Week 12-13: Special Counting Sequences

May 5, 2005

We have considered several special counting sequences. For instance, the sequence $n!$ counts the number of permutations of an n-set; the sequence D_n counts the number of derangements of an n-set; and the the Fibonacci sequence f_n counts the pairs of rabbits.

1 Catalan Numbers

Definition 1.1. The *Catalan sequence* is the sequence

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

The number C_n is called the *nth Catalan number.* The first few Catalan numbers are

$$\begin{array}{ll} C_0 = 1 & C_4 = 14 \\ C_1 = 1 & C_5 = 42 \\ C_2 = 2 & C_6 = 132 \\ C_3 = 5 & C_7 = 429 \end{array}$$

Theorem 1.2. *The number of words $a_1 a_2 \cdots a_{2n}$ of length $2n$ having exactly n positive ones* (+1's) *and exactly n negative ones* (−1's) *and satisfying*

$$a_1 + a_2 + \cdots + a_i \geq 0 \quad \text{for all} \quad 1 \leq i \leq 2n, \tag{1}$$

equals the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

Proof. We call a word of length $2n$ with exactly n positive ones (+1's) and n negative ones (−1's) *acceptable* if it satisfies (1) and *unacceptable* otherwise. Let A_n denote the set of acceptable words and let U_n denote the set of unacceptable words. Then $A_n \cup U_n$ is the set of all words of length $2n$ with exactly n positive ones and exactly n negative ones, and

$$|A_n| + |U_n| = \binom{2n}{n} = \frac{(2n)!}{n!n!}.$$

Let S_n be the set of words of length $2n$ with exactly $n+1$ ones and $n-1$ negative ones.

We define a map $f : U_n \longrightarrow S_n$ as follows: For each word $a_1 a_2 \cdots a_{2n}$ in U_n, since the word is unacceptable there is a smallest integer k such that

$$a_1 + a_2 + \cdots + a_k < 0.$$

Since the number k is smallest, we have $k \geq 1$, $a_1 + a_2 + \cdots + a_{k-1} = 0$, and $a_k = -1$. (If $k = 1$, then $a_0 = 0$.) Note that the integer k must be an odd number. Now switch the signs of the first k terms in the word $a_1 a_2 \cdots a_{2n}$ to get a new word $a'_1 a'_2 \cdots a'_k a_{k+1} \cdots a_{2n}$, where $a'_1 = -a_1$, $a'_2 = -a_2$, …, $a'_k = -a_k$. The new words $a'_1 a'_2 \cdots a'_k a_{k+1} \cdots a_{2n}$ has $n+1$ positive ones and $n-1$ negative ones. We set

$$f(a_1 a_2 \cdots a_{2n}) = a'_1 a'_2 \cdots a'_k a_{k+1} \cdots a_{2n}.$$

We define another map $g : S_n \longrightarrow U_n$ as follows: For each word $a_1'a_2'\cdots a_{2n}'$ in S_n, the word has exactly $n+1$ positive ones and exactly $n-1$ negatives ones. There is a smallest integer k such that

$$a_1' + a_2' + \cdots + a_k' > 0.$$

Then $k \geq 1$, $a_1' + a_2' + \cdots + a_{k-1}' = 0$, and $a_k' = 1$. Switch the signs of the first k terms in $a_1'a_2'\ldots a_{2n}'$ to get a new word $a_1a_2\cdots a_ka_{k+1}'\cdots a_{2n}'$, where $a_1 = -a_1'$, $a_2 = -a_2'$, …, $a_k = -a_k'$. The word $a_1a_2\cdots a_ka_{k+1}'\cdots a_{2n}'$ has exactly n ones and exactly n negative ones, and is unacceptable because $a_1 + a_2 + \cdots + a_k < 0$. We set

$$g(a_1'a_2'\cdots a_{2n}') = a_1a_2\cdots a_ka_{k+1}'\cdots a_{2n}'.$$

Now it is easy to see that the maps f and g are inverses of each other. Hence

$$|U_n| = |S_n| = \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!(n-1)!}.$$

It follows from $|A_n| + |U_n| = \frac{(2n)!}{n!n!}$ that

$$\begin{aligned} |A_n| &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\ &= \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{(2n)!}{n!(n-1)!} \cdot \frac{1}{n(n+1)} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

☐

Corollary 1.3. *The number of nondecreasing lattice paths from $(0,0)$ to (n,n) and above the straight line $x = y$ is equal to the nth Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

Proof. Viewing the $+1$ as a unit move upward and -1 as a unit move to the right, then each word of length $2n$ with exactly n positive ones ($+1$'s) and n negative ones (-1's) can be interpreted as a nondecreasing lattice path from $(0,0)$ to (n,n) and above the straight line $x = y$. ☐

Example 1.1. There are $2n$ people line to get into theater. Admission is 50 cents. Of the 2n people, n have a 50 cent piece and n has a 1 dollar bill. Assume the box office at the theater begin with empty cash register. In how many ways can the people line up so that whenever a person with a dollar bill buys a ticket and the box office has a 50 cent piece in order to make change?

