Euclid's Geometry

February 14, 2013

The first monument in human civilization is perhaps the Euclidean geometry, which was crystalized around 2000 years ago. The Copernican revolution is the next. In the twentieth century there are four revolutions: Darwinian theory of evolution, Marxian theory of communism, Einstein's theory of relativity, and Freud's interpretation of dreams.

1 Origins of Geometry

2 The Axiomatic Method

Requirement 1. Acceptance of certain statements called "axioms," or "postulates," without further justification.
Requirement 2. Agreement on how and when one statement "follows logically from another, i.e., agreement on certain rules of reasoning.

The following are the axioms listed in a school book of plane geometry, *New Plane Geometry* by Durell and Arnold, Charles E. Merrill Co. 1924.

Geometric figures: point, line, circle, surface, or solid, or any combination of these. (similar geometric figures, equivalent figures, congruent figures, shape=form)
Angle terminology: acute angle, obtuse angle, reflex angle, oblique angle (general term for acute, obtuse, and reflex angles); adjacent angle, vertical (opposite) angle; complementary angle, supplementary angle; corresponding angles, alternate interior angles, alternate exterior angles, consecutive interior angles.

General axioms:

Axiom 1. Things which are equal to the same thing, or to equal things, are equal to each other.
Axiom 2. If equals are added to (subtracted from) equals, the sums (remainders) are equal.
Axiom 3. If equals are multiplied (divided) by equals, the products (quotients) are equal.
Axiom 4. Like powers (roots) of equals are equal.
Axiom 5. The whole is equal to the sum of its parts.
Axiom 6. The whole is greater than any of its parts.
Axiom 7. A quantity may be substituted for its equal in any process.

Geometric axioms:

Geometric Axiom 1. Through or connecting two given points, only one straight line can be drawn.

Geometric Axiom 2. A geometric figure may be freely moved in space without any change in form or size.

Geometric Axiom 3. Through a given point outside a given straight line, one straight line and only one can be drawn parallel to the given line.

Geometric Axiom 4. Geometric figures which can be made to coincide are congruent.

Geometric Postulates:

Postulate 1. Through or connecting two points, a straight line may be drawn.
Postulate 2. A straight line may be extended indefinitely, or it may be limited at any point.

POSTULATE 3. A circle may be decsribed about any given point as a center, and with any given radius.

Logical postulates.

3 Undefined Terms

Requirement 0. Mutual understanding of the meaning of the words and symbols used in the disclosure.

Here are the five undefined geometric terms that are the basis for defining all other geometric terms in the plane Euclidean geometry.

point
line
lie on (as "two points *lie on* a unique line")
between (as "point C is between points A and B")
congruent (as "triangle ΔABC is congruent to triangle ΔDEF")

Two figures (points, lines, segments, rays, angles, right angles, triangles, circles, etc.) are said to be **congruent** if one can be moved (by translation and rotation) to coincide with the other.

Synonyms:

The statement "point P lies on line ℓ" can be said "line ℓ passes through point P" or "P is incident with ℓ"

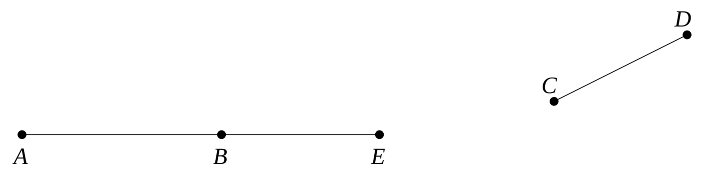
Sets and subsets. For example, "segment," "ray," and "circle," and other geometric terms are certain sets of points. However, a "line," is not a set of points in this book. The set of all points lying on a line "ℓ" is denoted by $\{\ell\}$.

4 Euclid's First Four Postulates

Euclid's Postulate I. For two distinct points P and Q there exists a unique line that passes through P and Q, denoted $\overleftrightarrow{PQ}$. (Two points determine a unique line.)

Definition 1 (segment)**.** Given two points A and B. The **segment** AB is the set whose members are the points A and B and all points that lie on the line $\overleftrightarrow{AB}$ and are between A and B.

