Week 10-11: Recurrence Relations and Generating Functions

April 20, 2005

1 Some Number Sequences

An *infinite sequence* (or just a *sequence* for short) is an ordered array

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_n,\ \ldots$$

of countably many real or complex numbers, and is usually abbreviated as $(a_n; n \geq 0)$ or just (a_n). A sequence (a_n) can be viewed as a function f from the set of nonnegative integers to the set of real or complex numbers, i.e.,

$$f(n) = a_n, \quad n = 0, 1, 2, \ldots$$

We call a sequence (a_n) an *arithmetic sequence* if it is of the form

$$a_0, \quad a_0 + q, \quad a_0 + 2q, \quad \ldots, \quad a_0 + nq, \quad \ldots$$

The general term satisfies the recurrence relation

$$a_n = a_{n-1} + q, \quad n \geq 1.$$

A sequence (a_n) is called a *geometric sequence* if it is of the form

$$a_0, \quad a_0 q, \quad a_0 q^2, \quad \ldots, \quad a_0 q^n, \quad \ldots$$

The general term satisfies the recurrence relation

$$a_n = q a_{n-1}, \quad n \geq 1.$$

The *partial sums* of a sequence (a_n) are the sums:

$$\begin{aligned} s_0 &= a_0, \\ s_1 &= a_0 + a_1, \\ s_2 &= a_0 + a_1 + a_2, \\ &\vdots \\ s_n &= a_0 + a_1 + \cdots + a_n, \\ &\vdots \end{aligned}$$

The partial sums form a new sequence $(s_n; n \geq 0)$.

For an arithmetic sequence $a_n = a_0 + nq$ $(n \geq 0)$, we have the partial sum

$$s_n = \sum_{k=0}^{n} (a_0 + kq) = (n+1)a_0 + \frac{qn(n+1)}{2}.$$

For a geometric sequence $a_n = a_0 q^n$ $(n \geq 1)$, we have

$$s_n = \sum_{k=0}^{n} a_0 q^n = \begin{cases} \frac{q^{n+1}-1}{q-1} a_0 & \text{if } q \neq 1 \\ (n+1)a_0 & \text{if } q = 1. \end{cases}$$

Example 1.1. Determine the number a_n of regions which are created by n mutually overlapping circles in general position on the plane. (By *mutually overlapping* we mean that each two circles intersect in two distinct points; thus non-intersecting or tangent circles are not allowed. By *general position* we mean that there are no three circles through a common point.)

We easily see that the first few numbers are given as

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4, \quad a_3 = 8.$$

It seems that we might have $a_4 = 16$. However, by try-and-error we quickly see that $a_4 = 14$.

Assume that there are n circles in general position on a plane. When we take one circle away, say the nth circle, there are $n-1$ circles in general position on the same plane. By induction hypothesis the $n-1$ circles divide the plane into a_{n-1} regions. Note that the nth circle intersects each of the $n-1$ circles in $2(n-1)$ distinct points, say the $2(n-1)$ points on the nth circle are ordered as $P_1, P_2, \ldots, P_{2(n-1)}$. Then each of the arcs

$$P_1P_2, \quad P_2P_3, \quad P_3P_4, \quad \ldots, \quad P_{2(n-2)+1}P_{2(n-1)}, \quad P_{2(n-1)}P_1$$

separate a region in the case $n-1$ circles into two regions. Then there are $2(n-1)$ more regions produced when the nth circle is drawn. We thus obtain the recurrence relation

$$a_n = a_{n-1} + 2(n-1), \quad n \geq 2.$$

Repeating the recurrence relation we have

$$\begin{aligned}
a_n &= a_{n-1} + 2(n-1) \\
&= h_{n-2} + 2(n-1) + 2(n-2) \\
&= h_{n-3} + 2(n-1) + 2(n-2) + 2(n-3) \\
&\vdots \\
&= h_1 + 2(n-1) + 2(n-2) + 2(n-3) + \cdots + 2 \\
&= h_1 + 2 \cdot \frac{(n-1)n}{2} = 2 + n(n-1) \\
&= n^2 - n + 2, \quad n \geq 2.
\end{aligned}$$

This formula is also valid for $n = 1$ (since $h_1 = 2$), although it doesn't hold for $n = 0$ (since $a_0 = 1$).

Example 1.2 (Fibonacci Sequence). A pair of newly born rabbits of opposite sexes is placed in an enclosure at the beginning of a year. Baby rabbits need one moth to grow mature. Beginning with the second month the female gives birth of a pair of rabbits of opposite sexes each month. Each new pair also gives birth to a pair of rabbits each month starting with their second month. Find the number of pairs of rabbits in the enclosure after one year?

Let f_n denote the number of pairs of rabbits at the beginning of the nth month. Some of these pairs are adult and some are babies. We denote by a_n the number of pairs of adult rabbits and denote by b_n the number of pairs of baby rabbits at the beginning of the nth month. Then the total number of pairs of rabbits at the beginning of the nth month is $f_n = a_n + b_n$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
a_n	0	1	1	2	3	5	8	13	21	34	55	89	144
b_n	1	0	1	1	2	3	5	8	13	21	34	55	89
f_n	1	1	2	3	5	8	13	21	34	55	89	144	233

At the beginning of the first month there is one pair of baby rabbits and no pair of adult rabbits. Then there is only one pair of rabbits, i.e., $a_1 = 0$, $b_1 = 1$, and $f_1 = 0 + 1 = 1$. At the beginning of the second month, the baby pair growing mature in the first month becomes an adult pair but did not give birth yet, we have $a_2 = 1$, $b_2 = 0$, and $f_2 = 1 + 0 = 1$. However, the female gives birth of a new pair of rabbits during the second month. At the beginning of the third month, there are two pairs of rabbits because the adult pair in the second month gives birth of a baby pair; so we have $a_3 = 1$, $b_3 = 1$, and $f_3 = 1 + 1 = 2$. At the beginning of the fourth month, the baby pair becomes an adult pair so that there are two adult pairs, but the the adult pair from the third month gives birth of a baby pair again; thus $a_4 = 2$, $b_4 = 1$, and $f_4 = 2 + 1 = 3$.

In general we have, (i) each pair in any month (no matter they are baby or adult) becomes an adult pair at the beginning of the next month, i.e., $a_n = f_{n-1}$, $n \geq 2$; (ii) each adult pair gives birth of a new baby pair during the month, but this new baby pair will be only counted at he beginning of the next month, i.e., $b_n = a_{n-1}$, $n \geq 2$. Thus

$$f_n = a_n + b_n = f_{n-1} + a_{n-1} = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

Let us define $f_0 = 0$. The sequence $f_0, f_1, f_2, f_3, \ldots$ satisfying the recurrence relation

$$\left\{ \begin{array}{rcl} f_n & = & f_{n-1} + f_{n-2}, \quad n \geq 2 \\ f_0 & = & 0 \\ f_1 & = & 1 \end{array} \right. \tag{1}$$

is called the *Fibonacci sequence*, and the terms in the sequence are called *Fibonacci numbers*.

Example 1.3. The partial sum of Fibonacci sequence is

$$s_n = f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1. \tag{2}$$

This can be verified by induction on n. For $n = 0$, we have $s_0 = f_2 - 1 = 0$. Now for $n \geq 1$, we assume that it is true for $n - 1$, i.e., $s_{n-1} = f_{n+1} - 1$. Then

$$\begin{array}{rcl} s_n & = & f_0 + f_1 + \cdots + f_n \\ & = & s_{n-1} + f_n \\ & = & f_{n+1} - 1 + f_n \quad \text{(by the induction hypothesis)} \\ & = & f_{n+2} - 1. \quad \text{(by the Fibonacci recurrence)} \end{array}$$

Example 1.4. The Fibonacci number f_n is even if and only if n is a multiple of 3.

Note that $f_1 = f_2 = 1$ is odd and $f_3 = 2$ is even. Assume that f_{3k} is even, f_{3k-2} and f_{3k-1} are odd. Then $f_{3k+1} = f_{3k} + f_{3k-1}$ is odd ($even + odd = odd$), and subsequently, $f_{3k+2} = f_{3k+1} + f_{3k}$ is also odd ($odd + even = odd$). It follows that $f_{3(k+1)} = f_{3k+2} + f_{3k+1}$ is even ($odd + odd = even$).

