What is combinatorics?

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Examples of combinatorial problems:

(1) Finding the number of games that n teams would play if each team played with every other team exactly once.

(2) Constructing a magic square.

(3) Attempting to trace through a network without removing your pencil from the paper and without tracing any part of the network more than once.

(4) Counting the number of poker hands which are full houses.

Historically, combinatorics has its roots in mathematical recreations and games. Many problems that were studied in the past, either for amusement or for aesthetic appeal, are today of great importance in pure and applied science. Now combinatorics is an important branch of mathematics, and its influence continues to expand. Part of the reason for the tremendous growth of combinatorics since the sixties has been the major impact that computers have had and continue to have in our society. Another reason for the recent growth of combinatorics is its applicability to disciplines that had previously had little serious contact with mathematics. It is often found that the ideas and techniques of combinatorics are being used not only in the traditional areas of mathematical application, namely, the physical sciences, but also in the social sciences, the biological sciences, information theory, and so on.

Combinatorics is concerned with arrangements of the objects of a set into patterns satisfying specified rules.

Combinatorial problems can be classified into following categories:

Existence of the arrangement.

Enumeration of the arrangements.

Classification of the arrangements.

Study of a known arrangement.

Construction of an optimal arrangement.

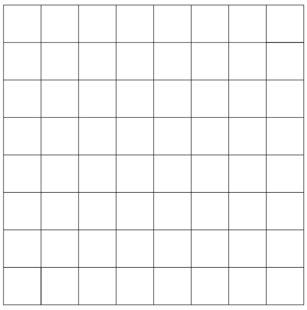
In other words, combinatorics is concerned with the existence, enumeration, analysis, and optimization of discrete structures.

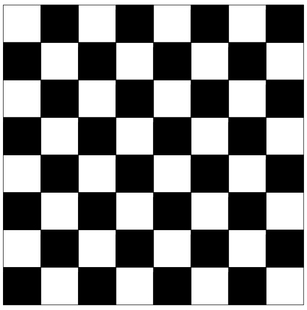
There are very few general methods in combinatorics that can apply to solve large number of combinatorial problems. The typical general methods in combinatorics: Induction; inclusion-exclusion principle, pigeonhole principle; bijective counting; methods of recurrence relations and generating function; Burnside's theorem and Pólya counting; and Möbius inversion formula.

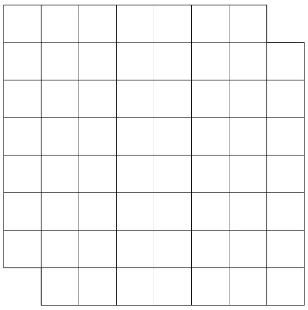
2 Examples

2.1 Perfect Cover of Chessboards

It is obvious that the chessboard can be covered by 32 dominoes so that no two dominoes overlap. Such a cover is called a *perfect cover*. However, the chessboard with two diagonal corners removed cannot be



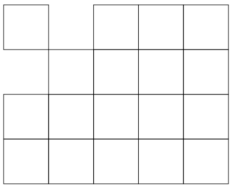


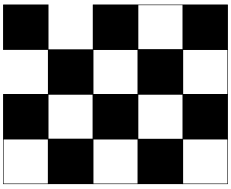


perfectly covered by 31 dominoes since

$$31BW \neq 32B + 30W,$$

The following board cannot be perfectly covered by dominoes.





A *perfect cover* of an m-by-n board by b-ominoes is an arrangement of b-ominoes on the board so that no two b-ominoes overlap. When does an $m \times n$-board have a perfect cover by b-ominoes? The answer is given by the following theorem.

Theorem 2.1. *An m-by-n-board have a perfect cover by b-ominoes if and only if b divides either m or n.*

Proof. "$\Leftarrow$:" The condition is obviously sufficient.