Euclid's Postulate II. For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and the segment CD is congruent to segment BE. (Any segment AB can be extended by a segment BE congruent to a given segment CD.)



Definition 2 (circle)**.** Given two points O and A. The set of all points P such that segment OP is congruent to segment OA is called a **circle** with O as **center**, and each of the segments OP is called a **radius** of the circle.

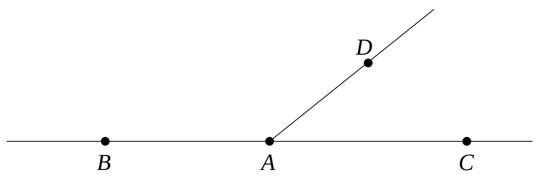
Euclid's Postulate III. For every point O and every point A not equal to O there exists a unique circle with center O and radius OA.

Definition 3 (ray)**.** The *ray* $\overrightarrow{AB}$ is the following set of points lying on the line $\overleftrightarrow{AB}$: those points that belong to the segment AB and all points C on $\overleftrightarrow{AB}$ such that B is between A and C. The ray $\overrightarrow{AB}$ is said to be *emanating from the vertex A through the point B.*

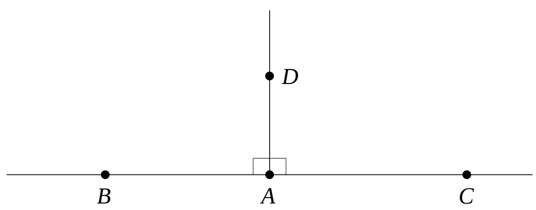
Definition 4 (opposite rays)**.** Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are *opposite* if they are distinct, if they emanate from the same point A, and if they are part of the same line $\overleftrightarrow{AB}=\overleftrightarrow{AC}$.

Definition 5 (angle with a vertex)**.** An *angle with a vertex* A is a point A together with two distinct rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (called the *sides* of the angle) emanating from A, denoted $\angle BAC$ or $\angle CAB$.

Definition 6 (supplementary angles)**.** If two angles $\angle BAD$ and $\angle CAD$ have a common side $\overrightarrow{AD}$, and other two sides $\overrightarrow{AB}$, $\overrightarrow{AC}$ form opposite rays, the angles are called *supplements* of each other, or *supplementary angles*.



Definition 7 (right angle)**.** An angle $\angle BAD$ is a *right angle* if it has a supplementary angle congruent to itself.



Euclid's Postulate IV. All right angles are congruent to each other.

This postulate expresses a sort of homogeneity – all right angles have the "same size," even they are "far away." The postulate provides a natural standard of measurement for angles.

5 The Parallel Postulate

Definition 8 (parallel)**.** Two lines l and m are *parallel* if they have no common point, denoted $l \parallel m$.

Euclid's Postulate V:
Euclidean Parallel Postulate (Popular Version) (EPP-P). For every line l and every point P not on the line l there exists a unique line m through P and parallel to l; see Figure 1.

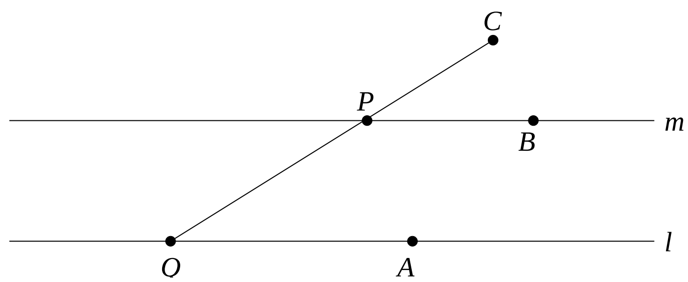


Figure 1: Unique parallel line to l through a point P outside l.

Since figures can be freely move in the space, when the line l is moved along the line $\overleftrightarrow{PQ}$ from point Q to point P, the line l is coincide with the line m and $\overleftrightarrow{PQ}$ is unchanged. Then the angle

$\angle AQP$ is congruent to the angle $\angle BPC$. Since $\angle BPC + \angle BPQ = \pi$, then $\angle AQP + \angle BPQ = \pi$. So

$$l \| m \;\Leftrightarrow\; \angle AQP = \angle BPC \;\Leftrightarrow\; \angle AQP + \angle BPQ = \pi.$$

Euclidean Parallel Postulate (Technical Version) (EPP-T). If a line that meets two lines makes the interior angles on the same side less than two right angles, then those two lines, if extended, will meet on that same side; see Figure 2.

