Cycle Space and Bond Space

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1 Flows and Tensions

Let $D = (V, A)$ be a digraph. Let be a function.

- A real-valued function $f : A \to \mathbb{R}$ is called a **flow** (or **circulation**) of D if

$$\sum_{a \in (v^c, v)} f(a) = \sum_{a \in (v, v^c)} f(a), \quad \text{i.e.,} \quad f^-(v) = f^+(v) \quad \text{for all} \quad v \in V.$$

  The set of all flows in D is a vector space, called the **flow space** of D, denoted $F(D)$.

- A real-valued function $g : A \to \mathbb{R}$ is called a **tension** if for each directed cycle C in $(V, \vec{E})$,

$$g^+(C) - g^-(C) = 0,$$

  where

$$g^+(C) := \sum_{a \in C} g(a), \quad g^-(C) := \sum_{-a \in C} g(a),$$

  The set of all tension of D is a vector space, called the **tension space** of D, denoted $T(D)$.

- If C is a directed in $(V, \vec{E})$, then $f_C : A \to \mathbb{R}$, defined by

$$f_C(a) = \begin{cases} 1 & \text{if } a \in C \\ -1 & \text{if } -a \in C, \\ 0 & \text{otherwise} \end{cases}$$

  is a flow of D, called the flow generated by the directed cycle C.

- If U is a directed cut in $(V, \vec{E})$, i.e., either $U = (X, X^c)$ or $U = (X^c, X)$ in $(V, \vec{E})$ for a nonempty proper subset $X \subset V$, then $g_U : A \to \mathbb{R}$, defined by

$$g_U(a) := \begin{cases} 1 & \text{if } a \in U \\ -1 & \text{if } -a \in U, \\ 0 & \text{otherwise} \end{cases}$$

  is a tension of D, called the flow generated by the directed cut U.

- Given a directed cycle C and a directed cut U in $(V, \vec{E})$. Then

$$\langle f_C, g_U \rangle = 0,$$

where $\langle f_C, g_U \rangle := \sum_{a \in A} f_C(a) g_U(a)$. In fact,

$$\langle f_C, g_U \rangle = |A \cap C \cap U| - |A \cap C^- \cap U| - |A \cap C \cap U^-| + |A \cap C^- \cap U^-| = 0,$$

since $|A \cap C \cap U| = |A \cap C \cap U^-|$ and $|A \cap C^- \cap U| = |A \cap C^- \cap U^-|$.

- Let $\mathbf{M}$ be incidence matrix of D, i.e., $\mathbf{M} = [m_{va}]$, where $v \in V$, $a \in A$, and

$$m_{va} = \begin{cases} 1 & \text{if } a \text{ has its tail at } v \\ -1 & \text{if } a \text{ has its head at } v. \\ 0 & \text{otherwise} \end{cases}$$

Then

$$F(D) = \ker \mathbf{M}, \quad \mathbf{T(D)} = \operatorname{Row} \mathbf{M},$$

and

$$\dim F(D) = |V(D)| - c(D), \quad \dim T(D) = |A(D)| - |V(D)| + c(D)|.$$

2 Basis Matrices

A **basis matrix** of an m-dimensional vector subspace of $\mathbb{R}^n$ is an $m \times n$ matrix whose row space is the given vector subspace. For a digraph D, we are interested in the integral basis matrix $\mathbf{B}$ of the tension space $T(D)$ and the basis matrix $\mathbf{C}$ of the flow space $F(D)$. For an edge subset $S \subseteq A(D)$, we denote by $\mathbf{B}|_S$ (or just $\mathbf{B}_S$) the submatrix of $\mathbf{B}$ consisting of the columns of $\mathbf{B}$ that are labeled by members of S.

Given a maximal spanning forest F of G. For each edge $e \in F$, $F^c \cup e$ contains a unique bond B_e, which must contain the edge e. For each edge $e' \in F^c$, $F \cup e'$ contains a unique cycle $C_{e'}$, which must contain the edge e'. It is well-know that the bond vectors g_{B_e}, $e \in F$, form an integral basis of the tension lattice of D, and the cycle vectors $f_{C_{e'}}$, $e' \in F^c$, form an integral basis of the flow lattice of G. Let the members of F be listed as $e_1, \ldots, m$, and the members of F^c as $e'_1, \ldots, e'_n$. We obtain integral basis matrices

$$\mathbf{B} = \begin{array}{c} \\ e_1 \\ e_2 \\ \vdots \\ e_m \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & \cdots & e_m & e'_1 & e'_2 & \cdots & e'_n \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & * & * & \cdots & * \\ 0 & 1 & \cdots & 0 & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & * & * & \cdots & * \end{pmatrix} \end{array},$$

$$\mathbf{C} = \begin{array}{c} \\ e'_1 \\ e'_2 \\ \vdots \\ e'_n \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & \cdots & e_m & e'_1 & e'_2 & \cdots & e'_n \end{array} \\ \begin{pmatrix} * & * & \cdots & * & 1 & 0 & \cdots & 0 \\ * & * & \cdots & * & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * & 0 & 0 & \cdots & 1 \end{pmatrix} \end{array},$$

called **integral basis matrices** of the tension lattice and flow the lattice relative to the maximal spanning forest F respectively.

Theorem 2.1. *Let $\mathbf{B}$ be a basis matrix of the tension $T(D)$ of a digraph D, and $\mathbf{C}$ the basis matrix of the flow space $F(D)$. Given a nonempty subset $S \subset A(D)$.*

(a) The columns of $\mathbf{B}|_S$ are linearly independent iff S does not contain cycle.

(b) The columns of $\mathbf{C}|_S$ are linearly independent iff S does not contain bond.