"$\Rightarrow$:" Let m and n be divided by b to have remainders r and s respectively; i.e.,

$$\begin{aligned} m &= pb + r, \quad 0 \leq r < b \\ n &= qb + s, \quad 0 \leq s < b. \end{aligned}$$

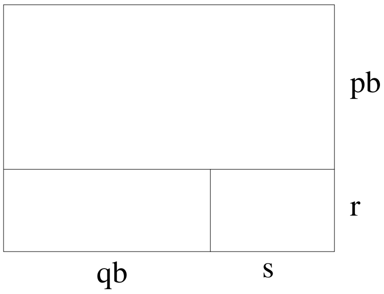
Without loss of generality we may assume $r \leq s$. We claim that $r = 0$.

1	2	3	$\cdots$	$b-1$	b
b	1	2	$\cdots$	$b-2$	$b-1$
$b-1$	b	1	$\cdots$	$b-3$	$b-2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
3	4	5	$\cdots$	1	2
2	3	4	$\cdots$	b	1

For example, for $m = 10$, $n = 15$, and $b = 4$, the 10-by-15 board can be colored as

1	2	3	4	1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2	3	4	1	2

Note that each b-omino of a perfect covering covers exactly one square of each of the b colors, no matter how the b-omino is placed. It follows that there are must be the same number of squares of each color of the board. Let the m-by-n board be divided into three parts as follows:



Obviously, the upper part is perfectly covered by b-ominoes; the lower left part is also perfectly covered by b-ominoes. It follows that the number of squares of each color in the r-by-s board is the same as the number of squares having the color 1. Since the number of squares of the color 1 in the r-by-s board is r, there are rb squares in the r-by-s board. Thus have $rs = rb$. If $r > 0$, then $s = b$, a contradiction. Hence, $r = 0$. This means that $b|m$. □

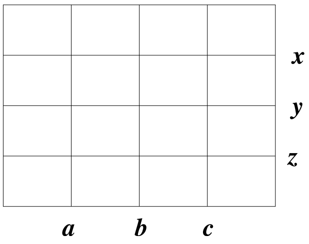
2.2 Cutting a cube

What is the minimal number of cuts to cut a cube of side length 3 into 27 small cubes of unit side length? Geometrically, it is easy to see that each cut must parallel to a face of the cube. It is also easy to see that 6 cuts are enough to do the job. The problem is whether 6 is minimal or not.

A *fault-line* for a perfect cover of a board is either a horizontal line or a vertical line that does not cut any domino in the cover.

Theorem 2.2. *Every perfect cover has a fault-line.*

We only prove the theorem for 4-by-4 board. Suppose the 4×4-board has a perfect cover which has no fault-line. Let a, b, c, x, y, z be the number of dominoes cut by the three vertical lines and the three horizontal lines, respectively. Since there is no fault line, the numbers a, b, c, x, y, z are positive. By try-and-error, we see that $a, b, c, x, y, z \geq 2$. It is clear that no dominoes can be cut by more than two lines. Then there are at least



$$a + b + c + x + y + z \geq 2 \cdot 6 = 12$$

dominoes in the perfect cover. However, the number of dominoes must be 8. This is a contradiction.

2.3 Magic Squares

A *magic square of order* n is an $n \times n$-array constructed out of the integers $1, 2, \ldots, n^2$ in such a way that the sum of the integers in each row, in each column, and in each of the two diagonals is the same number

s. The number s is called the *magic sum* of the magic square. For example,

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

are the magic squares of order 3 and 4 with magic sums 15 and 34, respectively.

Necessary condition:

$$1 + 2 + \cdots + n^2 = \frac{n^2(n^2+1)}{2}.$$

$$s = \frac{n(n^2+1)}{2}.$$

de la Loubère Method: This is only for constructing magic square of odd order n. A 1 is placed in the middle square of the top row. The successive integers are then placed in their natural order along a diagonal line which slopes upwards and to the right, with the following modifications:

1. When the top row is reached, the next integer is put in the bottom row as if it came immediately above the top row.
2. When the rightmost column is reached, the next integer is put in the leftmost column as if it immediately succeeded the rightmost column.
3. When a square is reached which has already been filled or when the top rightmost square is reached, the next integer is placed in the square immediately below the last square which was filled.