Theorem 1.1. *The general term of the Fibonacci sequence* (f_n) *is given by*

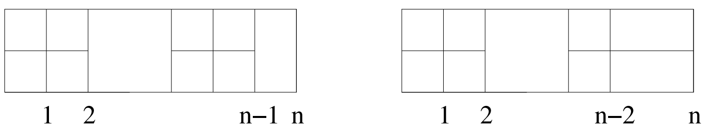
$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n, \quad n \geq 0. \tag{3}$$

Example 1.5. Determine the number h_n of ways to perfectly cover a 2-by-n board with dominoes. (Symmetries are not counted in counting the number of coverings.)

We assume $h_0 = 1$ since a 2-by-0 board is empty and it has exactly one perfect cover, namely, the empty cover. Note that the first few terms can be easily obtained such as

$$h_0 = 1, \quad h_1 = 1, \quad h_2 = 2, \quad h_3 = 3, \quad h_4 = 5.$$

Now for $n \geq 3$, the 2-by-n board can be covered by dominoes in two types:



There are h_{n-1} ways in the first type and h_{n-2} ways in the second type. Thus

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2.$$

Therefore the sequence $(h_n; n \geq 0)$ is the Fibonacci sequence $(f_n; n \geq 0)$ with $f_0 = 0$ deleted, i.e.,

$$h_n = f_{n+1}, \quad n \geq 0.$$

Example 1.6. Determine the number b_n of ways to perfectly cover a 1-by-n dominoes and monominoes.

Theorem 1.2. *The Fibonacci number f_n can be written as*

$$f_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}, \quad n \geq 0.$$

Proof. Let $g_0 = 0$ and

$$g_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}, \quad n \geq 1.$$

Note that $k > \lfloor \frac{n-1}{2} \rfloor$ is equivalent to $k > n-k-1$. Since $\binom{m}{p} = 0$ for any integers m and p such that $p > m$, we may write g_n as

$$g_n = \sum_{k=0}^{n-1} \binom{n-k-1}{k}, \quad n \geq 1.$$

To prove the theorem, it suffices to show that the sequence (g_n) satisfies the Fibonacci recurrence relation with the same initial values. In fact, $g_0 = 0$, $g_1 = \binom{0}{0} = 1$, and for $n \geq 0$,

$$\begin{aligned}
g_{n+1} + g_n &= \sum_{k=0}^{n} \binom{n-k}{k} + \sum_{k=0}^{n-1} \binom{n-k-1}{k} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n-k}{k} + \sum_{k=1}^{n} \binom{n-k}{k-1} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \left[\binom{n-k}{k} + \binom{n-k}{k-1} \right] \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n-k+1}{k} \quad \text{(By the Pascal formula)} \\
&= \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n-k+1}{k} + \binom{0}{n+1} \\
&= \sum_{k=0}^{n} \binom{(n+2)-k-1}{k} = g_{n+2}.
\end{aligned}$$

We conclude that the sequence (g_n) is the Fibonacci sequence (f_n). □

2 Linear Recurrence Relations

Definition 2.1. A sequence $(x_n; n \geq 0)$ of numbers is said to satisfy a *linear recurrence relation of order k* if

$$x_n = \alpha_1(n)x_{n-1} + \alpha_2(n)x_{n-2} + \cdots + \alpha_k(n)x_{n-k} + \beta_n, \quad \alpha_k(n) \neq 0, \quad n \geq k, \tag{4}$$

where the coefficients $\alpha_1(n)$, $\alpha_2(n)$, ..., $\alpha_k(n)$ are functions of n and β_n are constants. The linear recurrence relation (4) is called *homogeneous* if $\beta_n = 0$, and is said to have *constant coefficients* if $\alpha_1(n)$, $\alpha_2(n)$, ..., $\alpha_k(n)$ are constants. The recurrence relation

$$x_n = \alpha_1(n)x_{n-1} + \alpha_2(n)x_{n-2} + \cdots + \alpha_k(n)x_{n-k}, \quad \alpha_k(n) \neq 0, \quad n \geq k \tag{5}$$

is called the *corresponding homogeneous linear recurrence relation* of (4).

A solution of the linear recurrence relation (4) is any sequence (a_n) which satisfies (4). The *general solution* of (4) is a solution

$$x_n = a_n(c_1, c_2, \ldots, c_k) \tag{6}$$

with some parameters $c_1, c_2, \ldots, c_k$, such that for arbitrary initial values $x_0, x_1, \ldots, x_{k-1}$ there are constants $c_1, c_2, \ldots, c_k$ so that (6) is the unique sequence which satisfies both the recurrence relation (4) and the initial conditions.

Let S_∞ be the set of all sequences $(a_n; n \geq 0)$. It is clear that S_∞ is an infinite-dimensional vector space under the ordinary addition and scalar multiplication of sequences. Let N_k consist all solutions of the nonhomogeneous linear recurrence relation (4), and let H_k consist all solutions of the homogeneous linear recurrence relation (5). We shall see that H_k is a k-dimensional subspace of the vector space S_∞, and that N_k is a k-dimensional affine subspace of S_∞.

Theorem 2.2 (Structure Theorem for Linear Recurrence Relations). (a) *The solution space H_k is a k-dimensional subspace of the vector space S_∞ of sequences. Thus, if $(a_{n,1})$, $(a_{n,2})$, ..., $(a_{n,k})$ are linearly independent solutions of the homogeneous linear recurrence relation* (5)*, then the general solution of* (5) *is*

$$x_n = c_1 a_{n,1} + c_2 a_{n,2} + \cdots + c_k a_{n,k}, \quad n \geq 0.$$

(b) *Let (a_n) be a particular solution of the nonhomogeneous linear recurrence relation* (4)*. Then the general solution of* (4) *is*

$$x_n = a_n + h_n, \quad n \geq 0,$$

where (h_n) is the general solution of the corresponding homogeneous linear recurrence relation (5)*. In other words, N_k is a translate of H_k in S_∞, that is,*

$$N_k = (a_n) + H_k.$$

Proof. (a) To show that H_k is a vector subspace of S_∞, we need to show that H_k is closed under the addition and scalar multiplication of sequences. Let (a_n) and (b_n) be solutions of (5). Then

$$\begin{aligned} a_n + b_n &= [\alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}] + [\alpha_1(n)b_{n-1} + \alpha_2(n)b_{n-2} + \cdots + \alpha_k(n)b_{n-k}] \\ &= \alpha_1(n)(a_{n-1} + b_{n-1}) + \alpha_2(n)(a_{n-2} + b_{n-2}) + \cdots + \alpha_k(n)(a_{n-k} + b_{n-k}), \quad n \geq k; \end{aligned}$$

and for any scalars c,

$$\begin{aligned} ca_n &= c[\alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}] \\ &= \alpha_1(n)ca_{n-1} + \alpha_2(n)ca_{n-2} + \cdots + \alpha_k(n)ca_{n-k}, \quad n \geq k. \end{aligned}$$

This means that H_k is closed under the addition and scalar multiplication of sequences.

To show that H_k is k-dimensional, consider the projection $\pi : S_\infty \longrightarrow \mathbb{R}^k$ defined by

$$\pi(x_0, x_1, x_2, \ldots) = (x_0, x_1, \ldots, x_{k-1}).$$

We shall see that the restriction of π to H_k is a linear isomorphism. For any $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^k$, define a_n as

$$a_n = \alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k}, \quad n \geq k.$$

Obviously, we have $\pi(a_0, a_1, a_2, \ldots) = (a_0, a_1, \ldots, a_{k-1})$. This means that the restriction $\pi|H_k$ is from H_k onto $\mathbb{R}^k$. Now for any sequence $(x_n) \in H_k$, if $\pi(x_0, x_1, x_2, \ldots) = (0, 0, \ldots, 0)$, then $x_0 = x_1 = \cdots = x_{k-1} = 0$. Applying the recurrence relation (5) for $n = k$, we have $x_k = 0$; applying (5) again for $n = k+1$, we obtain $x_{k+1} = 0$. Continuing to apply (5), we have $x_n = 0$ for $n \geq k$. Thus (x_n) is the zero sequence. This means that π is one-to-one from H_k onto $\mathbb{R}^k$. We have finished the proof that π is a linear isomorphism from H_k to $\mathbb{R}^k$.