Proof. (a) Let $\mathbf{b}(a)$ denote the column vector of $\mathbf{B}$ corresponding to the arc $a \in A(D)$. We may write $\mathbf{b}(a) = [b_1(a), \ldots, b_m(a)]^T$, where $m = \dim T(D)$. The columns $\mathbf{b}(a)$ for $a \in S$ are linearly dependent iff there exists a nonzero function $f : A \to \mathbb{R}$ such that $f|_{A \smallsetminus S} = 0$ and

$$\sum_{a \in A} f(a)\mathbf{b}(a) = 0, \quad \text{i.e.,} \quad \langle f, b_i \rangle = 0 \;\text{ for }\; 1 \le i \le m,$$

which means that f is a flow of D and its support is contained in S. Now if there is such a flow f whose support is contained in S, then the support of f contains a cycle, so does S. If S contains a cycle C, then f_C is a nonzero flow whose support is C, which is contained in S.

(b) Let $\mathbf{c}(a)$ denote the column vector of $\mathbf{C}$ corresponding to the arc $a \in A(D)$. The columns $\mathbf{c}(a)$ for $a \in S$ are linearly independent iff there exists a nonzero function g on A such that $\sum_{a \in A} f(a)\mathbf{c}(a) = 0$, i.e., there exists a nonzero tension whose support is contained in S. Now if there is such a tension g whose support is contained in S, then the support of g contains a bond, so does S. If S contains a bond B, then g_B is a nonzero tension whose support is B, which is contained in S. □

A rectangular matrix is said to be **unimodular** if its full square submatrices have determinates 1, -1, or 0; and to be **totally unimodular** if its all square submatrices have determinates 1, -1, or 0.

Lemma 2.2. *Let $\mathbf{B}$ be a basis matrix of the tension of a connected digraph D, and $\mathbf{C}$ the basis matrix of the flow space. Given a maximal spanning forest F of D.*

(a) Then $\mathbf{B}$ is uniquely determined by $\mathbf{B}|_F$, and $\mathbf{C}$ is uniquely determined by $\mathbf{C}|_{F^c}$.

(b) If $\mathbf{B}, \mathbf{C}$ are basis matrices with respect to the maximal spanning forest F, then any basis matrices $\mathbf{B}', \mathbf{C}'$ of the tension and flow spaces respectively, we have

$$\mathbf{B}' = (\mathbf{B}'|_F)\mathbf{B}, \quad \mathbf{C}' = (\mathbf{C}'|_{F^c})\mathbf{C}.$$

Proof. (a) Since F contains no cycle, we see the columns of $\mathbf{B}|_F$ are linearly independent by Lemma 2.2(a). For each arc $a \in F^c$, the set $F \cup a$ contains a cycle, it follows that the columns of $\mathbf{B}|_{F \cup a}$ are linearly dependent; so the column $\mathbf{b}(a)$ is a unique linear combination of columns of $\mathbf{B}|_F$. So $\mathbf{B}$ is uniquely determined by $\mathbf{B}|_F$.

Analogously, the set F^c contains no bond, it follows from Lemma 2.2(b) that the columns of $\mathbf{C}|_{F^c}$ are linearly independent. Since $F^c \cup a$ contains a unique bond for each $a \in F$, the columns of $\mathbf{C}|_{F^c \cup a}$ are linearly dependent, so the column $\mathbf{c}(a)$ is a unique linear combination of the columns of $\mathbf{C}|_{F^c}$. So $\mathbf{C}$ is uniquely determined by $\mathbf{C}|_{F^c}$.

(b) We order the members of A as F, F^c. Since $\mathbf{B}, \mathbf{C}$ are respective to the maximal spanning forest F, we have the form

$$\mathbf{B} = \big[\mathbf{B}|_F \; \mathbf{B}|_{F^c}\big] = \big[\mathbf{I} \; \mathbf{A}\big], \quad \mathbf{C} = \big[\mathbf{C}|_F \; \mathbf{C}|_{F^c}\big] = \big[\mathbf{G} \; \mathbf{I}\big].$$

Write the basis matrices $\mathbf{B}', \mathbf{C}'$ in the same order F, F^c as the form

$$\mathbf{B}' = \big[\mathbf{B}'|_F \ \mathbf{B}'|_{F^c}\big], \quad \mathbf{C}' = \big[\mathbf{C}'|_F \ \mathbf{C}'|_{F^c}\big].$$

It is clear that there exist square matrices $\mathbf{P}, \mathbf{Q}$ such that

$$\mathbf{B}' = \mathbf{P}\mathbf{B}, \quad \mathbf{C}' = \mathbf{Q}\mathbf{C}.$$

Then

$$\mathbf{B}' = \mathbf{P}[\mathbf{I} \ \mathbf{A}] = [\mathbf{P}, \mathbf{P}\mathbf{A}], \quad \mathbf{C}' = \mathbf{Q}[\mathbf{G} \ \mathbf{I}] = [\mathbf{Q}\mathbf{G}, \mathbf{Q}]$$

It follows that $\mathbf{P} = \mathbf{B}'|_F$ and $\mathbf{Q} = \mathbf{C}'|_F$. Hence $\mathbf{B}' = (\mathbf{B}'|_F)\mathbf{B}$ and $\mathbf{C}' = (\mathbf{C}'|_F)\mathbf{C}$. □

Theorem 2.3. *Let $\mathbf{B}$ be an integral basis matrix of the tension space, and $\mathbf{C}$ an integral basis matrix of the flow space of a graph G. Then both $\mathbf{B}$ and $\mathbf{C}$ are unimodular.*

Proof. Given a maximal spanning forest F of G. Let $\mathbf{B}', \mathbf{C}'$ be basis matrices of the tension space and the flow space of G relative to F respectively. There exist unimodular matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{B}' = \mathbf{P}\mathbf{B}$ and $\mathbf{C}' = \mathbf{Q}\mathbf{C}$. Restrict both sides to F', F'^c respectively, we obtain

$$\mathbf{B}'|_{F'} = \mathbf{P}(\mathbf{B}|_{F'}), \quad \mathbf{C}'|_{F'^c} = \mathbf{Q}(\mathbf{C}'|_{F'^c}).$$

Since $\mathbf{B}'|_{F'}, \mathbf{C}'|_{F'^c}$ are identity matrices by definition, we see that

$$\det(\mathbf{P})\det(\mathbf{B}|_{F'}) = 1, \quad \det(\mathbf{Q})\det(\mathbf{C}'|_{F'^c}) = 1.$$

It follows that $\det(\mathbf{B}|_{F'}) = \pm 1$ and $\det(\mathbf{C}|_{F'^c}) = \pm 1$.