If the $2n$ people are considered indistinguishable, then the answer is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

If the $2n$ people are consider distinguishable, then answer is

$$\frac{1}{n+1}\binom{2n}{n} \cdot n!n! = \frac{(2n)!}{n+1}.$$

2 Difference Sequences and Stirling Numbers

Definition 2.1. The *first order difference sequence* (or just *difference sequence*) of a sequence a_n $(n \geq 0)$ is the sequence Δa_n defined by

$$\Delta a_n = a_{n+1} - a_n, \quad n \geq 0.$$

Example 2.1. The difference sequence of the sequence 3^n $(n \geq 0)$ is the sequence

$$\Delta 3^n = 3^{n+1} - 3^n = 2 \times 3^n, \quad n \geq 0.$$

The difference sequence of 2×3^n is

$$\Delta(2 \times 3^n) = 2 \times 3^{n+1} - 2 \times 3^n = 2^2 \times 3^n, \quad n \geq 0.$$

The difference sequence $\Delta(\Delta a_n)$ of the sequence (Δa_n) is called the *second order difference sequence* of a_n $(n \geq 0)$, and is denoted by $\Delta^2 a_n$ $(n \geq 0)$. More specifically,

$$\begin{aligned} \Delta^2 a_n &= \Delta(\Delta a_n) = \Delta(a_{n+1} - a_n) \\ &= (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) \\ &= a_{n+2} - 2a_{n+1} + a_n, \qquad n \geq 0. \end{aligned}$$

Similarly, the *pth order difference sequence* $\Delta^p a_n$ of a_n $(n \geq 0)$ is the difference sequence $\Delta(\Delta^{p-1} a_n)$ of $\Delta^{p-1} a_n$, namely,

$$\Delta^p a_n = \Delta(\Delta^{p-1} a_n), \quad n \geq 0.$$

For convenience, we define the 0*th-order difference sequence* $\Delta^0 a_n$ to be the sequence itself, namely,

$$\Delta^0 a_n = a_n, \quad n \geq 0.$$

Theorem 2.2. *For any sequence a_n $(n \geq 0)$, the pth order difference sequence $\Delta^p a_n$ is given by*

$$\Delta^p a_n = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} a_{n+k}, \quad n \geq 0.$$

Proof. For $p = 0$, it is clear that $\Delta^0 a_n = a_n$. For $p = 1$,

$$\sum_{k=0}^{1} (-1)^{1-k} \binom{1}{k} a_{n+k} = a_{n+1} - a_n = \Delta a_n.$$

For $p \geq 2$ we assume that it is true for $p - 1$; that is,

$$\Delta^{p-1} a_n = \sum_{k=0}^{p-1} (-1)^{p-1-k} \binom{p-1}{k} a_{n+k}.$$

The the difference of the sequence $\Delta^{p-1} a_n$ by definition is

$$\begin{aligned} \Delta(\Delta^{p-1} a_n) &= \sum_{k=0}^{p-1} (-1)^{p-1-k} \binom{p-1}{k} a_{n+1+k} - \sum_{k=0}^{p-1} (-1)^{p-1-k} \binom{p-1}{k} a_{n+k} \\ &= \sum_{k=1}^{p} (-1)^{p-k} \binom{p-1}{k-1} a_{n+k} + \sum_{k=0}^{p-1} (-1)^{p-k} \binom{p-1}{k} a_{n+k}. \end{aligned}$$

Applying the Pascal formula $\binom{p}{k} = \binom{p-1}{k-1} + \binom{p-1}{k}$, we obtain

$$\Delta^p a_n = \Delta(\Delta^{p-1} a_n) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} a_{n+k}, \quad n \geq 0.$$

□

Definition 2.3. The *difference table* of a sequence a_n $(n \geq 0)$ is the array

$$\begin{array}{ccccccccccc}
a_0 & & a_1 & & a_2 & & a_3 & & a_4 & & \cdots \\
& \Delta a_0 & & \Delta a_1 & & \Delta a_2 & & \Delta a_3 & & \cdots & \\
& & \Delta^2 a_0 & & \Delta^2 a_1 & & \Delta^2 a_2 & & \cdots & & \\
& & & \Delta^3 a_0 & & \Delta^3 a_1 & & \ddots & & & \\
& & & & \Delta^4 a_0 & & \ddots & & & & \\
& & & & & \ddots & & & & &
\end{array}$$

in which the pth row is the pth order difference sequence $\Delta^p a_n$, where $p \geq 0$.

Example 2.2. Let a_n be a sequence defined by

$$a_n = 2n^2 + 3n + 1, \quad n \geq 0.$$

Then its difference table is

$$\begin{array}{ccccccccccccc}
1 & & 6 & & 15 & & 28 & & 45 & & 66 & & \cdots \\
& 5 & & 9 & & 13 & & 17 & & 21 & & \cdots & \\
& & 4 & & 4 & & 4 & & 4 & & \cdots & & \\
& & & 0 & & 0 & & 0 & & \cdots & & & \\
& & & & 0 & & 0 & & \cdots & & & & \\
& & & & & 0 & & \cdots & & & & &
\end{array}$$