(b) For any solution (b_n) of (4), we claim that the sequence $h_n = b_n - a_n$ $(n \geq 0)$ is a solution of (5). So

$$b_n = a_n + h_n, \quad n \geq 0.$$

In fact, applying the recurrence relation (4), we have

$$\begin{aligned} h_n &= [\alpha_1(n)b_{n-1} + \alpha_2(n)b_{n-2} + \cdots + \alpha_k(n)b_{n-k} + \beta_n] \\ &\quad -[\alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k} + \beta_n] \\ &= \alpha_1(n)(b_{n-1} - a_{n-1}) + \alpha_2(n)(b_{n-2} - a_{n-2}) + \cdots + \alpha_k(n)(b_{n-k} - a_{n-k}) \\ &= \alpha_1(n)h_{n-1} + \alpha_2(n)h_{n-2} + \cdots + \alpha_k(n)h_{n-k}, \quad n \ge k. \end{aligned}$$

This means that (h_n) is a solution of (5). Conversely, for any solution (h_n) of (5), we have

$$\begin{aligned} a_n + h_n &= [\alpha_1(n)a_{n-1} + \alpha_2(n)a_{n-2} + \cdots + \alpha_k(n)a_{n-k} + \beta_n] \\ &\quad +[\alpha_1(n)h_{n-1} + \alpha_2(n)h_{n-2} + \cdots + \alpha_k(n)h_{n-k}] \\ &= \alpha_1(n)(a_{n-1} + h_{n-1}) + \alpha_2(n)(a_{n-2} + h_{n-2}) + \cdots + \alpha_k(n)(a_{n-k} + h_{n-k}) + \beta_n \end{aligned}$$

for $n \ge k$. This means that the sequence $(a_n + h_n)$ is a solution of (4).

□

Definition 2.3. The *Wronskian* $W_k(n)$ of k solutions $(a_{n,1})$, $(a_{n,2})$, …, $(a_{n,k})$ of the homogeneous linear recurrence relation (5) is the determinant

$$W_k(n) = \det \begin{bmatrix} a_{n,1} & a_{n,2} & \cdots & a_{n,k} \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,k} \\ \vdots & \vdots & & \vdots \\ a_{n+k-1,1} & a_{n+k-1,2} & \cdots & a_{n+j-1,k} \end{bmatrix}, \quad n \ge 0.$$

Theorem 2.4. *The solutions* $(a_{n,1})$, $(a_{n,2})$, …, $(a_{n,k})$ *of the homogeneous linear recurrence relation* (5) *are linearly independent if and only if there is a nonnegative integer* n_0 *such that the Wronskian*

$$W_k(n_0) \neq 0.$$

Proof. It suffices to show that the sequences $(a_{n,1})$, $(a_{n,2})$, …, $(a_{n,k})$ are linearly dependent if and only if $W_k(n) = 0$ for all $n \ge 0$. If $(a_{n,1})$, $(a_{n,2})$, …, $(a_{n,k})$ are linearly dependent, then for any $n \ge 0$ the columns of the matrix

$$\begin{bmatrix} a_{n,1} & a_{n,2} & \cdots & a_{n,k} \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,k} \\ \vdots & \vdots & & \vdots \\ a_{n+k-1,1} & a_{n+k-1,2} & \cdots & a_{n+j-1,k} \end{bmatrix}$$

are linearly dependent because the columns are part of the sequences $(a_{n,1})$, $(a_{n,2})$, …, $(a_{n,k})$ respectively. It follows from linear algebra that the Wronskian $W_k(n) = 0$ for all $n \ge 0$.

Conversely, if $W_k(n) = 0$ for all $n \ge 0$, in particular, $W_k(0) = 0$, then there are constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1 a_{i,1} + c_2 a_{i,2} + \cdots + c_k a_{i,k} = 0 \quad \text{for} \quad 0 \le i \le k-1.$$

Thus, applying the recurrence relation (5) for the sequences $(a_{n,1})$, $(a_{n,2})$, …, $(a_{n,k})$ respectively for $n = k$, we have

$$\begin{aligned} \sum_{j=1}^{k} c_j a_{k,j} &= \sum_{j=1}^{k} c_j \sum_{i=1}^{k} \alpha_i(k) a_{k-i,j} \\ &= \sum_{i=1}^{k} \alpha_i(k) \sum_{j=1}^{k} c_j a_{k-i,j} = 0. \end{aligned}$$

Continuing to apply the recurrence relation (5) for $n \ge k+1$, we conclude that for the same constants $c_1, c_2, \ldots, c_k$,

$$c_1 a_{n,1} + c_2 a_{n,2} + \cdots + c_k a_{n,k} = 0, \quad n \ge k+1.$$

This means that the sequences $(a_{n,1})$, $(a_{n,2})$, …, $(a_{n,k})$ are linearly dependent. □

3 Homogeneous Linear Recurrence Relations with Constant Coefficients

In this section we only consider linear recurrence relations of the form

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \cdots + \alpha_k x_{n-k}, \quad \alpha_k \neq 0, \quad n \geq k, \tag{7}$$

where α_1, α_2, ..., α_k are constants. We call this kinds of recurrence relations as *homogeneous linear recurrence relations of order k with constant coefficients.* Sometimes it is convenient to write (**??**) as of the form

$$\alpha_0 x_n + \alpha_1 x_{n-1} + \cdots + \alpha_k x_{n-k} = 0, \quad n \geq k \tag{8}$$

where $\alpha_0 \neq 0$ and $\alpha_{n-k} \neq 0$. The following polynomial equation

$$\alpha_0 x^k + \alpha_1 x^{k-1} + \cdots + \alpha_{k-1} x + \alpha_k = 0, \tag{9}$$

is called the *characteristic equation* associated with the recurrence relation (8). The polynomial on the left side of (9) is called the *characteristic polynomial* of (8).

Example 3.1. The Fibonacci sequence $(f_n; n \geq 0)$ satisfies the linear recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2$$

of order 2 with $\alpha_1 = \alpha_2 = 1$ in (7).

Example 3.2. The geometric sequence $(x_n; n \geq 0)$, where $x_n = q^n$, satisfies the linear recurrence relation

$$x_n = q x_{n-1}, \quad n \geq 1$$

of order 1 with $\alpha_1 = q$ in (7).

It is quite heuristic that solutions of the first order homogeneous linear recurrence relations are geometric sequences. This hints that the recurrence relation (7) may have solutions of the form $x_n = q^n$. The following theorem confirms the speculation.

Theorem 3.1. (a) *For any number $q \neq 0$, the geometric sequence*

$$x_n = q^n$$

is a solution of the kth order homogeneous linear recurrence relation (8) *with constant coefficients if and only if the number q is a root of the characteristic equation* (9).

(b) *If the characteristic equation* (9) *has k distinct roots q_1, q_2, ..., q_k, then the general solution of* (8) *is*

$$x_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_n q_k^n, \quad n \geq 0. \tag{10}$$

Proof. (a) Put $x_n = q^n$ into the recurrence relation (8); we have

$$\alpha_0 q^n + \alpha_1 q^{n-1} + \cdots + \alpha_k q^{n-k} = 0. \tag{11}$$

Since $q \neq 0$, dividing both sides of (11) by q^{n-k}, we obtain

$$\alpha_o q^k + \alpha_1 q^{k-1} + \cdots + \alpha_{k-1} q + \alpha_k = 0 \tag{12}$$

This means that (11) and (12) are equivalent. This finishes the proof of Part (a).