Given edge subsets $S \subset A(D)$. If $|S| = |V(D)| - 1$ and S is not a spanning tree, then S contains a cycle. Thus $\det(\mathbf{B}|_S) = 0$ by Lemma 2.2. If $|S| = |A(D)| - |V(D)| + 1$ and S^c is a not spanning tree, then S contains a bond, then $\det(\mathbf{C}|_S) = 0$ by Lemma 2.2. □

Proposition 2.4. *The incidence matrix $\mathbf{M}$ of an digraph $D = (V, A)$ is totally unimodular.*

Proof. Let $S \subseteq V$ and $F \subseteq E$ be such that $|S| = |F|$. If there exists a vertex $v \in S$ such that $v \notin V(F)$, then the v-row of $\mathbf{M}|_{S\times F}$ is a zero row; clearly, $\det(\mathbf{M}|_{S\times F}) = 0$. We may assume that $S \subseteq V(F)$. We see that $\mathbf{M}|_{S\times F}$ is the incidence matrix of the subgraph (S, F) with possible half-edges. If (S, F) contains a cycle, then the columns indexed by the edges of the cycle are linearly dependent; thus $\det(\mathbf{M}|_{S\times F}) = 0$. If (S, F) contains no cycles, we claim that S is a proper subset of $V(F)$. Otherwise, $S = V(F)$, then $(V(F), F)$ is a forest; thus $|F| = |V(F)| - c(F) = |S| - c(F) < |S|$, which is a contradiction.

Now let $e = uv \in F$ be an edge such that one of u, v is not in S, say, $v \notin S$. Then the e-column of $\mathbf{M}_{S\times F}$ has 1 or -1 at (u, e) and 0 elsewhere. Thus by the expansion along the e-column,

$$\det(\mathbf{M}|_{S\times F}) = \pm\det(\mathbf{M}|_{(S\smallsetminus u)\times(F\smallsetminus e)}) = \pm 1.$$

The second equality above follows from the fact that $(S \smallsetminus u, F \smallsetminus e)$ contains no cycles and by induction on the size of the matrix. □

3 The Matrix-Tree Theorem

In many occasions one needs to compute the determinant of a product matrix AB, where A is an $m \times n$ matrix and B an $n \times m$ matrix. If $m > n$, then $\det(AB) = 0$, since

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\} \leq n.$$

If $m \leq n$, we have the following Cauchy-Binet formula.

Proposition 3.1 (Cauchy-Binet Formula)**.** *Let A be an $m \times n$ matrix and B an $n \times m$ matrix. If $m \leq n$, then*

$$\det(A) = \sum_{S \subset [n],\, |S|=m} \det(A|_S) \det(B|_S), \tag{3.1}$$

where $A|_S$ is the $m \times m$ submatrix of A whose column index set is S, and $B|_S$ is the $m \times m$ submatrix of B whose row index set is S.

Proof. Let $A = [a_{ik}]_{m \times n}$, $B = [b_{kj}]_{n \times m}$, and $C = AB = [c_{ij}]_{m \times m}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Then

$$\begin{aligned}
\det(C) &= \det \begin{pmatrix} \sum_{k_1=1}^{n} a_{1k_1} b_{k_1 1} & \cdots & \sum_{k_m=1}^{n} a_{1k_m} b_{k_m m} \\ \vdots & \ddots & \vdots \\ \sum_{k_1=1}^{n} a_{mk_1} b_{k_1 1} & \cdots & \sum_{k_m=1}^{n} a_{mk_m} b_{k_m m} \end{pmatrix} \\
&= \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} \det \begin{pmatrix} a_{1k_1} b_{k_1 1} & \cdots & a_{1k_m} b_{k_m m} \\ \vdots & \ddots & \vdots \\ a_{mk_1} b_{k_1 1} & \cdots & a_{mk_m} b_{k_m m} \end{pmatrix} \\
&= \sum_{k_1,\ldots,k_m=1}^{n} \det \begin{pmatrix} a_{1k_1} & \cdots & a_{1k_m} \\ \vdots & \ddots & \vdots \\ a_{mk_1} & \cdots & a_{mk_m} \end{pmatrix} b_{k_1 1} \cdots b_{k_m m}.
\end{aligned}$$

Rewrite the nonzero terms in the above expansion of $\det(C)$, we obtain

$$\begin{aligned}
\det(C) &= \sum_{1 \leq k_1,\ldots,k_m \leq n,\, k_i \neq k_j} \det(A|_{\{k_1,\ldots,k_m\}})\, b_{k_1 1} \cdots b_{k_m m} \\
&= \sum_{1 \leq t_1 < \cdots < t_m \leq n} \sum_{\sigma \in S_m} \det(A|_{\{t_{\sigma(1)},\ldots,t_{\sigma(m)}\}})\, b_{t_{\sigma(1)} 1} \cdots b_{t_{\sigma(m)} m} \\
&= \sum_{1 \leq t_1 < \cdots < t_m \leq n} \det(A|_{\{t_1,\ldots,t_m\}}) \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) b_{t_{\sigma(1)} 1} \cdots b_{t_{\sigma(m)} m},
\end{aligned}$$

where $\mathfrak{S}_m$ is the set of all permutations of $\{1, \ldots, m\}$. Set $S = \{t_1, \ldots, t_m\}$, we have $\det(C) = \sum_{S \subset [n],\, |S|=m} \det(A|_S) \det(B|_S)$. $\square$