Theorem 2.4. *Let a_n be a polynomial sequence of degree p in the variable n, i.e.,*

$$a_n = \alpha_p n^p + a_{p-1} n^{p-1} + \cdots + \alpha_1 n + \alpha_0, \quad n \geq 0$$

where $\alpha_0, \alpha_1, \ldots, \alpha_p$ are constants. Then the $(p+1)$th order difference sequence is a zero sequence; that is,

$$\Delta^{p+1} a_n = 0, \quad n \geq 0.$$

Proof. We proceed by induction on p. For $p = 0$, the sequence $a_n = \alpha_0$ is a constant sequence and $\Delta a_n = \alpha_0 - \alpha_0 = 0$ is a zero sequence. Now for $p \geq 1$, we assume that $\Delta^p b_n = 0$ for any polynomial sequence b_n of degree $p - 1$. We compute the difference

$$\begin{aligned}
\Delta a_n &= \left[\alpha_p (n+1)^p + a_{p-1}(n+1)^{p-1} + \cdots + \alpha_1 (n+1) + \alpha_0\right] \\
&\quad -[\alpha_p n^p + a_{p-1} n^{p-1} + \cdots + \alpha_1 n + \alpha_0] \\
&= \left[\alpha_p \binom{p}{1} - \alpha_p + \alpha_{p-1}\right] n^{p-1} + \cdots .
\end{aligned}$$

The sequence Δa_n is a polynomial sequence of degree $p - 1$. By the induction hypothesis,

$$\Delta^{p+1} a_n = \Delta^p (\Delta a_n)$$

is a zero sequence. ☐

Theorem 2.5. *The difference table of a sequence a_n is determined by the* 0th diagonal,

$$\Delta^0 a_0, \quad \Delta^1 a_0, \quad \Delta^2 a_0, \quad \ldots, \quad \Delta^n a_0, \quad \ldots .$$

More precisely,

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k a_0, \quad n \geq 0.$$

Proof. By definition of difference operator Δ, we see that Δ is a linear isomorphism from the vector space Ω of sequences to Ω itself. We then have

$$\begin{aligned}
a_1 &= \Delta^0 a_0 + \Delta a_0 \\
&= \sum_{k=0}^{1} \binom{1}{k} \Delta^k a_0; \\
a_2 &= a_1 + \Delta a_1 \\
&= \left(\Delta^0 a_0 + \Delta a_0\right) + \Delta\left(\Delta^0 a_0 + \Delta a_0\right), \\
&= \Delta^0 a_0 + 2\Delta a_0 + \Delta^2 a_0 \\
&= \sum_{k=0}^{2} \binom{2}{k} \Delta^k a_0.
\end{aligned}$$

For the general term a_n, we have

$$\begin{aligned}
a_n &= a_{n-1} + \Delta a_{n-1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \Delta^k a_0 + \Delta \sum_{k=0}^{n-1} \binom{n-1}{k} \Delta^k a_0 \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \Delta^k a_0 + \sum_{k=0}^{n-1} \binom{n-1}{k} \Delta^{k+1} a_0 \\
&= \Delta^0 a_0 + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] \Delta^k a_0 + \Delta^n a_0 \\
&= \sum_{k=0}^{n} \binom{n}{k} \Delta^k a_0.
\end{aligned}$$

□

Corollary 2.6. *If the* 0*th diagonal of the difference table for a sequence* a_n *is*

$$c_0, \quad c_1, \quad c_2, \quad \ldots, \quad c_p\ (\neq 0), \quad 0, \quad 0, \quad \ldots,$$

then the sequence a_n *is a polynomial sequence of degree* p*, and is explicitly given by*

$$a_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + c_2 \binom{n}{2} + \cdots + c_p \binom{n}{p}, \quad n \geq 0.$$

In other words,

$$a_n = \sum_{k=0}^{p} \binom{n}{k} \Delta^k a_0, \quad n \geq 0.$$

Proof. It is obvious that the map $(a_n) \mapsto (\Delta^n a_0)$ is a linear isomorphism from the vector space of sequences to itself. For the 0th diagonal sequence

$$\Delta^0 a_0 = 0, \quad \Delta^1 a_0 = 0, \quad \ldots, \quad \Delta^{p-1} a_0 = 0, \quad \Delta^p a_0 = 1, \quad \ldots$$

It is easy to see that the first $p+1$ terms of the sequence a_n are given by

$$a_0 = 0, \quad a_1 = 0, \quad \ldots, \quad a_{p-1} = 0, \quad a_p = 1, \quad , a_{p+1}, \quad \ldots$$

This can be easily seen from the following difference table (for the case $p = 3$)

$$\begin{array}{ccccccccccccc}
0 & & 0 & & 0 & & 1 & & 4 & & 10 & & 20 \\
& 0 & & 0 & & 1 & & 3 & & 6 & & 10 & \\
& & 0 & & 1 & & 2 & & 3 & & 4 & & \\
& & & 1 & & 1 & & 1 & & 1 & & & \\
& & & & 0 & & 0 & & 0 & & & & \\
& & & & & 0 & & 0 & & & & & \\
& & & & & & 0 & & & & & &
\end{array}$$

Note that the term a_n are not constant for $n \geq p+1$. However, $\Delta^{p+1} a_n = 0$ for $n \geq 0$. Thus a_n is a polynomial sequence of degree p such that $a_0 = a_1 = \cdots = a_{p-1} = 0$ and $a_p = 1$. Therefore

$$a_n = \frac{n(n-1)(n-2)\cdots(n-p+1)}{p!} = \binom{n}{p}, \quad n \geq 0.$$

The formulas for a_n in general case follows from the linearity of Δ. □

Example 2.3. Consider the sequence

$$a_n = n^3 + 2n^2 - 3n + 2, \quad n \geq 0.$$

Computing the difference we obtain

$$\begin{array}{ccccccc} 2 & & 2 & & 12 & & 38 \\ & 0 & & 10 & & 26 & \\ & & 10 & & 16 & & \\ & & & 6 & & & \end{array}$$