(b) Since $q_1, q_2, \ldots, q_k$ are roots of the characteristic equation (9), then $x_n = q_i^n$ are solutions of the homogeneous linear recurrence relation (8) for all i $(1 \leq i \leq k)$. Since the solution space of (8) is a vector space, the linear combination

$$x_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_n q_k^n, \quad n \geq 0$$

are also solutions (8). Now given arbitrary values for $x_0, x_1, \dots, x_{k-1}$, the sequence (x_n) is uniquely determined by the recurrence relation (8). Set

$$c_1 q_1^i + c_2 q_2^i + \cdots + c_n q_k^i = x_i, \quad 0 \le i \le k-1.$$

The coefficients $c_1, c_2, \dots, c_k$ are uniquely determined by Cramer's rule as follows:

$$c_i = \frac{\det A_i}{\det A}, \quad 1 \le i \le k$$

where A is the *Vandermonde matrix*

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix},$$

and A_i is the matrix obtained from A by replacing its ith column by the column $[x_0, x_1, \dots, x_{k-1}]^T$. The determinant of A is given by

$$\det A = \prod_{1 \le i < j \le k} (q_j - q_i) \ne 0.$$

This finishes the proof of Part (b). □

Example 3.3. Find the sequence (x_n) satisfying the recurrence relation

$$x_n = 2x_{n-1} + x_{n-2} - 2x_{n-3}, \quad n \ge 3$$

and the initial conditions $x_0 = 1$, $x_1 = 2$, and $x_2 = 0$.

Solution. The characteristic equation of the recurrence relation is

$$x^3 - 2x^2 - x + 2 = 0.$$

Factorizing the equation, we have

$$(x-2)(x+1)(x-1) = 0.$$

There are three roots $x = 1, -1, 2$. By Theorem 3.1, we have the general solution

$$x_n = c_1(-1)^n + c_2 + c_3 2^n.$$

Applying the initial conditions,

$$\begin{cases} c_1 & +c_2 & +c_3 & = & 1 \\ c_1 & -c_2 & +2c_3 & = & 2 \\ c_1 & +c_2 & +4c_3 & = & 0 \end{cases}$$

Solving the linear system we have $c_1 = 2$, $c_2 = -2/3$, $c_3 = -1/3$. Thus

$$x_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}2^n.$$

Theorem 3.2. (a) *Let q be a root with multiplicity m of the characteristic equation* (9) *associated with the kth order homogeneous linear recurrence relation* (8) *with constant coefficients. Then the m sequences*

$$x_n = q^n, \quad nq^n, \quad \dots, \quad n^{m-1}q^n$$

are linearly independent solutions of the recurrence relation (8).

(b) *Let $q_1, q_2, \ldots, q_s$ be distinct roots with the multiplicities $m_1, m_2, \ldots, m_s$ respectively for the characteristic equation* (9)*. Then the sequences*

$$\begin{array}{rcllll} x_n & = & q_1^n, & nq_1^n, & \ldots, & n^{m_1-1}q_1^n; \\ & & q_2^n, & nq_2^n, & \ldots, & n^{m_2-1}q_2^n; \\ & & & & \vdots & \\ & & q_s^n, & nq_s^n, & \ldots, & n^{m_s-1}q_s^n; \quad n \geq 0 \end{array}$$

are linearly independent solutions of the homogeneous linear recurrence relation (8)*. Their linear combinations form the general solution of the recurrence relation* (8)*.*

4 Nonhomogeneous Linear Recurrence Relations with Constant Coefficients

Theorem 4.1. *Given a nonhomogeneous linear recurrence relation of the first order*

$$x_n = \alpha x_{n-1} + \beta_n. \tag{13}$$

(a) *Let $\beta_n = cq^n$ be an exponential function of n. Then* (13) *has a particular solution of the following form.*

- *If $q \neq \alpha$, then $x_n = Aq^n$.*
- *If $q = \alpha$, then $x_n = Anq^n$.*

(b) *Let $\beta_n = \sum_{i=0}^{k} b_i n^i$ be a polynomial function of n with degree k.*

- *If $\alpha \neq 1$, then* (13) *has a particular solution of the form*

$$x_n = A_0 + A_1 n + A_2 n^2 + \cdots + A_k n^k,$$

where the coefficients A_0, A_1, $\ldots$, A_k are be recursively determined as

$$\begin{aligned} A_k &= \frac{b_k}{1-\alpha}, \\ A_i &= \frac{1}{1-\alpha}\left[b_i + \alpha \sum_{j=i+1}^{k} (-1)^{j-i}\binom{j}{i} A_j\right], \quad 0 \leq i \leq k-1. \end{aligned}$$

- *If $\alpha = 1$, then the solution of* (13) *is given by*

$$x_n = x_0 + \sum_{i=1}^{n} \beta_i.$$

Proof. (a) We may assume $q \neq 0$; otherwise the recurrence (13) is homogeneous.
For the case $q \neq \alpha$, put $x_n = Aq^n$ in (13); we have

$$Aq^n = \alpha A q^{n-1} + cq^n.$$

The coefficient A is determined as $A = cq/(q-\alpha)$.
For the case $q = \alpha$, put $x_n = Anq^n$ in (13); we have

$$Anq^n = \alpha A(n-1)q^{n-1} + cq^n.$$

Since $q = \alpha$, then $\alpha A q^{n-1} = cq^n$. The coefficient A is determined as $A = cq/\alpha$.
(b) For the case $\alpha \neq 1$, put $x_n = \sum_{j=0}^{k} A_j n^j$ in (13); we obtain

$$\sum_{j=0}^{k} A_j n^j = \alpha \sum_{j=0}^{k} A_j (n-1)^j + \sum_{j=0}^{k} b_j n^j.$$

Then

$$\sum_{j=0}^{k} A_j n^j = \alpha \sum_{j=0}^{k} A_j \sum_{i=0}^{j} \binom{j}{i} n^i (-1)^{j-i} + \sum_{j=0}^{k} b_j n^j.$$

$$\sum_{i=0}^{k} A_i n^i = \alpha \sum_{i=0}^{k} n^i \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{i} A_j + \sum_{i=0}^{k} b_i n^i.$$

$$\sum_{i=0}^{k} \left[A_i - b_i - \alpha \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{i} A_j \right] n^i = 0.$$

The coefficients $A_0, A_1, \ldots, A_k$ are determined recursively as

$$\begin{aligned} A_k &= \frac{b_k}{1-\alpha}, \\ A_i &= \frac{1}{1-\alpha} \left[b_i + \alpha \sum_{j=i+1}^{k} (-1)^{j-i} \binom{j}{i} A_j \right], \quad 0 \leq i \leq k-1. \end{aligned}$$

As for the case $\alpha = 1$, iterate the recurrence relation (13); we have

$$\begin{aligned} x_n &= x_{n-1} + \beta_n = x_{n-2} + \beta_{n-1} + \beta_n \\ &= x_{n-1} + \beta_{n-2} + \beta_{n-1} + \beta_n = \cdots \\ &= x_0 + \beta_1 + \beta_2 + \cdots + \beta_n. \end{aligned}$$

□

Example 4.1. Solve the difference equation

$$\begin{cases} x_n = x_{n-1} + 3n^2 - 5n^3, & n \geq 1 \\ x_0 = 2. \end{cases}$$

Solution.

$$\begin{aligned} x_n &= x_0 + \sum_{i=1}^{n} b_i = 2 + \sum_{i=1}^{n} \left(3i^2 - 5i^3\right) \\ &= 2 + 3\sum_{i=1}^{n} i^2 - 5\sum_{i=1}^{n} i^3 \\ &= 2 + 3 \times \frac{n(n+1)(2n+1)}{6} - 5 \times \left(\frac{n(n+1)}{2}\right)^2. \end{aligned}$$

We have applied the following identities

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Example 4.2. Solve the equation

$$\begin{cases} x_n = 3x_{n-1} - 4n, & n \geq 1 \\ x_0 = 2. \end{cases}$$

Solution. Note that $x_n = 3^n c$ is the general solution of the corresponding homogeneous linear recurrence relation. Let $x_n = An + B$ be a particular solution. Then

$$An + B = 3[A(n-1) + B] - 4n$$

Comparing the coefficients of n^0 and$n$, it follows that $A = 2$ and $B = 3$. Thus the general solution is given by

$$x_n = 2n + 3 + 3^n c.$$

The initial condition $x_0 = 2$ implies that $c = -1$. Therefore the solution is

$$x_n = -3^n + 2n + 3.$$

Theorem 4.2. *Given a nonhomogeneous linear recurrence relation of the second order*

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + cq^n. \tag{14}$$

Let q_1 and q_2 be solutions of its characteristic equation

$$x^2 - \alpha_1 x - \alpha_2 = 0.$$

Then (14) has a particular solution of the following forms, where A is a constant to be determined.