Theorem 3.2 (Matrix-Tree Theorem)**.** *Let $\mathbf{B}$ be an integral basis matrix of the tension space, and $\mathbf{C}$ an integral basis matrix of the flow space of a graph G. Then the number of maximal spanning forests of G is*

$$\det(\mathbf{B}\mathbf{B}^T) = \det(\mathbf{C}\mathbf{C}^T).$$

Proof. Let m, n be the dimensions of the tension space and the flow space of D respectively, and $t(G)$ the number of maximal spanning forests of D. Note that an edge subset $S \subseteq E(G)$ is a maximal forest of G iff S contains no cycles and $|S|$ is the dimension of the tension space of G. By the Cauchy-Binet formula, we have

$$\det(\mathbf{B}\mathbf{B}^T) = \sum_{\substack{S\subset E,\, |S|=m \\ S \text{ is ayclic}}} \det(\mathbf{B}_S)\det(\mathbf{B}_S^T) = \sum_{\substack{S\subset E,\, |S|=m \\ S \text{ is ayclic}}} (\det \mathbf{B}_S)^2 = t(G),$$

$$\det(\mathbf{C}\mathbf{C}^T) = \sum_{\substack{S\subset E,\, |S|=n \\ S^c \text{ is ayclic}}} \det(\mathbf{C}_{S^c})\det(\mathbf{C}_{S^c}^T) = \sum_{\substack{S\subset E,\, |S|=m \\ S \text{ is ayclic}}} (\det \mathbf{C}_{S^c})^2 = t(G).$$

The second equality follows from the fact that an set $S \subset E(G)$ is a maximal edge set containing no bonds of G iff S^c is a maximal edge set containing no cycles. □

Corollary 3.3. *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be integral bases of the tension space and the flow space of a digraph* D*. Then the number of maximal spanning forests of* D *is the absolute value of*

$$\det \begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}.$$

Proof. It follows from the calculation

$$\left(\det \begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}\right)^2 = \det\left(\begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{B}^T \, \mathbf{C}^T \end{bmatrix}\right) = \det \begin{bmatrix} \mathbf{B}\mathbf{B}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\mathbf{C}^T \end{bmatrix} = t(G)^2.$$

□

4 Farkas' Lemma

Lemma 4.1 (Farkas' Lemma)**.** *Let* $\mathbf{A}$ *be a real* $m \times n$ *matrix and* $\mathbf{b} \in \mathbb{R}^n$*. Then exactly one of the following two statements is valid.*

(a) There exists a vector $\mathbf{x} \in \mathbb{R}^n$ *such that* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$*.*

(b) There exists a vector $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{A}^T\mathbf{y} \geq \mathbf{0}$ *and* $\mathbf{b}^T\mathbf{y} < 0$*, i.e., such that*

$$\mathbf{y}^T\mathbf{A} \geq \mathbf{0}^T, \quad \mathbf{y}^T\mathbf{b} < 0.$$

Proof. Farkas's Lemma is just the geometric interpretation: Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ denote the columns of A. Let $\mathrm{Cone}(\mathbf{A})$ denote the convex cone generated by $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Let $\mathbf{x}^T = (x_1, \ldots, x_n) \geq 0$. Then $\mathbf{A}\mathbf{x} = \mathbf{b}$ means that $\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$. The first statement means that $\mathbf{b} \in \mathrm{Cone}(\mathbf{A})$.

Let $\mathbf{y}^T = (y_1, \ldots, y_m)$. Consider the hyperplane $H = \{\mathbf{z} \in \mathbb{R}^m : \langle \mathbf{z}, \mathbf{y}\rangle = 0\}$. Then $\mathbf{y}^T\mathbf{A} \geq \mathbf{0}$ means that $\langle \mathbf{a}_i, \mathbf{y}\rangle \geq 0$, $i = 1, \ldots, n$, i.e., $\mathrm{Cone}(\mathbf{A})$ lies in one side of H. While the strictly inequality $\mathbf{b}^T\mathbf{y} < 0$ means that $\mathbf{b}$ lies in the other side of H. In other words, H separates the vector $\mathbf{b}$ and the cone $\mathrm{Cone}(\mathbf{A})$, which is equivalent to $\mathbf{b} \notin \mathrm{Cone}(\mathbf{A})$.

Assume that the first statement is true, i.e., there exists a vector $\bar{\mathbf{x}} \geq \mathbf{0}$ such that $\mathbf{A}\bar{\mathbf{x}} = \mathbf{b}$. Suppose the second statement is also true, i.e., there exists a vector $\bar{\mathbf{y}}$ such that $\mathbf{A}^T\bar{\mathbf{y}} \geq \mathbf{0}$ and $\mathbf{b}^T\bar{\mathbf{y}} < 0$. Then

$$0 > \mathbf{b}^T\bar{\mathbf{y}} = (\mathbf{A}\bar{\mathbf{x}})^T\bar{\mathbf{y}} = \bar{\mathbf{x}}\mathbf{A}^T\bar{\mathbf{y}} \geq 0,$$

which is a contradiction. □

Lemma 4.2 (Farkas' Lemma – variant version)**.** *Let* $\mathbf{A}$ *be a real* $m \times n$ *matrix and* $\mathbf{b} \in \mathbb{R}^n$*. Then exactly one of the following two statements is valid.*

(a) There exists a vector $\mathbf{x} \in \mathbb{R}^n$ *such that* $\mathbf{Ax} = \mathbf{0}$ *and* $\mathbf{x} \geq \mathbf{b}$*.*

(b) There exists a vector $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y}^T\mathbf{A} \geq \mathbf{0}$ *and* $\mathbf{y}^T\mathbf{Ab} > 0$*.*

Proof. Let $\mathbf{b}' = -\mathbf{Ab}$. The second statement becomes that there exists a vector $\mathbf{y}$ such that $\mathbf{y}^T\mathbf{A} \geq \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b}' < 0$. Let $\mathbf{x} = \mathbf{x}' + \mathbf{b}$. The first statement becomes that there exists a vector $\mathbf{x}'$ such that $\mathbf{Ax}' = \mathbf{b}'$ and $\mathbf{x}' \geq \mathbf{0}$. □