For instance, the magic squares of order 5 and 7 are constructed as follows:

$$\begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix}, \quad \begin{bmatrix} 30 & 39 & 48 & 1 & 10 & 19 & 28 \\ 38 & 47 & 7 & 9 & 18 & 27 & 29 \\ 46 & 6 & 8 & 17 & 26 & 35 & 37 \\ 5 & 14 & 16 & 25 & 34 & 36 & 45 \\ 13 & 15 & 24 & 33 & 42 & 44 & 4 \\ 21 & 23 & 32 & 41 & 43 & 3 & 12 \\ 22 & 31 & 40 & 49 & 2 & 11 & 20 \end{bmatrix}$$

A *magic cube* of order n is an $n \times n \times n$ cubical array constructed out of integers $1, 2, \ldots, n^3$ in such away that the sum s of the integers in the n cells of each of the following straight lines is the same:

1. lines parallel to an edge;
2. the two diagonals of each plane cross section;
3. the four space diagonals.

The number s is called the *magic sum* of the magic cube and has the value

$$\frac{n(n^3+1)}{2}.$$

There is no magic cube of order 3.

Suppose there is a magic cube of order 3. Its magic sum should be $\frac{3(3^3+1)}{2} = 42$.

For a magic cube of order 3, $1 \le i, j, k \le 3$, its magic sum should be 42. Consider any 3×3 plane cross section

$$\begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix}$$

Then

$$\begin{aligned} a+b+c &= 42 & (1) \\ x+y+z &= 42 & (2) \\ a+v+z &= 42 & (3) \\ b+v+y &= 42 & (4) \\ c+v+x &= 42 & (5) \end{aligned}$$

Do operation (3) + (4) + (5) − (1) − (2), we have $3v = 42$ and $v = 14$. This means that 14 has to be the center for any plane cross section of the magic cube. However, there are more than one such plane centers, and 14 can only occupy one place. This is a contradiction.

It is much more difficult to show that there is no magic cube of order 4. A magic cube of order 8 is given in an article by Gardner, "Mathematical games," *Scientific American,* January (1976), 118-123.

2.4 The Four-Color Problem

2.5 The problem of 36 officers

Given 36 officers of 6 ranks and from 6 regiments, can they be arranged in a 6×6 formation so that in each row and column there is one officer of each rank and one officer from each regiment? This problem can be stated as follows:

Can the 36 ordered pairs (i, j) $(i = 1, 2, \ldots, 6; j = 1, 2, \ldots, 6)$ be arranged in a 6×6 array so that in each row and each column the integers $1, 2, \ldots, 6$ occur in some order in the first positions and in some order in the second positions of the ordered pairs?

Such an array can be split into two 6×6 arrays, one corresponding to the first positions of the ordered pairs (the rank array) and the other to the second positions (the regiment array). Thus the problem can be stated:

Do there exist two 6×6 arrays whose entries are taken from the integers $1, 2, \ldots, 6$ such that (1) in each row and in each column of these arrays the integers $1, 2, \ldots, 6$ occur in some order, and (2) when the two arrays are juxtaposed all of the 36 ordered pairs (i, j) $(1 \le i, j \le 6)$ occur?

To make the problem concrete and easy, suppose instead that there 9 officers of 3 ranks and from 3 different regiments. Then a solution for the problem in this case is

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} (1,1) & (2,2) & (3,3) \\ (3,2) & (1,3) & (2,1) \\ (2,3) & (3,1) & (1,2) \end{bmatrix}$$

The rank and regiment arrays above are examples of what are called *Latin squares* of order 3; each of the integers 1, 2, and 3 occur once in each row and once in each column. The following are Latin squares of order 2 and 4:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}.$$

Two Latin squares of order 3 are called *orthogonal* because when they are juxtaposed there are all 9 possible ordered pairs (i, j) with $1 \le i \le 3$ and $1 \le j \le 3$.

