Connectivity

April 29, 2010

1 Vertex Connectivity

Let $G = (V, E)$ be a graph, $x, y \in V$.

- Tow (x, y)-path P and Q are said to be **internally disjoint** if they have no internal vertices in common.
- The **local connectivity** between two distinct vertices x and y is the maximum number of pairwise internally disjoint xy-paths, denoted $p(x, y)$ or $p_G(x, y)$.
- A nontrivial graph G is said to be k**-connected** if $p(u, v) \geq k$ for any two distinct vertices u, v. The **connectivity** of G is the maximum value of k for which G is k-connected.
- A trivial graph (i.e. a graph with a single vertex and no edges) is 0-connected and 1-connected, but not 2-connected.
- The complete graph K_n has $n - 2$ internally disjoint paths of length 2 and one path of length 1. So the connectivity of K_n is $n - 1$.
- Let G be a complete graph with multiple edges. Let $\mu(x, y)$ be the number edges between x and y. Then there $\mu(x, y)$ (x, y)-paths of length 1 and $n - 2$ internally disjoint (x, y)-paths. So the local connectivity between x and y is $n - 2 + \mu(x, y)$.
- Let μ be the minimum number of multiple edge between two distinct vertices of a complete graph G with multiple edges. Then the local connectivity G is
$$n - 2 + \mu.$$
- let x, y be two vertices nonadjacent in a graph G. An (x, y)**-vertex-cut** is a subset $S \subseteq V - \{x, y\}$ such that x, y belong to different components of $G - S$; we say that such a cut **separates** x and y. We denote by $c(x, y)$ the minimum size of an (x, y)-vertex-cut.
- A **vertex cut** of graph G is an (x, y)-vertex-cut for at least one pair of (x, y) of nonadjacent vertices. A vertex cut with k vertices is referred to a k**-vertex cut**.

Theorem 1.1 (Menger's Theorem)**.** *Let $G(x, y)$ be a graph with two nonadjacent vertices x, y . Then the maximum number of pairwise internally disjoint (x, y)-paths is equal to the minimum number of vertices in an (x, y)-vertex-cut, i.e.,*
$$p(x, y) = c(x, y).$$

Proof. Set $p := p_G(x, y)$, $k := c_G(x, y)$. There are p internally disjoint (x, y)-paths, and a vertex k-subset $K \subseteq V - \{x, y\}$ that separates x and y. Since every (x, y)-path meets S at an internal vertex, the p internally disjoint (x, y)-paths meet S at p vertices. Hence $p_G(x, y) \leq c_G(x, y)$. To prove $p_G(x, y) \geq c_G(x, y)$, we proceed by induction on the number of edges of G. We may assume that there is an edge e whose end-vertex is neither x nor y; otherwise, every (x, y)-path is of length 2, and the conclusion is obviously true.

Set $H := G\backslash e$. Since $|E(H)| < |E(G)|$ and $p_H(x,y) \leq c_H(x,y)$, we have $p_H(x,y) = c_H(x,y)$ by induction. Moreover, $c_G(x,y) \leq c_H(x,y) + 1$, since any (x,y)-vertex-cut of H, together with an end-vertex of e, is an (x,y)-vertex-cut of G. Hence

$$p_G(x,y) \geq p_H(x,y) = c_H(x,y) \geq c_G(x,y) - 1 = k - 1.$$

If $p_G(x,y) = k$, then there is nothing to prove. So we may assume that $p_G(x,y) = p_H(x,y) = c_H(x,y) = k-1$ and $c_G(x,y) = k$. Let $S := \{v_1, \ldots, v_{k-1}\}$ be a minimum (x,y)-vertex-cut of H. Let X be the set of vertices reachable from x in $H - S$, and Y the set of vertices reachable from y in $H - S$. Since $|S| = k-1$, the set S is not an (x,y)-vertex-cut of G; so there is an (x,y)-path in $G - S$. This path necessarily contains the edge e, and e must have end-vertices $u \in X$ and $v \in Y$.

Now consider the graph G/Y by contracting Y to y. It is clear that every (x,y)-vertex-cut in G/Y is an (x,y)-vertex-cut in G. Thus $c_{G/Y}(x,y) \geq k$. Note that $c_{G/Y}(x,y) \leq k$, because $S \cup \{u\}$ is an (x,y)-vertex-cut of G/Y. So $c_{G/Y}(x,y) = k$. Since $|E(G/Y)| < |E(G)$, by induction there are k internally disjoint (x,y)-paths $P_1, \ldots, P_k$ in G/Y, and each vertex of $S \cup \{u\}$ lies on one of them. Without loss of generality, we may assume that $v_i \in P_i$, $1 \leq i \leq k-1$, and $u \in P_k$. Likewise, there are k internally disjoint (x,y)-paths $Q_1, \ldots, Q_k$ in G/X such that $v_i \in Q_i$, $1 \leq i \leq k-1$, and $v \in Q_k$. Then there are k internally disjoint (x,y)-paths $P_i Q_i$ $(1 \leq i \leq k-1)$ and $P'_k e Q'_k$ in G, where $P_k = P' ev$, $Q_k = Q'_k eu$.

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