ldots, X_n)$ the number of permutations in $P(X_1, X_2, \ldots, X_n)$, i.e.,

$$p(X_1, X_2, \ldots, X_n) = |P(X_1, X_2, \ldots, X_n)|.$$

It is known that there is a one-to-one correspondence between permutations of $\{1, 2, \ldots, n\}$ and the placement of n non-attacking, indistinguishable rooks on an n-by-n board. The permutation $a_1a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$ corresponds to the placement of n rooks on the board in the squares with coordinates $(1, a_1)$, $(2, a_2)$, ..., (n, a_n). The permutations in $P(x_1, X_2, \ldots, X_n)$ corresponds to placements of n non-attacking rooks on an n-by-n board in which certain squares are not allowed to be put a rook.

Let S be the set of all placements of n non-attacking rooks on an $n \times n$-board. A rook placement in S is called to satisfy the property P_i provided that the rook in the ith row is in a column that belongs to X_i $(i = 1, 2, \ldots, n)$.

As usual let A_i denote the set of all rook placements satisfying the property P_i ($i = 1, 2, \ldots, n$). Then by the inclusion-exclusion principle we have

$$\begin{aligned} p(X_1, X_2, \ldots, X_n) &= |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| \\ &= |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|. \end{aligned}$$

Let r_k denote the number of ways to place k non-attacking rooks on an $n \times n$-board where each of the k rooks is in a forbidden position ($k = 1, 2, \ldots, n$). Then

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k (n-k)!.$$

Theorem 6.1. *The number of ways to place n non-attacking rooks on an $n \times n$-board with forbidden positions is given by*

$$p(X_1, X_2, \ldots, X_n) = \sum_{k=0}^{n} (-1)^k r_k (n-k)!.$$

Example 6.1. Let $n = 5$ and $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$, $X_3 = \{1, 5\}$, $X_4 = \{2, 3\}$, and $X_5 = \{4, 5\}$.

×	×			
		×	×	
×				×
	×	×		
			×	×

Note that $r_1 = 10$. Since

$$|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_4| = |A_4 \cap A_5| = |A_1 \cap A_5| = 4 \cdot 3!,$$

$$|A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_4| = |A_2 \cap A_5| = |A_3 \cap A_5| = 3 \cdot 3!,$$

then $r_2 = 5(4+3) = 35$. Using the symmetry between A_1, A_2, A_3, A_4, A_5 and A_5, A_4, A_3, A_2, A_1 respectively, we see that

$$\begin{aligned} |A_1 \cap A_2 \cap A_3| &= |A_3 \cap A_4 \cap A_5| = 6 \cdot 2!, \\ |A_1 \cap A_2 \cap A_4| &= |A_2 \cap A_4 \cap A_5| = 4 \cdot 2!, \\ |A_1 \cap A_2 \cap A_5| &= |A_1 \cap A_4 \cap A_5| = 4 \cdot 2!, \\ |A_1 \cap A_3 \cap A_4| &= |A_2 \cap A_3 \cap A_5| = 4 \cdot 2! \\ |A_1 \cap A_3 \cap A_5| &= 3 \cdot 2!, \\ |A_2 \cap A_3 \cap A_4| &= 6 \cdot 2!. \end{aligned}$$

These can be obtained by considering the following six patterns:

×	×			
		×	×	
×				×

×	×			
		×	×	
	×	×		

×	×			
		×	×	
			×	×

×	×			
×				×
	×	×		

×	×			
×				×
			×	×

		×	×	
×				×
	×	×		

We then have $r_3 = 2 \cdot 6 + 6 \cdot 4 + 3 + 6 = 45$. Using symmetric again, we see that

$$\begin{aligned} |A_1 \cap A_2 \cap A_3 \cap A_4| &= |A_1 \cap A_2 \cap A_3 \cap A_5| = |A_1 \cap A_2 \cap A_4 \cap A_5| \\ &= |A_1 \cap A_3 \cap A_4 \cap A_5| = |A_2 \cap A_3 \cap A_4 \cap A_4| = 5 \cdot 1!. \end{aligned}$$

Thus $r_4 = 5 \cdot 5 = 25$. Finally, $r_5 = |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| = 2$. The answer is given by

$$5! - 10 \cdot 4! + 35 \cdot 3! - 45 \cdot 2! + 25 \cdot 1! - 2 = 23.$$

A permutation of $\{1, 2, \ldots, n\}$ is called *nonconsecutive* if $12, 23, \ldots, (n-1)n$ do not occur. We denote by Q_n the number of nonconsecutive permutations of $\{1, 2, \ldots, n\}$.