Thus the sequence a_n can be written as

$$a_n = 2\binom{n}{0} + 10\binom{n}{2} + 6\binom{n}{3}, \quad n \geq 0.$$

Corollary 2.7. *For any sequence a_n $(n \geq 0)$, its partial sum can be written as*

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} \binom{n+1}{k+1} \Delta^k a_0, \quad n \geq 0. \tag{2}$$

Proof. Recall the identity

$$\sum_{k=i}^{n} \binom{k}{i} = \binom{n+1}{i+1}.$$

Then

$$\begin{aligned} \sum_{k=0}^{n} a_k &= \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} \Delta^i a_0 \\ &= \sum_{i=0}^{n} \left[\sum_{k=i}^{n} \binom{k}{i} \right] \Delta^i a_0 \\ &= \sum_{i=0}^{n} \binom{n+1}{i+1} \Delta^i a_0. \end{aligned}$$

□

Example 2.4. For the sequence $a_n = n^2$, computing the difference we have

$$\begin{array}{ccccc} 0 & & 1 & & 4 \\ & 1 & & 3 & \\ & & 2 & & \end{array}$$

Thus

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \binom{n+1}{2} + 2\binom{n+1}{3} = \frac{(n+1)n(2n+1)}{6}.$$

For the sequence $a_n = n^3$, computing the difference we obtain

$$\begin{array}{ccccccc} 0 & & 1 & & 8 & & 27 \\ & 1 & & 7 & & 21 & \\ & & 6 & & 14 & & \\ & & & 8 & & & \end{array}$$

Thus

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \binom{n+1}{2} + 6\binom{n+1}{3} + 8\binom{n+1}{4} = \frac{(n+1)n(2n^2+1)}{6}.$$

For $a_n = n^4$, computing the difference we have

$$\begin{array}{ccccccccc}
0 & & 1 & & 16 & & 81 & & 256 \\
& 1 & & 15 & & 65 & & 175 & \\
& & 14 & & 50 & & 110 & & \\
& & & 36 & & 60 & & & \\
& & & & 24 & & & &
\end{array}$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n} k^4 &= \binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5} \\
&= \frac{(n+1)n(6n^3+9n^2+n-1)}{30}.
\end{aligned}$$

Example 2.5. Consider the sequence $a_n = n^p$ with $p \geq 0$. Let

$$C(p,0), \quad C(p,1), \quad C(p,2), \quad \ldots, \quad C(p,p), \quad 0, \quad 0, \quad \ldots.$$

be the 0th diagonal of its difference table. Then

$$n^p = \sum_{k=0}^{p} C(p,k)\binom{n}{k}$$

Definition 2.8. The *falling factorial* of n with length k is the number

$$\begin{aligned}
[n]_0 &= 1, \\
[n]_k &= n(n-1)\cdots(n-k+1) \quad \text{for} \quad k \geq 1.
\end{aligned}$$

We call the numbers

$$S(p,k) = \frac{C(p,k)}{k!}, \quad 0 \leq k \leq p$$

the *Stirling numbers of the second kind.*

It is easy to see that the falling factorial satisfies the relation

$$[n]_{k+1} = (n-k)[n]_k;$$

$$\binom{n}{k} = \frac{[n]_k}{k!}$$

Corollary 2.9. *For any integer $p \geq 0$,*

$$n^p = \sum_{k=0}^{p} S(p,k)[n]_k \tag{3}$$

Theorem 2.10. *The Stirling number $S(p,k)$ of the second kind are integers and satisfy the recurrence relation:*

$$\begin{cases}
S(0,0) = S(p,p) = 1 & \text{for} \quad p \geq 0 \\
S(p,0) = 0 & \text{for} \quad p \geq 1 \\
S(p,1) = 1 & \text{for} \quad p \geq 1 \\
S(p,k) = S(p-1,k-1) + kS(p-1,k) & \text{for} \quad p-1 \geq k \geq 1
\end{cases} \tag{4}$$

Proof. For $p = 0$, we have $a_n = n^0 = 1$; it follows that $S(0,0) = 1$. Since $[n]_k$ is a polynomial of degree k in n, it follows from (3) that $S(p,p) = 1$. For $p \geq 1$, the constant term of the polynomial n^p is zero. Since $[n]_k$ is a polynomial of degree k, then the constant term of $[n]_k$ is zero for all $k \geq 1$. It follows from (3) that $S(p,0) = 0$.

Now notice that

$$n^p = \sum_{k=0}^{p} S(p,k)[n]_k \quad \text{and} \quad n^{p-1} = \sum_{k=0}^{p-1} S(p-1,k)[n]_k.$$

It follows that

$$\begin{aligned} n^p = n \times n^{p-1} &= n \sum_{k=0}^{p-1} S(p-1,k)[n]_k = \sum_{k=0}^{p-1} S(p-1,k) n [n]_k \\ &= \sum_{k=0}^{p-1} S(p-1,k)(n-k+k)[n]_k \\ &= \sum_{k=0}^{p-1} S(p-1,k)(n-k)[n]_k + \sum_{k=0}^{p-1} S(p-1,k) k [n]_k \\ &= \sum_{k=0}^{p-1} S(p-1,k)[n]_{k+1} + k \sum_{k=0}^{p-1} S(p-1,k)[n]_k. \end{aligned}$$