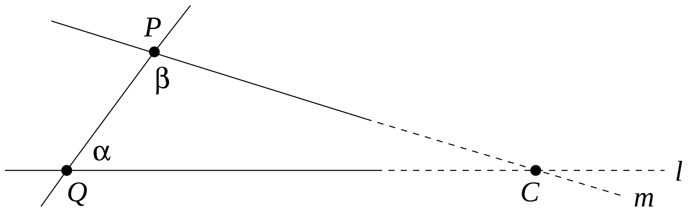


Figure 2: $\alpha + \beta < \pi$ implies that l and m meet on the same side.

Theorem 6. EPP-P *and* EPP-T *are logically equivalent.*

Proof. "EPP-P $\Rightarrow$ EPP-T": Let l, m, and n be three distinct lines such that n meets m at P and l at Q. Let $\overrightarrow{PD}$ be a ray parallel to l on the other side of n not containing α, β. See Figure 3. Assume $\alpha + \beta < \pi$. Then $\gamma + \angle BPQ > \pi$ because $\alpha + \beta + \gamma + \angle BPQ = 2\pi$. Since $\gamma + \angle DPQ = \pi$, then $\delta := \angle BPQ - \angle DPQ > 0$. So $\overrightarrow{PD}$ is not a ray on m and is between the rays $\overrightarrow{PB}$ and $\overrightarrow{PQ}$. Thus the ray $\overrightarrow{PB}$ does not meet l.

We need to show that the ray $\overrightarrow{PC}$ meets l. Suppose $\overrightarrow{PC}$ does not meet l. Then the line m is parallel to l and pass through P. Thus the ray $\overrightarrow{PD}$ must be on m by EPP-P. This is a contradiction.

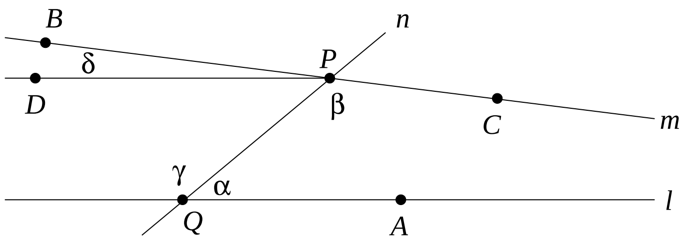


Figure 3: Equivalence of EPP-P and EPP-T.

"EPP-P $\Leftarrow$ EPP-T". Let P be a point outside line l. Let m be a line through P and parallel to l (assume lines through P and parallel to l exist). Fix a point Q on line l and make line $\overleftrightarrow{PQ}$; see Figure 4. We claim that $\alpha + \beta = \pi$. If $\alpha + \beta < \pi$, then m meets l by EPP-T, which is contradictory

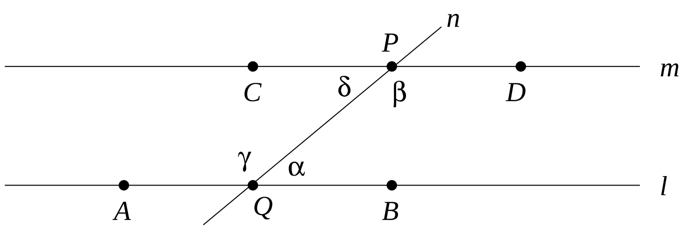


Figure 4: Equivalence of EPP and EPPA.

to that m is parallel to l. So $\alpha + \beta \geq \pi$. Likewise, $\gamma + \delta \geq \pi$. Since $\alpha + \beta + \gamma + \delta = 2\pi$, it follows that $\alpha + \beta = \gamma + \delta = \pi$.