Theorem 6.2. *For $n \geq 1$, the number of nonconsecutive permutations of $\{1, 2, \ldots, n\}$ is given by*

$$Q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!.$$

Proof. Let S be the set of all permutations of $\{1, 2, \ldots, n\}$. Let P_i be the property that in a permutation the pattern $i(i+1)$ does occur, and let A_i be the set of all permutations satisfying the property P_i, $1 \leq i \leq n-1$. Then Q_n is equal to the number of permutations that satisfy none of the properties, i.e., $Q_n = |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n|$. Note that

$$|A_i| = (n-1)!, \quad i = 1, 2, \ldots, n-1.$$

Similarly,

$$|A_i \cap A_j| = (n-2)!, \quad i < j.$$

More generally,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!.$$

Thus by the inclusion-exclusion principle,

$$Q_n = |S| + \sum_{k=1}^{n-1} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n-1} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!.$$

□

Example 6.2. Suppose 8 persons line up in one column in such that way every person except the first one has a person in front. What is the chance when the 8 persons reline up after a break so that everyone has a different person in his front?

We assign numbers $1, 2, \ldots, 8$ to the 8 boys so that the number i is assigned to the ith boy (counted from the front). Then the problem becomes to find the number of permutations of $\{1, 2, \ldots, 8\}$ in which the patterns 12, 23, ..., 78 do not occur. For instance, 31542876 is an allowed permutation, while 83475126 is not.

The answer is given by

$$P = \frac{Q_8}{8!} = \sum_{k=0}^{7} (-1)^k \binom{7}{k} \frac{(8-k)!}{8!}.$$

Example 6.3. There are n persons seating at a round table. The n persons left the table and reseat after a break. How many seating plans can be made in the second time so that each person has a different person seating on his/her left comparing to the person before break.

This is equivalent to finding the number of circular nonconsecutive permutations of $\{1, 2, \ldots, n\}$. A *circular nonconsecutive* permutation of $\{1, 2, \ldots, n\}$ is circular permutation of $\{1, 2, \ldots, n\}$ such that 12, 23, ..., $(n-1)n$, $n1$ do not occur in the counterclockwise direction.

Let S be the set of all circular permutation of $\{1, 2, \ldots, n\}$. Let A_i denote the subset of all circular permutations of $\{1, 2, \ldots, n\}$ such that $i(i+1)$ does not occur, $1 \leq i \leq n$. We understand that A_n is the subset of all circular permutations that $n1$ does not occur. Then the answer is $|\bar{A}_1 \cap \bar{A}_1 \cap \cdots \cap \bar{A}_n|$. Note that

$$|A_i| = (n-1)!/(n-1) = (n-2)!.$$

More generally,

$$|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!/(n-k) = (n-k-1)!.$$

$$|\bar{A}_1 \cap \cdots \cap \bar{A}_n| = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k-1)! + (-1)^n.$$

Theorem 6.3.

$$Q_n = D_n + D_{n-1}, \quad n \geq 2.$$

Proof.

$$\begin{aligned} D_n + D_{n-1} &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \\ &= (n-1)! \left(n + n \sum_{k=1}^{n} \frac{(-1)^k}{k!} + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(k-1)!} \right) \\ &= n! + (n-1)! \sum_{k=1}^{n} \frac{(-1)^k}{k!} (n-k) \\ &= n! + \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} (n-k)! \\ &= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)! = Q_n. \end{aligned}$$

□

Definition 6.4. Let C be a board. Let $r_k(C)$ be the number of ways to arrange k rooks on the board C so that no one can take another; $r_0(C) = 1$. The polynomial

$$R(C, x) = \sum_{k=0}^{\infty} r_k(C) x^k$$

is called the *rook polynomial* of C.

Proposition 6.5. *Let C be a board. Fix a square σ. Let C_σ denote the board obtained from C by deleting all squares on the row and column that contains the square σ. Let $C - \sigma$ denote the bard obtained from C by deleting the square σ. Then*

$$r_k(C) = r_k(C - \sigma) + r_{k-1}(C_\sigma).$$

Equivalently,

$$R(C, x) = R(C - \sigma, x) + xR(C_\sigma, x).$$

Proof. The k rook arrangements on the board C can be divided into two kinds: the rook arrangements that the square σ is occupied and the rook arrangements that the square is not occupied, i.e., the the rook arrangements on the board $C - \sigma$ and the rook arrangements on the board C_σ. Thus $r_k(C) = r_k(C - \sigma) + r_{k-1}(C_\sigma)$. □

Two chessboards C_1 and C_2 are called *independent* if they have no common rows and common columns. If so the boards C_1 and C_2 must be disjoint.

Proposition 6.6. *Let C_1 and C_2 be independent chessboards, then*

$$r_k(C) = \sum_{i=0}^{k} r_i(C_1) r_{k-i}(C_2).$$

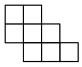
Equivalently,

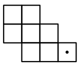
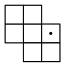
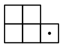
$$R(C_1 + C_2, x) = R(C_1, x) R(C_2, x).$$

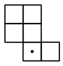
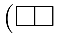
Proof. Since C_1 and C_2 have disjoint rows and columns, then each i rook arrangement of C_1 and each j rook arrangement of C_2 will constitute a $i + j$ rook arrangement of $C_1 + C_2$, and vice versa. Thus

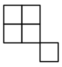
$$r_k(C_1 + C_2) = \sum_{\substack{i+j=k \\ i,j \geq 0}} r_i(C_1) r_j(C_2).$$

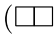
□

Example 6.4. Find the rook polynomial of the board  . We use ⊡ (a square with dot) to denote a selected square when applying the recurrence formula of rook polynomial.