Thus

$$\sum_{k=0}^{p} S(p,k)[n]_k = S(p-1,p-1)[n]_p + \sum_{k=1}^{p-1} [S(p-1,k-1) + k(S(p-1,k)][n]_k.$$

Therefore

$$S(p,k) = S(p-1,k-1) + kS(p-1,k), \quad 1 \leq k \leq p-1.$$

In particular, for $p \geq 2$ and $k = 1$, since $S(p-1,0) = 0$, we have

$$S(p,1) = S(p-1,0) + S(p-1,1) = S(p-1,1) = \cdots = S(2,1) = S(1,1) = 1.$$

With the recurrence relation we conclude that $S(p,k)$ are integers for all $0 \leq k \leq p$. □

(p,k)	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	1	1						
3	0	1	3	1					
4	0	1	7	6	1				
5	0	1	15	25	10	1			
6	0	1	31	90	65	15	1		
7	0	1	63	301	350	140	21	1	
8	0	1	127	966	1701	1050	266	28	1

Theorem 2.11.

$$\sum_{k=1}^{n} k^p = \sum_{k=0}^{p} \binom{n+1}{k+1} C(p,k) = \sum_{k=0}^{p} \frac{S(p,k)}{k+1} [n+1]_{k+1}.$$

Definition 2.12. A partition of a set S is a collection $\mathcal{P}$ of disjoint subsets of S such that

$$S = \bigcup_{A \in \mathcal{P}} A.$$

The cardinality $|\mathcal{P}|$ is called the number of parts (or blocks) of the partition $\mathcal{P}$. We denote by $S_{n,k}$ the number of partitions of an n-set into k parts. A partition of a set S into k parts can be viewed as a placement of S into k indistinguishable boxes so that each box is nonempty. The number of partitions of an n-set into k parts is denoted by $S_{n,k}$.

Example 2.6. (a) For an n-set S and $k = 1$, the number of ways to partition S into one part is always 1, i.e.,

$$S_{n,1} = 1.$$

(b) For $S = \{1, 2\}$, there is only one partition of S into two parts, namely, $\{\{1\}, \{2\}\}$. There is only one partition of S into one part.

(c) For $S = \{1, 2, 3\}$, there are three partitions of S into two parts:

$$\{\{1\}, \{2, 3\}\}, \quad \{\{1, 2\}, \{3\}\}, \quad \{\{1, 3\}, \{2\}\}.$$

There is only one partition of S into one part.

(d) For $S = \{1, 2, 3, 4\}$, there are 7 partitions of S into two parts:

$$\{\{1\}, \{2, 3, 4\}\}, \quad \{\{1, 3, 4\}, \{2\}\}, \quad \{\{1, 2, 4\}, \{3\}\}, \quad \{\{1, 2, 3\}, \{4\}\},$$

$$\{\{1, 2\}, \{3, 4\}\}, \quad \{\{1, 3\}, \{2, 4\}\}, \quad \{\{1, 4\}, \{2, 3\}\}.$$

There are 6 partitions of S into three parts:

$$\{\{1\}, \{2\}, \{3, 4\}\}, \quad \{\{1\}, \{3\}, \{2, 4\}\}, \quad \{\{1\}, \{4\}, \{2, 3\}\},$$

$$\{\{2\}, \{3\}, \{1, 4\}\}, \quad \{\{2\}, \{4\}, \{1, 3\}\}, \quad \{\{3\}, \{4\}, \{1, 2\}\}.$$

Theorem 2.13. *The number $S_{n,k}$ satisfy the recurrence relation:*

$$\begin{cases} S_{0,0} = S_{n,n} = 1 & \text{for} \quad n \geq 0 \\ S_{n,0} = 0 & \text{for} \quad n \geq 1 \\ S_{n,1} = 1 & \text{for} \quad n \geq 1 \\ S_{n,k} = S_{n-1,k-1} + kS_{n-1,k} & \text{for} \quad n-1 \geq k \geq 1 \end{cases} \tag{5}$$

Proof. Obviously, $S_{0,0} = S_{n,n} = 1$. For $n \geq 1$, its also clear that $S_{n,0} = 0$ and $S_{n,1} = 1$.

Let S be a set of n elements, $n > k \geq 1$. Fix an elements $a \in S$. The partitions of S into k parts can be divided into two categories: partitions in which $\{a\}$ is a single part, and the partitions that $\{a\}$ is not a single part. The formal partitions can be viewed as partitions of $S - \{a\}$ into $k - 1$ parts; there are $S_{n-1,k-1}$ such partitions. The latter partitions can be obtained by partitions of $S - \{a\}$ into k parts and joining the element a in one of the k parts; there are $kS_{n-1,k}$ such partitions. Thus

$$S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}.$$

□

Corollary 2.14.

$$S(p, k) = S_{p,k}, \quad 0 \leq k \leq p.$$

Theorem 2.15.

$$C(p, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^p$$

$$S(p, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^p.$$

Definition 2.16. The *pth Bell number B_p* is the number of partitions of a p-set into nonempty indistinguishable boxes, i.e.,

$$B_p = \sum_{k=0}^{p} S_{p,k}.$$

The first few Bell numbers are

$$\begin{array}{ll} B_0 = 1 & B_4 = 15 \\ B_1 = 1 & B_5 = 52 \\ B_2 = 2 & B_6 = 203 \\ B_3 = 5 & B_7 = 877 \end{array}$$