Lemma 4.3 (Farkas' Lemma – Another Variation)**.** *Let* $\mathbf{A}$ *be a real* $m \times n$ *matrix and* $\mathbf{b} \in \mathbb{R}^n$*. Then exactly one of the following two statements is valid.*

(a) There exists a vector $\mathbf{x} \in \mathbb{R}^n$ *such that* $\mathbf{Ax} = \mathbf{0}$ *and* $\mathbf{x} \leq \mathbf{b}$*.*

(b) There exists a vector $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y}^T\mathbf{A} \geq \mathbf{0}^T$ *and* $\mathbf{y}^T\mathbf{Ab} < 0$*.*

Proof. Let $\mathbf{b}' = \mathbf{Ab}$. The second statement becomes that there exists a vector $\mathbf{y}$ such that $\mathbf{y}^T\mathbf{A} \geq \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b}' < 0$. Let $\mathbf{x} = -\mathbf{x}' + \mathbf{b}$. The first statement becomes that there exists a vector $\mathbf{x}'$ such that $\mathbf{Ax}' = \mathbf{b}'$ and $\mathbf{x}' \geq \mathbf{0}$. □

Lemma 4.4 (A Variant Farkas' Lemma)**.** *Let* $\mathbf{A}$ *be a real* $m \times n$ *matrix and* $\mathbf{b} \in \mathbb{R}^n$*. Then exactly one of the following two statements is valid.*

(a) There exists a vector $\mathbf{x} \in \mathbb{R}^n$ *such that* $\mathbf{Ax} = \mathbf{0}$*,* $\mathbf{x} \geq \mathbf{0}$*, and* $\mathbf{b}^T\mathbf{x} > 0$*.*

(b) There exists a vector $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y}^T\mathbf{A} \geq \mathbf{b}^T$*, i.e.,* $\mathbf{A}^T\mathbf{y} \geq \mathbf{b}$*.*

Proof. It is easy to see that (a) and (b) cannot be valid simultaneously, since the contradiction

$$0 = \mathbf{y}^T\mathbf{Ax} \geq \mathbf{b}^T\mathbf{x} > 0.$$

Let C be the convex set of vectors $\mathbf{c} \in \mathbb{R}^m$ such that $\mathbf{c} \geq \mathbf{b}$. The statement (b) is equivalent to $\mathrm{C} \cap \mathrm{Row}\,(\mathbf{A}) \neq \varnothing$. The statement (a) is equivalent to that there exists a vector $\mathbf{x} \in \mathbb{R}^m_{\geq 0}$ such that $\mathrm{Row}\,(\mathbf{A}) \subseteq \mathbf{x}^\perp$ and $\langle \mathbf{x}, \mathbf{b} \rangle > 0$. The lemma is obviously true when $\mathrm{Row}\,(\mathbf{A}) = \mathbb{R}^m$, since (b) is obviously valid and (a) is not. Consider the case that $\mathrm{Row}\,(\mathbf{A})$ is a proper vector subspace of $\mathbb{R}^m$. Assume that $\mathrm{C} \cap \mathrm{Row}\,(\mathbf{A}) = \varnothing$. Let $\mathrm{Row}\,(\mathbf{A})$ be extended into a hyperplane H with normal vector $\mathbf{x}$ such that C is on the one side of H, i.e., $\mathbf{c} \cdot \mathbf{x} > 0$ for all $\mathbf{c} \in \mathrm{C}$. Then obviously, $\mathbf{Ax} = \mathbf{0}$, $\mathbf{b}^T\mathbf{x} > 0$. We claim that $\mathbf{x} \geq \mathbf{0}$. Suppose $x_i < 0$ for a fixed i. Let $\mathbf{c} \in \mathrm{C}$ be such that c_i is large enough. Then $\mathbf{c} \cdot \mathbf{x} < 0$, which is a contradiction. □

5 Feasible Flows and Tensions

Let $D = (V, A)$ be a digraph wit two weight functions $b, c : A \to \mathbb{R}$ such that $b(a) \leq c(a)$ for all $a \in A$. A flow or a tension f of D is said to be **feasible** (with respect to b and c) if $b(a) \leq f(a) \leq c(a)$ for all $a \in A$. If f is a flow, then $f^+(X) = f^-(X)$ for each vertex nonempty proper subset $X \subseteq V(D)$. By definition of feasibility, we have

$$f^+(X) \leq c^+(X), \quad f^-(X) \geq b^-(X).$$

It follows that $b^-(X) \leq c^+(X)$, i.e.,

$$b(X^c, X) \leq c(X, X^c). \tag{5.1}$$

Proposition 5.1. *Let D be a digraph with a capacity lower bound function b. Then there exists either a flow $\mathbf{f}$ of D such that $\mathbf{f} \geq \mathbf{b}$, or a nonnegative tension $\mathbf{g}$ such that $\mathbf{g}^T\mathbf{b} > 0$, but not both.*

Proof. Let $\mathbf{M}$ be the incidence matrix of D and write the capacity lower function b as the column vector $\mathbf{b}$. By Farkas' Lemma, exactly one of the two systems

$$\mathbf{M}\mathbf{f} = \mathbf{0}, \quad \mathbf{f} \geq \mathbf{b}; \qquad \mathbf{p}^T\mathbf{M} \geq \mathbf{0}^T, \quad \mathbf{p}^T\mathbf{M}\mathbf{b} > 0$$

has a solution. Note that $\mathbf{f}$ is a flow, and $\mathbf{g}^T := \mathbf{p}^T\mathbf{M}$ is a tension, since rows of $\mathbf{M}$ are tensions of D. □

Corollary 5.2. *Let D be a digraph and $\mathbf{b}$ a real-valued function on $A(D)$. Then there exists either a flow $\mathbf{f}$ of D such that $\mathbf{f} \geq \mathbf{b}$, or a directed bond B such that $g_B^T\mathbf{b} > 0$, but not both.*