Now the parallel line m can be constructed as follows: Draw the ray $\overrightarrow{QP}$. Draw angle β at P with sides PQ and PD such that $\beta = \gamma$. Draw the opposite ray $\overrightarrow{PC}$ of $\overrightarrow{PD}$. Then all lines m through P and parallel l must be equal to $\overleftrightarrow{CD}$. This is exactly the EPP-P. $\square$

7 The Power of Diagram

Theorem 8 (Pythagoras Theorem)**.** *For a right triangle with sides a, b and hypotenuse c,*

$$a^2 + b^2 = c^2.$$

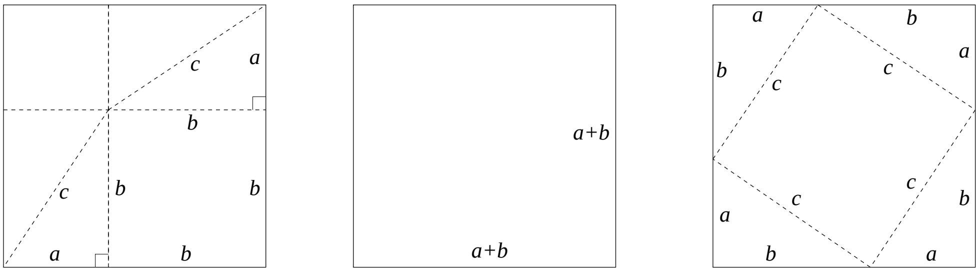


Figure 5: Equivalence of EPP and EPPA.

$$(a+b)^2 = a^2 + b^2 + 2ab,\ (a+b)^2 = c^2 + 2ab \ \Rightarrow\ a^2 + b^2 = c^2.$$

Practice is the unique standard to test truth. The first four Euclid's Postulates were easily accepted for they can be practiced by straightedge and compass. However, the Euclid's Fifth Postulate was controversial (though it seems obvious to us) for more than thousand years, since it cannot be practiced as there is no real infinite straightedge.

Proposition 9 (Side-Angle-Side Congruence)**.** *If two triangles have two sides and the included angle of one respectively equal to two sides and the included angle of the other, then the triangles are congruent.* (Align the side-angle-side equal part of the two triangles.)

Proposition 10 (Angel-Side-Angle Congruence)**.** *If two triangles have a side and the two adjacent angles of one respectively equal to a side and the adjacent angles of the other, then the triangles are congruent.* (Align the angle-side-angle equal part of the two triangles.)

Proposition 11 (Isosceles Triangle Theorem)**.** *If two sides of a triangle are equal, then the angels opposite these sides are equal.* (Flip over the isosceles triangle to coincide with its original one.)

Proposition 12 (Equal Three Sides Congruence)**.** *If two triangles have three sides of one respectively equal to three sides of the other, then the triangles are congruent.*

Proof. Given triangles ΔABC and ΔDEF such that $AB = DE$, $AC = DF$ and $BC = EF$. Align the longest sides of the two triangles such that B and E on the opposite sides of the line $\overleftrightarrow{DF}$ (flip over ΔABC if the two triangles are symmetric about a line) and make $DB = AB$. Draw segment EB. See Figure 6.

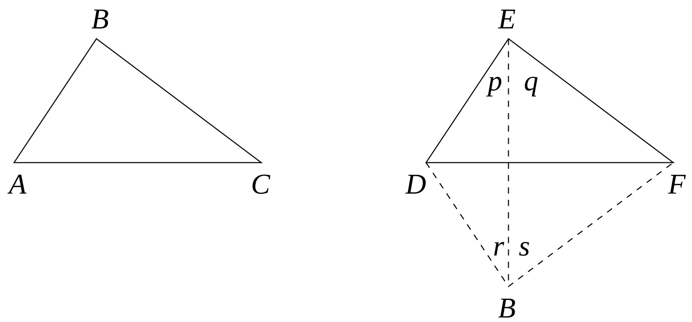


Figure 6: Corresponding Equal Sides Triangles

Then ΔDEB and ΔEFB are isosceles triangles. By the isosceles triangle theorem, $\angle p = \angle r$ and $\angle q = \angle s$; subsequently, $\angle E = \angle p + \angle q = \angle r + \angle s = \angle B$. Hence $\Delta ABC \simeq \Delta DEF$. $\square$