(a) *If $q \neq q_1$, $q \neq q_2$, then $x_n = Aq^n$.*

(b) *If $q = q_1$, $q_1 \neq q_2$, then $x_n = Anq^n$.*

(c) *If $q = q_1 = q_2$, then $x_n = An^2 q^n$.*

Proof. The homogeneous linear recurrence relation corresponding to (14) is

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2}, \quad n \geq 2. \tag{15}$$

We may assume $q \neq 0$. Otherwise (14) is homogeneous.

(a) Put $x_n = Aq^n$ into (14); we have

$$Aq^n = \alpha_1 A q^{n-1} + \alpha_2 A q^{n-2} + cq^n.$$

Then

$$A(q^2 - a_1 q - a_2) = cq^2.$$

Since q is not a root of the characteristic equation $x^2 = \alpha_1 x + \alpha_2$, that is, $q^2 - \alpha_1 q - \alpha_2 \neq 0$, the coefficient A is determined as

$$A = \frac{cq^2}{q^2 - \alpha_1 q - \alpha_2}.$$

(b) Since $q = q_1 \neq q_2$, then $x_n = q^n$ is a solution of (15) but $x_n = nq^n$ is not, that is,

$$q^2 - \alpha_1 q - \alpha_2 = 0 \quad \text{and} \quad nq^n \neq \alpha_1 (n-1) q^{n-1} + \alpha_2 (n-2) q^{n-2}.$$

It follows that

$$\begin{aligned} nq^2 - \alpha_1 (n-1) q - \alpha_2 (n-2) &= n(q^2 - \alpha_1 q - \alpha_2) + \alpha_1 q + 2\alpha_2 \\ &= \alpha_1 q + 2\alpha_2 \neq 0. \end{aligned}$$

Put $x_n = Anq^n$ into (14); we have

$$Anq^n = \alpha_1 A(n-1) q^{n-1} + \alpha_2 A(n-2) q^{n-2} + cq^n.$$

Then

$$A\left[nq^2 - \alpha_1 (n-1) q - \alpha_2 (n-2)\right] = cq^2.$$

Since $\alpha_1 q + 2\alpha_2 \neq 0$, the coefficient A is determined as

$$A = \frac{cq^2}{\alpha_1 q + 2\alpha_2}.$$

(c) Since $q = q_1 = q_2$, then both $x_n = q^n$ and $x_n = nq^n$ are solutions of (15), but $x_n = n^2 q^n$ is not. It then follows that

$$q^2 - \alpha_1 q - \alpha_2 = 0, \quad \alpha_1 q + 2\alpha_2 = 0, \quad \text{and}$$

$$\begin{aligned} n^2q^2 - \alpha_1(n-1)^2 q - \alpha_2(n-2)^2 &= n^2(q^2 - \alpha_1 q - \alpha_2) + 2n(\alpha_1 q + 2\alpha_2) - \alpha_1 q - 4\alpha_2 \\ &= -\alpha_1 q - 4\alpha_2 \neq 0. \end{aligned}$$

Put $x_n = An^2q^n$ into (14); we have

$$Aq^{n-2}\left[n^2q^2 - \alpha_1(n-1)^2 q - \alpha_2(n-2)^2\right] = cq^n.$$

The coefficient A is determined as

$$A = -\frac{cq^2}{\alpha_1 q + 4\alpha_2}.$$

□

Example 4.3. Solve the equation

$$\begin{cases} x_n &= 10x_{n-1} - 25x_{n-2} + 5^{n+1}, \quad n \geq 2 \\ x_0 &= 5 \\ x_1 &= 15. \end{cases}$$

Put $x_n = An^2 \times 5^n$ into the recurrence relation; we have

$$An^2 \times 5^n = 10A(n-1)^2 \times 5^{n-1} - 25A(n-2)^2 \times 5^{n-2} + 5^{n+1}.$$

Dividing both sides we further have

$$An^2 = 2A(n-1)^2 - A(n-2)^2 + 5.$$

Thus $A = 5/2$. The general solution is given by

$$x_n = \frac{5}{2}n^2 5^n + c_1 5^n + c_2 n 5^n.$$

Applying the initial conditions $x_0 = 5$ and $x_1 = 15$, we have $c_1 = 5$ and $c_2 = -9/2$. Hence

$$x_n = \left(\frac{5}{2}n^2 - \frac{9}{2}n + 5\right) 5^n.$$

Theorem 4.3. *Given a nonhomogeneous linear recurrence relation of the second order*

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \beta_n, \quad n \geq 2, \tag{16}$$

where β_n is a polynomial function of n with degree k.

(a) *If $\alpha_1 + \alpha_2 \neq 1$, then (16) has a particular solution of the form*

$$x_n = A_0 + A_1 n + \cdots + A_k n^k,$$

where A_0, A_1, …, A_k are constants to be determined. If $k \leq 2$, then a particular solution has the form

$$x_n = A_0 + A_1 n + A_2 n^2.$$

(b) *If $\alpha_1 + \alpha_2 = 1$, then (16) can be reduced to a first order recurrence relation*

$$y_n = (\alpha_1 - 1)y_{n-1} + \beta_n, \quad n \geq 2,$$

where $y_n = x_n - x_{n-1}$ for $n \geq 1$.

Proof. (a) Let $\beta_n = \sum_{j=0}^k b_j n^j$. Put $x_n = \sum_{j=0}^k A_j n^j$ into the recurrence relation (16); we obtain

$$\sum_{j=0}^{k} A_j n^j = \alpha_1 \sum_{j=0}^{k} A_j (n-1)^j + \alpha_2 \sum_{j=0}^{k} A_j (n-2)^j + \sum_{j=0}^{k} b_j n^j;$$

$$\sum_{j=0}^{k} A_j n^j = \alpha_1 \sum_{j=0}^{k} A_j \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} n^i + \alpha_2 \sum_{j=0}^{k} A_j \sum_{i=0}^{j} (-2)^{j-i} \binom{j}{i} n^i + \sum_{j=0}^{k} b_j n^j;$$

$$\sum_{j=0}^{k} A_j n^j = \alpha_1 \sum_{i=0}^{k} n^i \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{i} A_j + \alpha_2 \sum_{i=0}^{k} n^i \sum_{j=i}^{k} (-2)^{j-i} \binom{j}{i} A_j + \sum_{j=0}^{k} b_j n^j.$$

Collecting the coefficients of n^i, we have

$$\sum_{i=0}^{k} \left[A_i - \alpha_1 \sum_{j=i}^{k} (-1)^{j-i} \binom{j}{i} A_j - \alpha_2 \sum_{j=i}^{k} (-2)^{j-i} \binom{j}{i} A_j - \sum_{i=0}^{k} b_i \right] n^i = 0.$$

Since $\alpha_1 + \alpha_2 \neq 1$, the coefficients $A_0, A_1, \ldots, A_k$ are determined as

$$\begin{aligned} A_k &= \frac{b_k}{1 - \alpha_1 - \alpha_2}, \\ A_i &= \frac{1}{1 - \alpha_1 - \alpha_2} \left[b_i + \sum_{j=i+1}^{k} (-1)^{j-i} \binom{j}{i} \left(\alpha_1 + 2^{j-i} \alpha_2 \right) A_j \right], \quad 0 \leq i \leq k-1. \end{aligned}$$

(b) The recurrence relation (16) becomes

$$x_n = \alpha_1 x_{n-1} + (1 - \alpha_1) x_{n-2} + \beta_n, \quad n \geq 2.$$

Set $y_n = x_n - x_{n-1}$ for $n \geq 1$; recurrence (16) reduces to the required first order recurrence relation. □