$R\Big($  $, x\Big) = R\Big($  $, x\Big) + xR\Big($  $, x\Big)$

$= \Big[R\Big($  $, x\Big) + xR\Big($  $, x\Big)\Big] + x\Big[R\Big($  $, x\Big) + xR\big($  $, x\big)\Big]$

$= \Big\{\Big[R\Big($  $, x\Big) + xR\Big($  $, x\Big)\Big] + x\Big[R\Big($  $, x\Big) + xR(\emptyset, x)\Big]\Big\}$

$+x\Big[R\Big($  $, x\Big) + xR\big($  $, x\big)\Big]$

$$\begin{aligned}
&= \Big\{\Big[(1+4x+2x^2)(1+x)+x(1+2x)\Big]+x\left[(1+x)^2+x\right]\Big\}\\
&\quad +x\Big[(1+4x+2x^2)+x(1+2x)\Big]\\
&= 1+8x+16x^2+7x^3.
\end{aligned}$$

7 Weighted Version of Inclusion-Exclusion Principle

Let X be a set. The *characteristic function* of a subset A of X is a real-valued function 1_A defined on X by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For (real-valued) functions f, g and a real number c, we define functions $f+g$, cf, and fg as follows: For $x \in X$,

$$(f+g)(x) = f(x) + g(x),$$
$$(af)(x) = af(x),$$
$$fg(x) = f(x)g(x).$$

The *size* of a function f on S is the value

$$|f| = \sum_{x\in X} f(x).$$

Clearly, for any functions f_i and constants c_i $(1 \le i \le n)$, we have

$$\left|\sum_{i=1}^{n} c_i f_i\right| = \sum_{i=1}^{n} c_i |f_i|.$$

Let A and B are subsets of X. Note that

1. $1_{A\cap B} = 1_A 1_B$,
2. $1_{\bar{A}} = 1_S - 1_A$,
3. $1_{A\cup B} = 1_A + 1_B - 1_{A\cap B}$,
4. $1_\emptyset f = 1_\emptyset$ and $1_X f = f$ for any function f on S.

Proposition 7.1. *Let P_i be some properties about the elements of a set S, and let A_i be the set of all elements of S that satisfy the property P_i, $1 \le i \le n$. Then the inclusion-exclusion principle can be stated as*

$$1_{\bar{A}_1\cap\bar{A}_2\cap\cdots\cap\bar{A}_n} = 1_S + \sum_{k=1}^{n}(-1)^k \sum_{1\le i_1<i_2<\cdots<i_k\le n} 1_{A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}}. \tag{4}$$

Proof.

$$
\begin{aligned}
1_{\bar{A}_1\cap\bar{A}_2\cap\cdots\cap\bar{A}_n} &= 1_{\bar{A}_1}1_{\bar{A}_2}\cdots 1_{\bar{A}_n} = (1_S - 1_{A_1})(1_S - 1_{A_2})\cdots(1_S - 1_{A_n}) \\
&= \sum f_1 f_2 \cdots f_n \quad (\text{where } f_i = 1_S \text{ or } -1_{A_i},\ 1\le i\le n) \\
&= \underbrace{1_S\cdots 1_S}_{n} + \sum_{\substack{i_1<\cdots<i_k \\ 1\le k\le n}} \underbrace{1_S\cdots 1_S}_{n-k}(-1_{A_{i_1}})\cdots(-1_{A_{i_k}}) \\
&= 1_S + \sum_{k=1}^{n}(-1)^k \sum_{i_1<i_2<\cdots<i_k} 1_{A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}}.
\end{aligned}
$$

□

Let w be a real-valued weight function on a set X. Then w can be extended to a function on the power set $\mathcal{P}(X)$ of X by

$$w(A) = \sum_{x\in A} w(x), \quad A \subseteq X.$$

Let $S = \{1, 2, \ldots, n\}$. We introduce two functions α and β on the power set $\mathcal{P}(S)$ of S as follows: For $I \subseteq S$,

$$\alpha(I) = \begin{cases} w\left(\bigcap_{i\in I} A_i\right) & \text{if } I \ne \emptyset \\ 0 & \text{if } I = \emptyset, \end{cases}$$

$$\beta(I) = \begin{cases} w\left(\bigcup_{i\in I} A_i\right) & \text{if } I \ne \emptyset \\ 0 & \text{if } I = \emptyset. \end{cases}$$

Theorem 7.2. *Let α and β be functions defined above. Then*

$$\beta(J) = \sum_{I\subseteq J} (-1)^{|I|-1}\alpha(I),$$

if and only if

$$\alpha(J) = \sum_{I\subseteq J} (-1)^{|I|-1}\beta(I).$$