Proposition 13. *To construct an angle equal to a given angle.*

Constructing an angle congruent to a given angle $\angle BAC$: Draw a line $\overleftrightarrow{DE}$. Draw a partial circle of radius AB centered at D. Draw a partial circle of radius BC centered at E. Name the

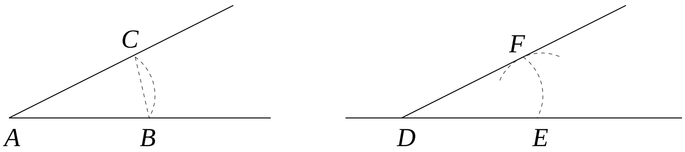


Figure 7: Constructing a given angle

intersection of the two partial circle as F. Draw the ray $\overrightarrow{DF}$. Then the angle $\angle EDF$ is congruent to $\angle BAC$ since $\Delta DEF \simeq \Delta ABC$ by Proposition 12.

Proposition 14. *To bisect a given angle by straightedge and compass.*

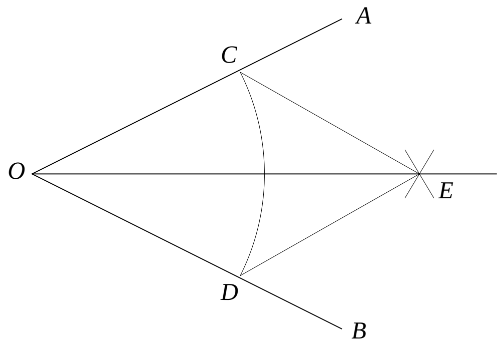


Figure 8: Bisecting an angle

Mark a point C on OA and a point D on OB such that $OC = OD$. Take a radius larger than CD. Make partial circles of the given radius centered at C and D, intersecting at a point E. Draw the segment OE, which is the bisector of the angle $\angle AOB$. See Figure 8.

Proposition 15. *To construct midpoint of a given segment.*

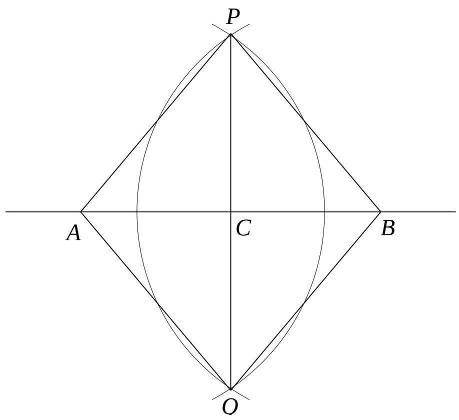


Figure 9: Bisecting an angle

Draw to partial circles of the same radius larger than AB, centered at A and B. Mark the two intersections as P and Q. Draw the segment PQ, intersecting AB at C. Then $\Delta APQ \simeq \Delta BPQ$ by three-sides congruence theorem. Hence $\angle APC = \angle AQC = \angle BPC = \angle BQC$. Thus $\Delta APC \simeq \Delta BPC$ by side-angle-side congruence theorem. We then have $AC = BC$ (i.e., C is the midpoint of AB), and $\angle ACP = \angle BCP$. Since $\angle ACP + \angle BCP = \pi$, we see that $\angle ACP = \angle BCP = \pi/2$ (i.e., $PC \perp AB$).

Proposition 16 (Exterior Angle Theorem)**.** *The exterior angle of triangle is greater than each of opposite interior angles.*

Proof. Given a triangle ΔABC and extend the segment AC to a point D. Find the midpoint E of BC. Draw segment DE and extend it to point F such that $EF \simeq DE$. Draw segment CE. Note that points E and F are on the same side of line $\overleftrightarrow{AD}$. Then the ray $\overrightarrow{CF}$ is between the rays $\overrightarrow{CB}$ and $\overrightarrow{CD}$.