Theorem 2.17. *For $p \geq 1$,*

$$B_p = \sum_{k=0}^{p-1} \binom{p-1}{k} B_k.$$

Proof. Let S be a set of p elements and fix an element $a \in S$. For each partition $\mathcal{P}$ of S, there is a part (or block) A which contains a. Then $A' = A - \{a\}$ is a subset of $S - \{a\}$. The other blocks of $\mathcal{P}$ except the block A form a partition $\mathcal{P}'$ of $S - A$. Let $k = |A'|$. Then $0 \leq k \leq p-1$. Conversely, for any subset $A' \subset S - \{a\}$ and any partition $\mathcal{P}'$ of $S - A' \cup \{a\}$, the collection $\mathcal{P}' \cup \{A \cup \{a\}\}$ forms a partition of S. If $|A'| = k$, then there are $\binom{p-1}{k}$ ways to select A'; there are B_{p-1-k} partitions for the set $S - A' \cup \{a\}$. Thus

$$B_p = \sum_{k=0}^{p-1} \binom{p-1}{k} B_{p-1-k} = \sum_{k=0}^{p-1} \binom{p-1}{k} B_k.$$

□

The falling factorial $[n]_k$ is a polynomial of degree k in n, and so can be written as a linear combination of the monomials $1, n, n^2, \ldots, n^k$. Let

$$[n]_p = n(n-1)\cdots(n-p+1) = \sum_{k=0}^{p} s(p,k)n^k = \sum_{k=0}^{p} (-1)^{p-k} c(p,k) n^k.$$

The integers $s(p,k)$ are called the *Stirling numbers of the first kind.* For variables $x_1, x_2, \ldots, x_p$, the elementary symmetric polynomials $s_0, s_1, s_2, \ldots, s_p$ are defined as follows:

$$\begin{array}{rcl} s_0(x_1, x_2 \ldots, x_p) & = & 1, \\ s_1(x_1, x_2 \ldots, x_p) & = & \displaystyle\sum_{i=1}^{p} x_i, \\ s_2(x_1, x_2 \ldots, x_p) & = & \displaystyle\sum_{i<j} x_i x_j, \\ & \vdots & \\ s_p(x_1, x_2 \ldots, x_p) & = & x_1 x_2 \cdots x_p, \end{array}$$

Since

$$[n]_p = n(n-1)\cdots(n-p+1) = \sum_{k=0}^{p} (-1)^{p-k} s_{p-k}(0, 1, \ldots, p-1) n^k,$$

we have

$$s(p,k) = (-1)^{p-k} s_{p-k}(0, 1, \ldots, p-1).$$

Theorem 2.18. *The integers $c(p,k)$ satisfy the recurrence relation:*

$$\begin{cases} c(0,0) = c(p,p) = 1 & \text{for} \quad p \geq 0 \\ c(p,0) = 0 & \text{for} \quad p \geq 1 \\ c(p,k) = c(p-1,k-1) + (p-1)c(p-1,k) & \text{for} \quad p-1 \geq k \geq 1 \end{cases} \tag{6}$$

Proof. It follows from the definition that $c(0,0) = c(p,p) = 1$ and $c(p,0) = 0$ for $p \geq 1$.

Let $1 \leq k \leq p-1$. Note that

$$[n]_p = \sum_{k=0}^{p} (-1)^{p-k} c(p,k) n^k,$$

$$[n]_{p-1} = \sum_{k=0}^{p-1} (-1)^{p-1-k} c(p-1,k) n^k,$$

$$[n]_p = (n-(p-1))[n]_{p-1}.$$

Then

$$\begin{aligned} [n]_p &= \sum_{k=0}^{p-1} (-1)^{p-1-k} (n-(p-1)) c(p-1,k) n^k \\ &= \sum_{k=0}^{p-1} (-1)^{p-1-k} c(p-1,k) n^{k+1} - \sum_{k=0}^{p-1} (-1)^{p-1-k} (p-1) c(p-1,k) n^k \\ &= \sum_{k=1}^{p} (-1)^{p-k} c(p-1,k-1) n^k + (p-1) \sum_{k=0}^{p-1} (-1)^{p-k} c(p-1,k) n^k. \end{aligned}$$

Comparing the coefficients of n^k, we obtain

$$c(p,k) = c(p-1,k-1) + (p-1)c(p-1,k), \quad 1 \leq k \leq p-1.$$

☐

Proposition 2.19. *Let $c_{n,k}$ denote the number of arrangements of n objects into k nonempty circular permutations. Then $c_{n,k}$ satisfy the recurrence relation:*

$$\begin{cases} c_{0,0} = c_{n,n} = 1 & \text{for} \quad n \geq 0 \\ c_{n,0} = 0 & \text{for} \quad n \geq 1 \\ c_{n,k} = c_{n-1,k-1} + (n-1)c_{n-1,k} & \text{for} \quad n-1 \geq k \geq 1 \end{cases} \tag{7}$$

Proof. Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of n objects. It is clear that

$$c_{n,n} = 1 \quad \text{for} \quad n \geq 0$$

because when S is divided into n circles then each circle contains exactly one object. We also have

$$c_{n,0} = 0 \quad \text{for} \quad n \geq 1$$

because when S is nonempty then any arrangement contains at least one circle.