Euler investigated the more general problem of orthogonal Latin squares of order n. It is easy to see that there is no pair of Latin squares of order 2.

Euler showed that how to construct a pair of orthogonal Latin squares of order n when n is odd or divisible by 4. Notice that this does not include $n = 6$. On the basis of many trials he concluded, but did not prove, that there is no pair of orthogonal Latin squares of order 6, and he conjectured that there is no such pair existed for any of integers $4k + 2$ with $k \geq 1$.

By exhaustive enumeration Terry in 1901 proved that Euler's conjecture is true for $n = 6$. Around 1960 Bose, Parker, and Shrikhande succeeded in proving that Euler's conjecture was false for all $n > 6$, i.e., for $4k + 2$ with $k \geq 2$.

2.6 Shortest path problem

2.7 The game of Nim

Nim is a game played by two players with heaps of coins. Suppose there are $k \geq 1$ heaps of coins which contain, respectively, $n_1, \ldots, n_k$ coins. The object of the game is to select the last coin. The rule of the game are the following:

1. The players alternate turns (let us call the player who makes the first move I and then call the other player II)

2. Each player, when it is their turn, selects one of the heaps and removes one or more coins from the selected heap. (The player may take all of the coins from the selected heap, thereby leaving an empty heap, which is now "out of play.")

The game ends when all the heaps are empty. The last player to make a move, that is, the player who takes the last coin(s), is the winner.

When $k = 1$, obviously, player I wins the game.

When $k = 2$, if $n_1 \neq n_2$, say, $n_1 > n_2$, player I may remove $n_1 - n_2$ coins from the fist heap to make the two heaps having the same amount of coins; such a move is called *balancing*. No matter how player II moves, the player I may adopt the *winning strategy* to remove the same amount of coins that player II moves. This guarantees that player I wins the game. If $n_1 = n_2$, the game is already balanced, no matter how player I moves at beginning, player II can take the winning strategy to win the game.

For instance,

$$(9,7) \xrightarrow{I} (7,7) \xrightarrow{II} (4,7) \xrightarrow{I} (4,4) \xrightarrow{II} (3,4) \xrightarrow{I} (3,3) \xrightarrow{II} (3,2) \xrightarrow{I} (2,2) \xrightarrow{II} (0,2) \xrightarrow{II} (0,0).$$

When there k heaps of coins, say $(n_1, n_2, \ldots, n_k)$, we write the numbers $n_1, n_2, \ldots, n_k$ as base 2 numerals:

$$\begin{aligned} n_1 &= a_s a_{s-1} \cdots a_2 a_1 a_0, \\ n_2 &= b_s b_{s-1} \cdots b_2 b_1 b_0, \\ &\vdots \\ n_k &= c_s c_{s-1} \cdots c_2 c_1 c_0. \end{aligned}$$

Dividing each heap into s subheaps, the the game becomes a game having total ks subheaps as the follows:

$$(a_s, \ldots, a_1, a_0; b_s, \ldots, b_1, b_0; \ldots; c_s, \ldots, c_1, c_0).$$

A Nim game is called *balanced* if the integers

$$a_s + b_s + \cdots + c_s, \quad \ldots, \quad a_1 + b_1 + \cdots + c_1, \quad a_0 + b_0 + \cdots + c_0$$

are even; otherwise, it is called *unbalanced.*

When the game is unbalanced, it is always possible to move certain amount of coins in the largest heap so that the game becomes balanced. When the game is unbalanced, player I wins the game since player I can always balance the game. When the game is balanced, player II wins the game. For example,

Size of heaps	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
6	0	1	1	0
10	1	0	1	0
13	1	1	0	1
15	1	1	1	1

We can take 6 coins away from the heap 2 to balance the game as

Size of heaps	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
6	0	1	1	0
4	0	1	0	0
13	1	1	0	1
15	1	1	1	1

The game can be also balanced by removing 10 coins from the heap 3 or removing 14 coins from the heap 4.