Proof. Let $\mathbf{g}$ be the nonnegative tension of D in Proposition 5.1 such that $\mathbf{g}^T\mathbf{b} > 0$. Since $\mathbf{g} \geq \mathbf{0}$, we have $\mathbf{g} = \sum_i a_i\mathbf{g}_{B_i}$ with $a_i > 0$, where B_i are some directed bonds contain in $\operatorname{supp}(g)$. Since $\sum_i a_i\mathbf{g}_{B_i}^T\mathbf{b} > 0$, there exists at least one i such that $\mathbf{g}_{B_i}^T\mathbf{b} > 0$. □

Let $D' = (V', A')$ be a digraph obtain from D by subdivide each arc $a = uv$ into two arcs $a_1 = uw$ and $a_2 = vw$., i.e.,

$$V' = V(D) \cup \{w(a) : a \in A\}, \quad A' = \{(t(a), w(a)), (h(a), w(a)) : a \in A\},$$

where $t(a)$ denote the tail of a, $h(a)$ the head of a, and $w(a)$ the middle point of a. The lower bound function b' on A' is defined by

$$b'(t(a), w(a)) = b(a), \quad b'(h(a), w(a)) = -c(a).$$

For each function $f : A \to \mathbb{R}$, we associate a function $f' : A' \to \mathbb{R}$ defined by

$$f'((t(a), w(a)) = f(a), \quad f'(h(a), w(a)) = -f(a).$$

Then f is a flow of D iff f' is a flow of D'. Moreover, $b(a) \leq f(a) \leq c(a)$ is equivalent to

$$f'(t(a), w(a)) \geq b'(t(a), w(a)), \quad f'(h(a), w(a)) \geq b'(h(a), w(a)).$$

By Corollary 5.2, there exists either a flow $\mathbf{f}'$ of D' such that $\mathbf{f}' \geq \mathbf{b}'$, or a bond tension $\mathbf{g}_{B'}$ of D' such that $\mathbf{g}_{B'}^T\mathbf{b}' > 0$, but not both. We claim that the latter case cannot be happen.

In fact, let $B' = [X', X'^c]$ with $X'^c = V(D') \smallsetminus X'$, where $X' \subset V(D')$. If B' contains both half-arcs from a common arc $a = \overrightarrow{uv}$ of D, then $[X', X'^c] = \{(u, w), (v, w)\}$, since B' is a bond. Thus

$$\mathbf{g}_{B'}^T\mathbf{b}' = b'(u, w) + b'(v, w) = b(a) - c(a) > 0, \quad \text{i.e.,} \quad b(a) > c(a),$$

which is a contradictory to $b(a) \leq c(a)$. If B' contains at most one half-arc from each arc a of D, then we must have $X \subseteq V(D)$ and

$$[X', X'^c] = \{(h(a), w(a)),\ ((t(a'), w(a')) : a \in (X, X^c),\ a' \in (X^c, X)\},$$

where $X^c = V(D) \smallsetminus X$. Thus

$$\begin{aligned}
\mathbf{g}_{B'}^T \mathbf{b}' &= \sum_{a \in (X, X^c)} b'(t(a), w(a)) + \sum_{a' \in (X^c, X)} b'(h(a'), w(a')) \\
&= \sum_{a \in (X, X^c)} b(a) - \sum_{a' \in (X^c, X)} c(a') \\
&= b(X, X^c) - c(X^c, X) > 0,
\end{aligned}$$

which is contradictory to $b(X, X^c) \le c(X^c, X)$. We have proved the following theorem.

Theorem 5.3 (Hoffman's Feasible Flow Theorem)**.** *A digraph D has a feasible flow with respect to bounds b and c iff for each nonempty proper vertex subset X,*

$$b(X^c, X) \le c(X, X^c). \tag{5.2}$$

Moreover, if b and c are integer-valued, then D has integer-valued flows.

Proof. (Constructive) Let $f : A \to \mathbb{R}$ be such that $b(a) \le f(a) \le c(a)$ for all $a \in A$. Note that

$$\begin{aligned}
\sum_{v \in V} (f^+(v) - f^-(v)) &= \sum_{v \in V} \Bigg(\sum_{a \in A,\, t(a)=v} f(a) - \sum_{a \in A,\, h(a)=v} f(a) \Bigg) \\
&= \sum_{a \in A} (f(a) - f(a)) = 0.
\end{aligned}$$

The **access** $\eta(f)$ of f is defined by

$$\eta(f) = \sum_{v \in V} |f^+(v) - f^-(v)|.$$

Then f is a flow iff $\eta(f) = 0$. Whenever $\eta(f) > 0$, choose a vertex x such that $f^-(x) > f^+(x)$ (and also a vertex y such that $f^-(y) < f^+(y)$). We must have either $f^-(v) > b^-(v)$ or $f^+(v) < c^+(v)$. Otherwise, $f^-(v) = b^-(v)$ and $f^+(v) = c^+(v)$ imply that $b^-(v) > c^+(v)$, contradicting to $b^-(v) \le c^+(v)$. If $f^-(v) > b^-(v)$, there is an arc $a = \overrightarrow{vu}$ such that $f(a) > b(a)$; if $f^+(v) < c^+(v)$, there is an arc $a = \overrightarrow{uv}$ such that $f(a) < c(a)$.