Now fix the object a_n of S. Then the arrangements of S into k circles can be divided into two kinds: the arrangements that the singleton $\{a_n\}$ is a circle, and the arrangements that the singleton $\{a_n\}$ is not a circle. In the former case, deleting the circle $\{a_n\}$ the arrangements become the arrangements of $n-1$ objects into $k-1$ circles; there are $c_{n-1,k-1}$ such arrangements. In the latter case, deleting the element a from the circle that a_n is contained, the arrangements S become arrangements of $S - \{a_n\}$ into k circles; for each such arrangement S into k circles can be obtained by petting a_n into the left of elements $a_1, a_2, \ldots, a_{n-1}$, and there are $n-1$ such ways; there are $(n-1)c_{n-1,k}$ arrangements of the second type. Thus we obtain the recurrence relation:

$$c_{n,k} = c_{n-1,k-1} + (n-1)c_{n-1,k} \quad \text{for} \quad 1 \leq k \leq n-1.$$

☐

Corollary 2.20.

$$c(p,k) = c_{p,k}.$$

3 Partition Numbers

A *partition of a positive integer* n is a representation of n as an unordered sum of one or more positive integers (called *parts*). The number of partitions of n is denoted by p_n. For instance,

$$\begin{aligned} 2 &= 1+1, \\ 3 &= 2+1 = 1+1+1, \\ 4 &= 3+1 = 2+2 = 2+1+1 = 1+1+1+1. \end{aligned}$$

Thus $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $p_4 = 5$. The *partition sequence* is the sequence of numbers

$$p_0 = 1, \quad p_1, \quad p_2, \quad \ldots, \quad p_n, \quad \ldots.$$

A partition of n is sometimes symbolically written as

$$\lambda = 1^{a_1} 2^{a_2} \cdots n^{a_n}$$

where a_k is the number of parts equal to k. If i is not a part of the partition λ then $a_k = 0$, and in this case the term k^{a_k} is usually omitted. For instance, the partitions

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$$

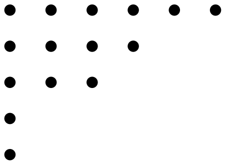
can be written as

$$5^1, \quad 1^1 4^1, \quad 2^1 3^1, \quad 1^2 3^1, \quad 1^1 2^2, \quad 1^3 2^1, \quad 1^5.$$

Let λ be the partition

$$n = n_1 + n_2 + \cdots + n_k$$

of n with $n_1 \geq n_2 \geq \cdots \geq n_k$. The *Ferrers diagram* of λ is a left-justified array of dots which has$k$ rows with n_i dots in the ith row. For instance, the Ferrers diagram of the partition $15 = 6+4+3+1+1$ is



Theorem 3.1.

$$\sum_{n=0}^{\infty} p_n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{8}$$

Proof. Note that the right side of (8) is the product of the series

$$\frac{1}{1-x^k} = 1 + x^k + x^{2k} + x^{3k} + \cdots$$

for $1 \leq k \leq \infty$. The term x^n arises in the product by choosing a term $x^{a_1 1}$ form the first factor, a term $x^{a_2 2}$ from the second factor, a term $x^{a_3 3}$ from the third factor, and so on, with

$$a_1 1 + a_2 2 + a_3 3 + \cdots + a_k k + \cdots = n.$$

Such choices are in one-to-one correspondent with the partitions

$$\lambda = 1^{a_1} 2^{a_2} 3^{a_3} \cdots (n-1)^{a_{n-1}} n^{a_n}$$

of the integer n. □

Definition 3.2. Let λ and μ be partitions of a positive integer n, and

$$\begin{aligned} \lambda: &\quad n=\lambda_1+\lambda_2+\cdots+\lambda_k, &\quad \lambda_1\ge\lambda_2\ge\cdots\ge\lambda_k,\\ \mu: &\quad n=\mu_1+\mu_2+\cdots+\mu_k, &\quad \mu_1\ge\mu_2\ge\cdots\ge\mu_k. \end{aligned}$$

The partition λ is called *majorized by* the partition μ (or μ *majorizes* λ), denoted by $\lambda \le \mu$, if

$$\lambda_1+\lambda_2+\cdots+\lambda_i \le \mu_1+\mu_2+\cdots+\mu_i \quad \text{for} \quad i=1,2,\ldots,k.$$

Example 3.1. Consider the three partitions of 9:

$$\begin{aligned} \lambda: &\quad 9=5+1+1+1+1,\\ \mu: &\quad 9=4+2+2+1,\\ \nu: &\quad 9=4+4+1. \end{aligned}$$

The partition μ is majorized by the partition ν because

$$\begin{aligned} &4\le 4,\\ &4+2\le 4+4,\\ &4+2+2\le 4+4+1,\\ &4+2+2+1\le 4+4+1. \end{aligned}$$

However, the partitions λ and μ are incomparable because

$$\begin{aligned} &5>4,\\ &4+2+2>5+1+1. \end{aligned}$$

Similarly, λ and ν are incomparable.