Example 4.4. Solve the following recurrence relation

$$\begin{cases} x_n &= 6x_{n-1} - 9x_{n-2} + 8n^2 - 24n \\ x_0 &= 5 \\ x_1 &= 5. \end{cases}$$

Solution. Put $x_n = A_0 + A_1 n + A_2 n^2$ into the recurrence relation; we obtain

$$A_0 + A_1 n + A_2 n^2 = 6[A_0 + A_1(n-1) + A_2(n-1)^2] - 9[A_0 + A_1(n-2) + A_2(n-2)^2] + 8n^2 - 24n.$$

Collecting the coefficients of n^2, n, and the constant, we have

$$(4A_2 - 8)n^2 + (4A_1 - 24A_2 + 24)n + (4A_0 - 12A_1 + 30A_2) = 0.$$

We conclude that $A_2 = 2$, $A_1 = 6$, and $A_0 = 3$. So $x_n = 2n^2 + 6n + 3$ is a particular solution. Then the general solution of the recurrence is

$$x_n = 2n^2 + 6n + 3 + 3^n c_1 + 3^n n c_2.$$

Applying the initial condition $x_0 = x_1 = 5$, we have $c_1 = 2$, $c_2 = -4$. The sequence is finally obtained as

$$x_n = 2n^2 + 6n + 3 + 2 \times 3^n - 4n \times 3^n.$$

5 Generating Functions

The (ordinary) *generating function* of an infinite sequence

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_n,\ \ldots$$

is the infinite series

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots .$$

A finite sequence

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_n$$

can be regarded as the infinite sequence

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_n,\ 0,\ 0,\ \ldots$$

and its generating function

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

is a polynomial.

Example 5.1. The generating function of the constant infinite sequence

$$1,\ 1,\ \ldots,\ 1,\ \ldots$$

is the function

$$A(x) = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

Example 5.2. For any positive integer n, the generating function for the binomial coefficients

$$\binom{n}{0},\ \binom{n}{1},\ \binom{n}{2},\ \ldots,\ \binom{n}{n}$$

is the function

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

Example 5.3. For any real number α, the generating function for the infinite sequence of binomial coefficients

$$\binom{\alpha}{0},\ \binom{\alpha}{1},\ \binom{\alpha}{2},\ \ldots,\ \binom{\alpha}{n},\ \ldots$$

is the function

$$\sum_{n=0}^{n} \binom{\alpha}{n} x^n = (1+x)^\alpha.$$

Example 5.4. Let k be a positive integer and let

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_n,\ \ldots$$

be the infinite sequence whose general term a_n is the number of nonnegative integral solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

Then the generating function of the sequence (a_n) is

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} \left(\sum_{i_1+\cdots+i_k=n} 1 \right) x^n = \sum_{n=0}^{\infty} \sum_{i_1+\cdots+i_k=n} x^{i_1+\cdots+i_k} \\ &= \left(\sum_{i_1=0}^{\infty} x^{i_1} \right) \cdots \left(\sum_{i_k=0}^{\infty} x^{i_k} \right) = \frac{1}{(1-x)^k} \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{-k}{n} x^n = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n. \end{aligned}$$

Example 5.5. Let a_n be the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = n,$$

where $0 \le x_1 \le 3$, $0 \le x_2 \le 2$, $x_3 \ge 2$, and $3 \le x_4 \le 5$. The generating function of the sequence (a_n) is

$$\begin{aligned} A(x) &= \left(1 + x + x^2 + x^3\right)\left(1 + x + x^2\right)\left(x^2 + x^3 + \cdots\right)\left(x^3 + x^4 + x^5\right) \\ &= \frac{x^5\left(1 + x + x^2 + x^3\right)\left(1 + x + x^2\right)^2}{1 - x}. \end{aligned}$$

Example 5.6. Determine the generating function for the number of n-combinations of apples, bananas, oranges, and pears where in each n-combination the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and the number of pears is at least two.

The required generating function is

$$\begin{aligned} A(x) &= \left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{2i+1}\right)\left(\sum_{i=0}^{4} x^{i}\right)\left(\sum_{i=1}^{\infty} x^{i}\right) \\ &= \frac{x^3(1 - x^5)}{(1 - x^2)^2(1 - x)^2}. \end{aligned}$$

Example 5.7. Determine the number a_n of bags with n pieces of fruit (apples, bananas, oranges, and pears) such that the number of apples is even, the number bananas is a multiple of 5, the number oranges is at most 4, and the number of pears is either one or zero.

The generating function of the sequence (a_n) is

$$\begin{aligned} A(x) &= \left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{5i}\right)\left(\sum_{i=0}^{4} x^{i}\right)\left(\sum_{i=0}^{1} x^{i}\right) \\ &= \frac{(1 + x + x^2 + x^3 + x^4)(1 + x)}{(1 - x^2)(1 - x^5)} \\ &= \frac{(1 + x)(1 - x^5)/(1 - x)}{(1 + x)(1 - x)(1 - x^5)} \\ &= \frac{1}{(1 - x)^2} = \sum_{n=0}^{\infty} (-1)^n \binom{-2}{n} x^n \\ &= \sum_{n=0}^{\infty} \binom{n+1}{n} x^n = \sum_{n=0}^{\infty} (n+1)x^n. \end{aligned}$$

Thus $a_n = n + 1$.

Example 5.8. Find a formula for the number $a_{n,k}$ of integral solutions $(i_1, i_2, \ldots, i_k)$ of the equation

$$x_1 + x_2 + \cdots + x_k = n$$

such that $i_1, i_2, \ldots, i_k$ are nonnegative odd numbers.

The generating function of the sequence (a_n) is

$$\begin{aligned} A(x) &= \left(\sum_{i=0}^{\infty} x^{2i+1}\right)\cdots\left(\sum_{i=0}^{\infty} x^{2i+1}\right) = \frac{x^k}{(1 - x^2)^k} \\ &= x^k \sum_{i=0}^{\infty} \binom{i+k-1}{i} x^{2i} = \sum_{i=0}^{\infty} \binom{i+k-1}{i} x^{2i+k} \\ &= \begin{cases} \sum_{j=r}^{\infty} \binom{j+r-1}{j-r} x^{2j} & \text{for} \quad k = 2r \\ \sum_{j=r}^{\infty} \binom{j+r}{j-r} x^{2j+1} & \text{for} \quad k = 2r + 1. \end{cases} \end{aligned}$$

We then conclude that $a_{2s,2r} = \binom{s+r-1}{s-r}$, $a_{2s+1,2r+1} = \binom{s+r}{s-r}$, and $a_{n,k} = 0$ otherwise. We may combine the three case as two cases:

$$a_{n,k} = \binom{\lfloor \frac{n}{2} \rfloor + \lceil \frac{k}{2} \rceil - 1}{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor} \quad \text{if} \quad n - k = even,$$

and $a_{n,k} = 0$ if $n - k = odd$.

Example 5.9. Let a_n denote the number of nonnegative integral solutions of the equation

$$2x_1 + 3x_2 + 4x_3 + 5x_4 = n.$$

Then the generating function of the sequence (a_n) is

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} \left(\sum_{\substack{i,j,k,l \geq 0 \\ 2i+3j+4k+5l=0}} 1 \right) x^n \\ &= \left(\sum_{i=0}^{\infty} x^{2i} \right) \left(\sum_{j=0}^{\infty} x^{3j} \right) \left(\sum_{k=0}^{\infty} x^{4k} \right) \left(\sum_{l=0}^{\infty} x^{5l} \right) \\ &= \frac{1}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}. \end{aligned}$$

Theorem 5.1. *Let s_n be the number of nonnegative integral solutions of the equation*

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = n.$$

Then the generating function of the sequence (s_n) is

$$A(x) = \frac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_k})}.$$

6 Recurrence and Generating Functions

Since

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k, \quad |x| < 1;$$

then

$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-ax)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k x^k, \quad |x| < \frac{1}{|a|}.$$

Example 6.1. Determine the generating function of the sequence

$$0,\ 1,\ 2^2,\ \ldots,\ n^2,\ \ldots.$$

Since $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, then

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} \frac{d}{dx}\left(x^k\right) = \sum_{k=0}^{\infty} kx^{k-1}.$$