An x-path P is said to be **f-improving** provided that $f(a) < c(a)$ if a is a forward arc in P, and $f(a) > b(a)$ if a is a reverse arc in P. Let X be the set of vertices reachable from x by an f-improving path of positive length. Note that for each $a \in [X, X^c]$, we have $f(a) = c(a)$ if $a \in (X, X^c)$, and $f(a) = b(a)$ if $a \in (X^c, X)$. Then

$$\sum_{v \in X} (f^+(v) - f^-(v)) = f^+(X) - f^-(X) = c(X, X^c) - b(X^c, X) \ge 0.$$

Since $f^+(x) - f^-(x) < 0$, there exists a vertex $y \in X$ such that $f^+(y) - f^-(y) > 0$. Let P be an f-improving path from x to y in D. Let $\epsilon(P)$ denote the minimum of the positive numbers

$$f^-(x) - f^+(x), \quad f^+(y) - f^-(y), \quad c(a) - f(a), \quad f(a') - b(a')$$

where $a, a' \in P$ are forward and reverse arcs respectively. We modify f into the function $f' : A \to \mathbb{R}$ by

$$f'(a) = \begin{cases} f(a) + \epsilon(P) & \text{if } a \text{ is a forward arc of } P \\ f(a) - \epsilon(P) & \text{if } a \text{ is a reverse arc of } P \\ f(a) & \text{otherwise.} \end{cases}$$

Clearly, $b(a) \leq f'(a) \leq c(a)$ for all $a \in A$. For each vertex $v \in P$ with $v \neq x, y$, consider the local subpath $uava'w$ of P near v, we have

$$\begin{aligned} f'^{+}(v) - f'^{-}(v) \quad &= \begin{cases} (f^{+}(v) + \epsilon) - (f^{-}(v) + \epsilon) & \text{if } a = \overrightarrow{uv} \text{ and } a' = \overrightarrow{vw} \\ (f^{+}(v) - \epsilon + \epsilon) - f^{-}(v) & \text{if } a = \overleftarrow{uv} \text{ and } a' = \overrightarrow{vw} \\ (f^{+}(v) - (f^{-}(v) + \epsilon - \epsilon) & \text{if } a = \overrightarrow{uv} \text{ and } a' = \overleftarrow{vw} \\ (f^{+}(v) - \epsilon) - (f^{-}(v) - \epsilon) & \text{if } a = \overleftarrow{uv} \text{ and } a' = \overleftarrow{vw} \end{cases} \\ &= \; f^{+}(v) - f^{-}(v), \qquad\qquad\qquad \text{where } \; \epsilon = \epsilon(P). \end{aligned}$$

For the initial and terminal vertices x, y of P and near their local subpaths xau and $wa'y$ of P, we have

$$f'^{+}(x) - f'^{-}(x) = \begin{cases} (f^{+}(x) + \epsilon) - f^{-}(x) & \text{if } a = \overrightarrow{xu} \\ f^{+}(x) - (f^{-}(x) - \epsilon) & \text{if } a = \overleftarrow{xu} \end{cases} = f^{+}(x) - f^{-}(x) + \epsilon,$$

$$f'^{+}(y) - f'^{-}(y) = \begin{cases} f^{+}(y) - (f^{-}(y) + \epsilon)) & \text{if } a' = \overrightarrow{wy} \\ (f^{+}(y) - \epsilon) - f^{-}(y) & \text{if } a' = \overleftarrow{wy} \end{cases} = f^{+}(y) - f^{-}(y) - \epsilon.$$

It is clear that the access $\eta(f')$ of f', given by

$$\eta(f') = \eta(f) - 2\epsilon \geq 0,$$

is less than the access $\eta(f)$ of f. Continue this procedure if $\eta(f') > 0$, we obtain a feasible function on A with zero access, i.e., a feasible flow of D. □

Theorem 5.4 (Ghouila-Houri's Theorem)**.** *Let D be a digraph with capacity bound functions b and c. Then D has feasible tensions g iff the bound functions b and c satisfy the condition: for all cycles C of D,*

$$b(C^{-}) \leq c(C^{+}), \quad b(C^{+}) \leq c(C^{-}). \tag{5.3}$$

where C^{+} is the set of forward arcs of C with respect to one of its two directions, C^{-} is the set of reverse arcs of C with respect to the same direction, and

$$b(C^{-}) = \sum_{a \in C^{-}} b(a), \qquad c(C^{+}) = \sum_{a \in C^{+}} c(a).$$

Proof. Let D' be the digraph obtained from D by adding a new arc a' with $t(a') = v$ and $h(a') = u$ for each arc a of D with $t(a) = u$ and $h(a) = v$. Let b' be a lower bound capacity function on $A(D')$, given by

$$b'(a) = b(a), \quad b'(a') = -c(a).$$

A function g on $A(D)$ corresponds to a function g' on $A(D')$ defined by

$$g'(a) = b(a), \quad g'(a') = -g(a).$$

Then $b(a) \le g(a) \le c(a)$ for all $a \in A(D)$ iff $g'(a) \ge b'(a)$ for all $a \in A(D')$, and g is a tension of D iff g' is tension of D'.

Let $\mathbf{M}'$ be the incidence matrix of D'. By Farkas' Lemma, there exists either a vector $\mathbf{f}'$ such that

$$\mathbf{M}'\mathbf{f}' = \mathbf{0}, \quad \mathbf{f}' \ge \mathbf{0}, \quad \mathbf{b}'^T\mathbf{f}' > 0,$$

or a vector $\mathbf{p}'$ such that

$$\mathbf{p}'^T\mathbf{M}' \ge \mathbf{b}'^T,$$

but not both, i.e., there exists either a nonnegative flow $\mathbf{f}'$ such that $\mathbf{b}'^T\mathbf{f}' > 0$, or a tension $\mathbf{g}' = \mathbf{M}'^T\mathbf{p}'$ such that $\mathbf{g}' \ge \mathbf{b}'$, but not both. We claim that the existence of a flow $\mathbf{f}' \ge \mathbf{0}$ such that $\mathbf{b}'^T\mathbf{f}' > 0$ is a contradiction.