Theorem 3.3. *The lexicographic order is a linear extension of the partial order of majorization on the set $\mathcal{P}_n$ of partitions of a positive integer n.*

Proof. Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ and $\mu=(\mu_1,\mu_2,\ldots,\mu_k)$ be distinct partitions of n. We need to show that if λ is majorized by μ then there exists an i such that

$$\lambda_1=\mu_1,\quad \lambda_2=\mu_2,\quad \ldots,\quad \lambda_{i-1}=\mu_{i-1},\quad \text{and}\quad \lambda_i<\mu_i.$$

In fact we choose the smallest integer i such that $\lambda_j=\mu_j$ for all $j<i$ but $\lambda_i\ne\mu_i$. Since

$$\lambda_1+\lambda_2+\cdots+\lambda_i\le\mu_1+\mu_2+\cdots+\mu_i,$$

we conclude that $\lambda_i<\mu_i$, and hence λ precedes μ in the lexicographic order. □

4 A Geometric Problem

In this section we shall give a geometric and combinatorial interpretation for the numbers

$$h_n^{(k)}=\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{k},\quad k\ge 0,\ n\ge 0.$$

For each fixed $k\ge 0$, we obtain a sequence

$$h_0^{(k)},\quad h_1^{(k)},\quad h_2^{(k)},\quad \ldots,\quad h_n^{(k)},\quad \ldots.$$

For each fixed n,

$$2^n=h_n^{(n)})=h_n^{(n+1)}=h_n^{(n+2)}=\cdots.$$

For $k=0$, we have

$$h_n^{(0)}=\binom{n}{0}=1,\quad n\ge 0,$$

which is the constant sequence 1. For $k = 1$, we obtain

$$h_n^{(1)} = \binom{n}{0} + \binom{n}{1} = n + 1, \quad n \geq 0.$$

For $k = 2$, we have

$$h_n^{(2)} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} = \frac{n^2 + n + 2}{2}, \quad n \geq 0.$$

Using Pascal's formula $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$, the difference of the sequence $h_n^{(k)}$ can be computed as

$$\begin{aligned} \Delta h_n^{(k)} &= h_{n+1}^{(k)} - h_n^{(k)} \\ &= \sum_{i=0}^{k} \binom{n+1}{i} - \sum_{i=0}^{k} \binom{n}{i} \\ &= \sum_{i=0}^{k} \left[\binom{n+1}{i} - \binom{n}{i} \right] \\ &= \sum_{i=0}^{k} \binom{n}{i-1} \\ &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k-1} = h_n^{(k-1)}. \end{aligned}$$

Theorem 4.1. *The number $h_n^{(k)}$ counts the number of regions into which a k-dimensional space is divided by n hyperplanes in general position.*

Hyperplanes of a k-dimensional space are in general position means that any l $(1 \leq l \leq k)$ hyperplanes meet in a $(k - l)$-dimensional plane, but no $l + 1$ hyperplanes meet in a $(k - l)$-dimensional plane.

Proof. For $k = 1$, a 1-dimensional space is a straight line, and a 0-dimensional plane is a point. If n distinct points are inserted on the straight line, the line gets divided into $n + 1$ parts (called regions). Thus the number of regions of a line divided by n distinct points is

$$h_n^{(1)} = \binom{n}{0} + \binom{n}{1}.$$

For $k = 2$, consider n lines in a plane in general position. General position means in this case that the lines are distinct and not parallel (so that each pair of lines intersects in exactly one point) and no three lines meet in the same point.

Suppose n lines are in general position and we insert a new line so that the the total $n + 1$ lines are in general position. The first n lines intersect the $(n + 1)$th line at n distinct points, and the $(n + 1)$th line is divided into

$$h_n^{(1)} = n + 1$$

parts. Each of these $h_n^{(1)}$ parts divides a region formed by the first n lines into two regions. Thus the number of regions formed by $n + 1$ lines is increased by $h_n^{(1)} = \Delta h_n^{(2)}$ from the number of regions formed by n lines. Note that

$$\Delta h_n^{(2)} = h_{n+1}^{(2)} - h_n^{(2)} = n + 1.$$

Since $h_0^{(2)} = 1$ is the number of regions of a plane divided by zero lines, we conclude that the number of regions of a plane divided by n lines in general position is

$$h_n^{(2)} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}.$$

For $k = 3$, the number regions of 3-dimensional space divided by zero planes is $h_0^{(3)} = 1$. Consider $n + 1$ planes in general position, the first n planes intersect the $(n+1)$th plane in n lines, and these n lines on the $(n+1)$th plane are in general position. Then there are $h_n^{(2)}$ regions on the $(n + 1)$th plane, and each of these regions divides a solid region formed by the first n planes into two solid regions. Thus the number of solid regions formed by $n + 1$ planes in general position is increased by $h_n^{(2)} = \Delta h_n^{(3)}$ from the number of solid regions formed by n planes in general position. By induction we conclude that the number of solid regions formed by n planes in general position is

$$h_n^{(3)} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$

More generally, the number of regions of a k-dimensional space divided by 0 number of $(k-1)$-dimensional planes is $h_0^{(k+1)} = 1$. Consider $n+1$ planes of dimension $k-1$, the first n planes intersect the $(n+1)$th plane in n distinct $(k-2)$-dimensional planes in general position. These n planes of dimension $k-2$ divide the $(n+1)$th plane into $h_n^{(k-1)}$ regions of dimension $k-1$, and each of these $(k-1)$-dimensional regions divides a k-dimensional region (formed by the first n planes of dimension $k-1$) into two k-dimensional regions. Then the number of k-dimensional regions formed by $n+1$ planes of dimension $k-1$ is increased by $h_n^{(k-1)} = \Delta h_n^{(k)}$ from the number of regions formed by the first n planes of dimension $k-1$. By induction we conclude that

$$h_n^{(k)} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}.$$

□