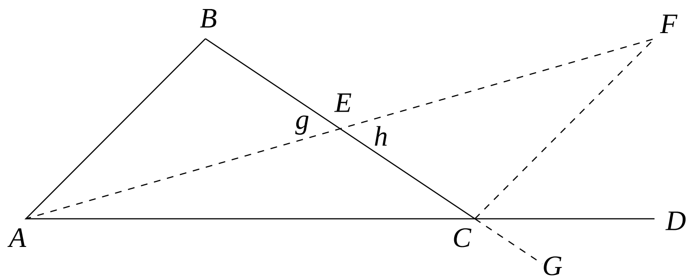


Figure 10: Exterior angle is larger than each of two opposite angles

Since $EB = EC$, $EA = EF$ and $\angle g = \angle h$, then $\Delta ABE \simeq \Delta FEC$ by Proposition 12. So $\angle BCF = \angle B$. Since $\angle BCD > \angle BCF$, we have $\angle BCD > \angle B$.

Extend segment BC to a point G. Likewise, $\angle ACG > \angle A$. Since $\angle ACG = \angle BCD$, we have $\angle BCD > \angle A$. □

Proposition 17. *If two lines are cut by a transversal so that a pair of corresponding angles are equal, then the two lines are parallel.*

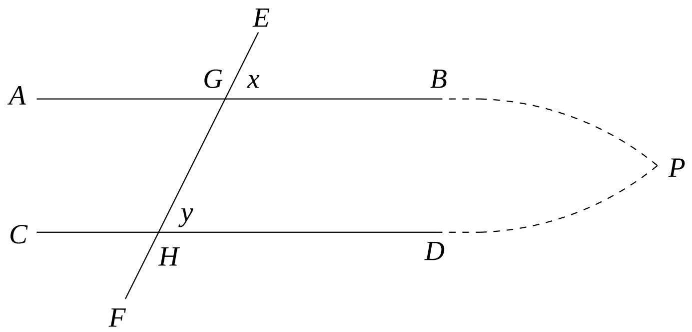


Figure 11: Corresponding angles are equal

Proof. Suppose the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet at a point P on the side of x and y (see Figure 11). Then x is an exterior angle of the triangle ΔGHP. Then $\angle x > \angle y$ by the exterior-angle theorem. This is contradictory to $x = y$. □

Corollary 18. *If two lines are cut by a transversal so that a pair of alternate interior angles are equal, then the two lines are parallel.*

Corollary 19. *If two lines are cut by a transversal so that two consecutive interior angles are supplementary, the two lines are parallel.*

Proposition 20. *To draw a line through a given point and parallel to a given line.*

Proposition 21. *If two parallel lines are cut by a transversal, then the alternate interior angles are equal.*

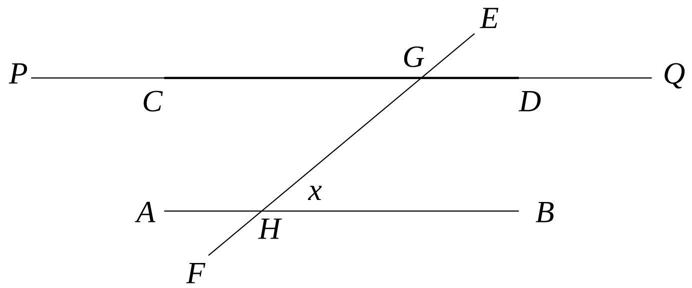


Figure 12: Interior alternate angles are equal

Proof. Let parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be cut by a transversal line $\overleftrightarrow{EF}$ at points G and H; see Figure 12. Draw a line $\overleftrightarrow{PQ}$ through G such that $\angle PGH = \angle x$ by Proposition 13. Then $\overleftrightarrow{PQ}$ is parallel to $\overleftrightarrow{AB}$ by Corollary 18. EPP-P implies $\overleftrightarrow{PQ}$=$\overleftrightarrow{CD}$. Hence $\angle CGH = \angle x$. $\square$

Theorem 22. *The sum of three angles of a triangle is equal to two right angles.*

Corollary 23. *If two angles have their sides respectively parallel, then they are either equal or supplementary.*

Corollary 24. *If two angles have their sides respectively perpendicular, then they are either equal or supplementary.*

Proposition 25. *If two angles of triangle are equal, then the opposite sides of these angles are equal.*

Proposition 26. *If two right angles have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, then the triangles are congruent.*