Homework – Non-Euclidean Geometry

Deadline: May 16, 2013

1. Given a circle γ with center O in Euclidean plane, and chord T_1T_2 of γ, parallel to a diameter NS of γ. Let two tangents of γ at T_1, T_2 meet at P. See Figure 1. Let

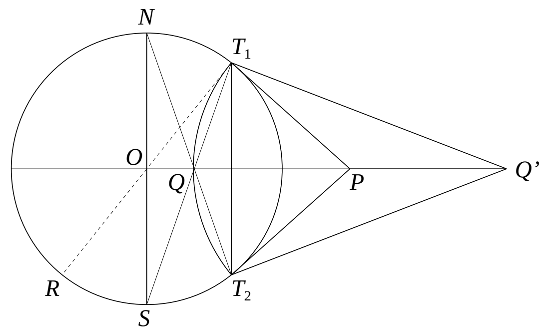

Figure 1: Construction of Poincaré line

   Q be the intersection of segments NT_2 and ST_1. Show that $|PT_1| = |PQ| = |PT_2|$. Subsequently, the open circular arc T_1QT_2 is a Poincaré line in the disk bounded by γ.

2. Let Σ be a sphere with the North Pole N and the South Pole S, and Π the Euclidean plane tangent to Σ at the South Pole S. Consider the stereographic projection

$$\mathrm{Proj} : \Sigma - \{N\} \to \Pi$$

   at the North Pole N. For each chord AB of equator γ of Σ, let $\widetilde{AB}$ denote the circular arc which is the intersection of the lower open hemisphere and the plane that contains AB and are perpendicular to Π. Show that the image $\mathrm{Proj}(\widetilde{AB})$ is a Poincaré line inside the Poincaré disk Δ bounded by the circle $\mathrm{Proj}(\gamma)$.

3. Given a Poincaré line l in the Poincaré disk bounded by a circle γ. For each point A not on l, let m be be the uniuqe Poincaré line through A perpendicular to l, with foot B on l. The unique point A' on m such that $d(A, B) = d(A', B)$ with $A * B * A'$ is called the **Poincaré reflection** about P-line l. Show that A, A' are inverse each other under the inversion in circle δ that contains the P-line l. (Hint: Applying cross-ratio.)