Thus $\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$. Taking the derivative with respect to x we have

$$\frac{1+x}{(1-x)^3} = \sum_{k=0}^{\infty} k^2 x^{k-1}.$$

Therefore the desired generating function is

$$g(x) = \frac{x(1+x)}{(1-x)^3}.$$

Example 6.2. Solve the recurrence relation

$$\begin{cases} a_n &= 5a_{n-1} - 6a_{n-2}, \quad n \geq 2 \\ a_0 &= 1 \\ a_1 &= -2 \end{cases}$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Applying the recurrence relation, we have

$$\begin{aligned} A(x) &= a_0 + a_1 x + \sum_{n=2}^{\infty} (5a_{n-1} - 6a_{n-2})\, x^n \\ &= a_0 + a_1 x - 5xa_0 + 5xA(x) - 6x^2 A(x). \end{aligned}$$

Applying the initial values and collecting the coefficient functions of $A(x)$, we further have

$$\left(1 - 5x + 6x^2\right) A(x) = 1 - 7x.$$

Thus the function $g(x)$ is solved as

$$A(x) = \frac{1 - 7x}{1 - 5x + 6x^2}.$$

Observing that $1 - 5x + 6x^2 = (1 - 2x)(1 - 3x)$ and applying partial fraction,

$$\frac{1 - 7x}{1 - 5x + 6x^2} = \frac{A}{1 - 2x} + \frac{B}{1 - 3x}.$$

The constants A and B can be determined by

$$A(1 - 3x) + B(1 - 2x) = 1 - 7x.$$

Then

$$\begin{cases} A + B = 1 \\ -3A - 2B = -7 \end{cases}$$

Thus $A = 5$, $B = -4$. Hence

$$\frac{1 - 7x}{1 - 5x + 6x^2} = \frac{5}{1 - 2x} - \frac{4}{1 - 3x}.$$

Since

$$\frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n \quad \text{and} \quad \frac{1}{1 - 3x} = \sum_{n=0}^{\infty} 3^n x^n$$

We obtain the sequence

$$a_n = 5 \times 2^n - 4 \times 3^n, \quad n \geq 0.$$

Theorem 6.1. *Let $(a_n; n \geq 0)$ be a sequence satisfying the homogeneous linear recurrence relation*

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_k a_{n-k}, \quad \alpha_k \neq 0, \quad n \geq k \tag{17}$$

of order k with constant coefficients. Then its generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is of the form

$$A(x) = \frac{P(x)}{Q(x)} \tag{18}$$

where $Q(x)$ is a polynomial of degree k with a nonzero constant term and $P(x)$ is a polynomial of degree less than k.

Conversely, given such polynomials $P(x)$ and $Q(x)$, there is a unique sequence (a_n) that satisfies the linear homogeneous recurrence relation (17) *and its generating function is the rational function in* (18).

Proof. The generating function $A(x)$ of the sequence (a_n) can be written as

$$
\begin{aligned}
A(x) &= \sum_{i=0}^{k-1} a_i x^i + \sum_{n=k}^{\infty} a_n x^n = \sum_{i=0}^{k-1} a_i x^i + \sum_{n=k}^{\infty} \left(\sum_{i=1}^{k} \alpha_i a_{n-i} \right) x^n \\
&= \sum_{i=0}^{k-1} a_i x^i + \sum_{i=1}^{k} \alpha_i \sum_{n=k}^{\infty} a_{n-i} x^n = \sum_{i=0}^{k-1} a_i x^i + \sum_{i=1}^{k} \alpha_i \sum_{n=k-i}^{\infty} a_n x^{n+i} \\
&= \sum_{i=0}^{k-1} a_i x^i + \alpha_k x^k \sum_{n=0}^{\infty} a_n x^n + \sum_{i=1}^{k-1} \alpha_i x^i \left(\sum_{n=0}^{\infty} a_n x^n - \sum_{j=0}^{k-i-1} a_j x^j \right) \\
&= \sum_{i=0}^{k-1} a_i x^i + g(x) \sum_{i=1}^{k} \alpha_i x^i - \sum_{i=1}^{k-1} \alpha_i x^i \sum_{j=0}^{k-i-1} a_j x^j \\
&= \sum_{i=0}^{k-1} a_i x^i + g(x) \sum_{i=1}^{k} \alpha_i x^i - \sum_{l=1}^{k-1} x^l \sum_{i=1}^{l} \alpha_i a_{l-i}.
\end{aligned}
$$

Then

$$
A(x) \left(1 - \sum_{i=1}^{k} \alpha_i x^i \right) = \sum_{i=0}^{k-1} a_i x^i - \sum_{l=1}^{k-1} x^l \sum_{i=1}^{l} \alpha_i a_{l-i} = a_0 + \sum_{i=1}^{k-1} \left(a_i - \sum_{j=1}^{i} \alpha_j a_{i-j} \right) x^i.
$$

Thus

$$
\begin{aligned}
P(x) &= a_0 + \sum_{i=1}^{k-1} \left(a_i - \sum_{j=1}^{i} \alpha_j a_{i-j} \right) x^i, \\
Q(x) &= 1 - \sum_{i=1}^{k} \alpha_i x^i.
\end{aligned}
$$

Conversely, let (a_n) be the sequence whose generating function is $A(x)$. Let

$$
A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad P(x) = \sum_{i=0}^{k} b_i x^i, \quad Q(x) = 1 - \sum_{i=1}^{k} \alpha_i x^i.
$$

Then $g(x) = p(x)/q(x)$ is equivalent to

$$
\left(1 - \sum_{i=1}^{k} \alpha_i x^i \right) \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{i=0}^{k} b_i x^i.
$$

The polynomial $q(x)$ can be viewed as an infinite series with $\alpha_i = 0$ for $i > k$. Thus

$$
\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n} \alpha_i a_{n-i} \right) x^n = \sum_{i=0}^{k} b_i x^i.
$$

Equating the coefficients of x^n, we have the recurrence relation

$$
a_n = \sum_{i=1}^{k} \alpha_i a_{n-i}, \quad n \ge k.
$$

□

Proposition 6.2 (Partial Fractions). (a) *If $P(x)$ is a polynomial of degree at most k, then*

$$
\frac{P(x)}{(1-ax)^k} = \frac{A_1}{1-ax} + \frac{A_2}{(1-ax)^2} + \cdots + \frac{A_k}{(1-ax)^k},
$$

where $A_1, A_2, \ldots, A_k$ are constants to be determined.

(b) *If $P(x)$ is a polynomial of degree at most $p+q+r$, then*

$$\frac{P(x)}{(1-ax)^p(1-bx)^q(1-cx)^r} = \frac{A_1(x)}{(1-ax)^p} + \frac{A_2(x)}{(1-bx)^q} + \frac{A_3(x)}{(1-cx)^r},$$

where $A_1(x)$, $A_2(x)$, and $A_3(x)$ are polynomials of degree $q+r$, $p+r$, and $p+q$, respectively.

7 A Geometry Example

A polygon P in $\mathbb{R}^2$ is called *convex* if the segment joining any two points in P is also contained in P. Let C_n denote the number ways to divide a labelled convex polygon with $n+2$ sides into triangles. The first a few such numbers are $C_1 = 1$, $C_2 = 2$, $C_3 = 5$.

We first establish a recurrence relation between C_{n+1} and $C_0, C_1, \ldots, C_n$. Let $P(v_1, v_2, \ldots, v_{n+3})$ denote a convex polygon with the vertices $v_1, v_2, \ldots, v_{n+3}$. In each triangular decomposition of $P(v_1, v_2, \ldots, v_{n+3})$ into triangles, the segment $v_1 v_{n+3}$ is one side of a triangle Δ in the decomposition; the third vertex of the triangle Δ is one of the vertices $v_2, v_3, \ldots, v_{n+2}$. Let v_{k+2} be the third vertex of Δ other than v_1 and v_{n+3} $(0 \le k \le n)$; see Figure 1 below. Then

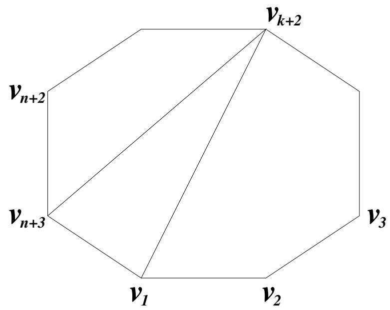


Figure 1: v_{k+2} is the third vertex of the triangle with the side $v_1 v_{n+2}$

we have one convex polygon $P(v_1, v_2, \ldots, v_{k+2})$ of $(k+2)$ sides and another convex polygon $P(v_{k+2}, v_{k+3}, \ldots, v_{n+3})$ of $(n-k+2)$ sides. Then there are C_k ways to divide $P(v_1, v_2, \ldots, v_{k+2})$ into triangles and there are C_{n-k} ways to divide $P(v_{k+2}, v_{k+3}, \ldots, v_{n+3})$ into triangles. We thus have the following recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad \text{with} \quad C_0 = 1.$$

Consider the generating function $F(x) = \sum_{n=0}^{\infty} C_n x^n$. Then

$$\begin{aligned} F(x)F(x) &= \left(\sum_{n=0}^{\infty} C_n x^n\right)\left(\sum_{n=0}^{\infty} C_n x^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} C_k C_{n-k}\right) x^n \\ &= \sum_{n=0}^{\infty} C_{n+1} x^n = \frac{1}{x}\sum_{n=1}^{\infty} C_n x^n \\ &= \frac{F(x)}{x} - \frac{1}{x}. \end{aligned}$$

We thus obtain the functional equation

$$xF(x)^2 - F(x) + 1 = 0.$$

Solving for $F(x)$, we have

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Note that

$$\begin{aligned}
\sqrt{1-4x} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n \\
&= 1+\sum_{n=1}^{\infty} \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} 2^{2n}(-1)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)(-3)(-5)\cdots(-2(n-1)+1)}{n!} 2^n(-1)^n x^n \\
&= -\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2(n-1)-1)}{n!} 2^n x^n \\
&= 1-2\sum_{n=1}^{\infty} \frac{(2(n-1))!}{n!(n-1)!} x^n \\
&= 1-2\sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^{n+1}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
F(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.
\end{aligned}$$

Hence the sequence (C_n) is given by the binomial coefficients:

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

The sequence (C_n) is known as the *Catalan sequence* and the numbers C_n as the *Catalan numbers*.

Example 7.1. Let C_n be the number of ways to evaluate a matrix product $A_1A_2\cdots A_{n+1}$ $(n \geq 0)$ by adding various parentheses. For instance, $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, and $C_3 = 5$. In general the formula is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Note that each way of evaluating the matrix product $A_1A_2\cdots A_{n+2}$ will be finished by multiplying of two matrices at the end. There are exactly $n+1$ ways of multiplying the two matrices at the end:

$$A_1A_2\cdots A_{n+2} = (A_1\cdots A_{k+1})(A_{k+2}\cdots A_{n+2}), \quad 0 \leq k \leq n.$$

This yields the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

Thus $C_n = \frac{1}{n+1}\binom{2n}{n}$, $n \geq 0$.

8 Exponential Generating Functions

The ordinary generating function method is a powerful algebraic tool for finding unknown sequences, especially when the sequences are certain binomial coefficients or some the order is not material. However, when the sequences are not binomial type or the order is material in defining the sequences, we may need to consider a different type of generating functions. For example, for the sequence $a_n = n!$, counting the number of permutations of n distinct objects. It is not easy at all to figure out the ordinary generating function

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n! x^n.$$

However, the generating function

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is obviously figured out.

The *exponential generating function* of a sequence $(a_n; n \geq 0)$ is the infinite series

$$E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Example 8.1. The exponential generating function of the sequence

$$P(n,0), \quad P(n,1), \quad \ldots, \quad P(n,n)$$

is given by

$$\begin{aligned} E(x) &= \sum_{k=0}^{n} \frac{P(n,k)}{k!} x^k \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k \\ &= (1+x)^n. \end{aligned}$$

Example 8.2. The exponential generating function of the constant sequence $(a_n = 1; n \geq 0)$ is

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

The exponential generating function of the geometric sequence $(a_n = a^n; n \geq 0)$ is

$$E(x) = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} = e^{ax}.$$

Theorem 8.1. *Let $M = \{n_1 a_1, n_2 a_2, \ldots, n_k a_k\}$ be a multiset over the set $S = \{a_1, a_2, \ldots, a_k\}$ with n_1 a_1's, n_2 a_2's, $\ldots$, n_k a_k's. Let a_n be the number of permutations of the multiset M. Then the exponential generating function of the sequence $(a_n; n \geq 0)$ is given by*

$$E(x) = \left(\sum_{i=0}^{n_1} \frac{x^i}{i!} \right) \left(\sum_{i=0}^{n_2} \frac{x^i}{i!} \right) \cdots \left(\sum_{i=0}^{n_k} \frac{x^i}{i!} \right). \tag{19}$$

Proof. Note that $a_n = 0$ for $n > n_1 + \cdots + n_k$. Thus $E(x)$ is a polynomial. The right side of (19) can be expanded to the form

$$\sum_{i_1, i_2, \ldots, i_k = 0}^{n_1, n_2, \ldots, n_k} \frac{x^{i_1 + i_2 + \cdots + i_k}}{i_1! i_2! \cdots i_k!} = \sum_{n=0}^{n_1 + n_2 + \cdots + n_k} \frac{x^n}{n!} \sum_{\substack{i_1 + i_2 + \cdots + i_k = n \\ 0 \leq i_1 \leq n_1, \ldots, 0 \leq i_k \leq n_k}} \frac{n!}{i_1! i_2! \cdots i_k!}.$$

Note that the number of permutation of M with exactly i_1 a_1's, i_2 a_2's, ..., and i_k a_k's such that

$$i_1 + i_2 + \cdots + i_k = n$$

is the multinomial coefficient

$$\binom{n}{i_1, i_2, \ldots, i_k} = \frac{n!}{i_1! i_2! \cdots i_k!}.$$

It turns out that the sequence (a_n) is given by

$$a_n = \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ 0\le i_1\le n_1,\ldots,0\le i_k\le n_k}} \frac{n!}{i_1! i_2! \cdots i_k!}, \quad n \ge 0.$$

□

Example 8.3. Determine the number of ways to color the squares of a 1-by-n chessboard using the colors, red, white, and blue, if an even number of squares are colored red.

Let a_n denote the number of ways of such colorings and set $a_0 = 1$. Each such coloring can be considered as a permutation of three objects r (for red), w (for white), and b (for blue) with repetition allowed, and the element r appears even number of times. The exponential generating function of the sequence (a_n) is

$$\begin{aligned} E(x) &= \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 \\ &= \frac{e^x + e^{-x}}{2} e^{2x} = \frac{1}{2}\left(e^{3x} + e^x\right) \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty} (3^n + 1) \cdot \frac{x^n}{n!}. \end{aligned}$$

Thus the sequence is given by

$$a_n = \frac{3^n + 1}{2}, \quad n \ge 0.$$

Example 8.4. Determine the number a_n of n digit (under base 10) numbers with each digit odd where the digit 1 and 3 occur an even number of times.

Let $a_0 = 1$. The number a_n equals the number of $n$permutations of the multiset $M = \{\infty 1, \infty 3, \infty 5, \infty 7, \infty 9\}$, in which 1 and 3 occur an even number of times. The exponential generating function of the sequence a_n is

$$\begin{aligned} E(x) &= \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right)^2 \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^3 \\ &= \left(\frac{e^x + e^{-x}}{2}\right)^2 e^{3x} \\ &= \frac{1}{4}\left(e^{5x} + 2e^{3x} + e^x\right) \\ &= \frac{1}{4}\left(\sum_{n=0}^{\infty} \frac{5^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \left(\frac{5^n + 2 \times 3^n + 1}{4}\right) \frac{x^n}{n!}. \end{aligned}$$

Thus

$$a_n = \frac{5^n + 2 \times 3^n + 1}{4}, \quad n \ge 0.$$