Suppose the existence of such a flow $\mathbf{f}'$, and let it be a positive linear combination of directed cycle flows of D'. Then one of such directed cycles of D', say, C', satisfies $\mathbf{b}'^T\mathbf{f}_{C'} > 0$. If C' is of the form $\{a, a'\}$ with $a = \overrightarrow{uv}$ and $a' = \overrightarrow{vu}$, then $\mathbf{b}'^T\mathbf{f}_{C'} > 0$ is $b(a) - c(a) > 0$, which contradicts $b(a) \le c(a)$. If C' is directed cycle of length at least three, let $C_1 = C' \cap A(D)$ and $C_2 = C' \cap (A(D') \smallsetminus A(D))$. Then the directed cycle C' of D' corresponds to a cycle C of D, with each arc $a' = \overrightarrow{vu} \in C_2$ replaced by $a = \overrightarrow{uv}$ in D. Now $\mathbf{b}'^T\mathbf{f}_{C'} > 0$ is

$$\sum_{a \in C_1} b(a) + \sum_{a' \in C_2} b'(a') = \sum_{a \in C^+} b(a) - \sum_{a \in C^-} c(a) > 0, \quad \text{i.e.,} \quad b(C^+) > c(C^-),$$

which is contradictory to the feasible condition (5.3). $\square$

6 Graph Laplacian

Let $D = (V, A)$ be a connected loopless digraph in which half arcs are allowed, and $\mathbf{M}$ the incidence matrix of D. For each vertex $v \in V$, let $\mathbf{M}_v$ denote the matrix obtained from $\mathbf{M}$ by deleting the row corresponding to the vertex v. A **Kirchhoff matrix** of D is the matrix $\mathbf{K} := \mathbf{M}_v$ for a vertex v of D. The **Laplace matrix** of the underlying graph G of the digraph D is the matrix

$$\mathbf{L} := \mathbf{M}\mathbf{M}^T.$$

Let $\mathbf{A}$ be the adjacency matrix of G, whose (u, v)-entry is the number edges between u and v, each loop is counted twice. Let $\mathbf{D}$ be the diagonal matrix whose diagonal (v, v)-entry is $\deg_G(v)$, which is the number of edges at v with loops counted twice. Then

$$\mathbf{L} = \mathbf{D} - \mathbf{A}.$$

In fact, let $e_1, \ldots, e_k$ be the links and $f_1, \ldots, f_l$ the loops at v. Recall that (v, e_i)-entry in $\mathbf{M}$ is either 1 or -1, and (v, f_j)-entry in $\mathbf{M}$ is always 0. So the (v, v)-entry of $\mathbf{L}$ is

$$\sum_{i=1}^{k} \omega(v, e_i)^2 + \sum_{j=1}^{l} \omega(v, f_j)^2 = k = \text{number of links at } v.$$

The (v, v)-entry of $\mathbf{D}$ is $\deg(v)$, which is the number of edges incident with v, where each loop is counted twice. The (v, v)-entry of $\mathbf{A}$ is the twice number of loops at v. Then the (v, v)-entry of $\mathbf{D} - \mathbf{A}$ is also the number of links at v.

For distinct vertices $u, v \in V$, let $g_1, \ldots, g_m$ be the edges between u and v. The (u, v)-entry of $\mathbf{L}$ is

$$\sum_{i=1}^{m} \omega(u, g_i)\omega(v, g_i) = -m = -a_{uv} = \text{number of edge between } u \text{ and } v.$$

We have seen that $\mathbf{L} = \mathbf{D} - \mathbf{A}$.

Lemma 6.1. *Let G be a graph with n vertices and c connected components. Then the Laplacian $\mathbf{L}(G)$ has rank $n - c$.*

Proof. Since $\operatorname{rank}(\mathbf{M}) = n - c$, it suffices to show that $\operatorname{rank}(\mathbf{L}) = \operatorname{rank}(\mathbf{M})$. Given a vector $\boldsymbol{v}$. If $\mathbf{M}\mathbf{M}^T\mathbf{v} = \mathbf{0}$, then $\mathbf{v}^T\mathbf{M}\mathbf{M}^T\mathbf{v} = \mathbf{0}$, i.e., $\|\mathbf{M}^T\mathbf{v}\| = 0$, thus $\mathbf{M}^T\mathbf{v} = \mathbf{0}$. Clearly, $\mathbf{M}^T\mathbf{v} = \mathbf{0}$ implies $\mathbf{M}\mathbf{M}^T\mathbf{v} = \mathbf{0}$. So $\mathbf{L}$ and $\mathbf{M}^T$ have the same kernel. Hence $\operatorname{rank}\mathbf{L} = \operatorname{rank}\mathbf{M}^T = \operatorname{rank}\mathbf{M}$. $\square$

Since $\mathbf{L}$ is a symmetric square matrix, all eigenvalues of $\mathbf{L}$ are real. Since $\mathbf{v}^T\mathbf{L}\mathbf{v} = \|\mathbf{M}^T\mathbf{v}\|^2 \geq 0$ for each vector $\mathbf{v} \in \mathbb{R}^{E(G)}$. We see that $\mathbf{L}$ is semi-positive definite. If $\mathbf{v}$ is an eigenvector for the eigenvalue λ, i.e., $\mathbf{L}\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{v}^T\mathbf{L}\mathbf{v} = \lambda\mathbf{v}^T\mathbf{v} = \lambda\|\mathbf{v}\|^2 \geq 0$, thus $\lambda \geq 0$. So all eigenvalues $\mathbf{L}$ are nonnegative, and 0 is always an eigenvalue, since $\mathbf{L}$ is singular. It is easy to see that the multiplicity of the zero eigenvalue is $c(G)$, the number of components of G. Let $G_1, \ldots, G_k$ be the connected components of G. Then the eigenspace of $\mathbf{L}$ for the eigenvalue 0 is the vector space generated by $\mathbf{1}_{V(G_i)}$, $1 \leq i \leq k$. Let $\lambda_2(G)$ denote the smallest positive eigenvalue of G, called the **second smallest eigenvalue** of $\mathbf{L}$. The eigenvalues of $\mathbf{L}(G)$ are ordered as

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \quad n = |V(G)|.$$

Lemma 6.2. *For two $n \times n$ matrices A and B, the determinant of $A + B$ is given by*

$$\det(A + B) = \sum_{S \subseteq [n]} \det(A_S \cup B_{S^c}),$$

where $A_S \cup B_{S^c}$ is the matrix obtained from A by replacing the rows with indices not in S with the corresponding rows of B.

Proof. Write the rows of A as $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, and the rows of B as $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$. The formula follows from the following direct calculation:

$$\begin{aligned}
\det(A + B) &= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (a_{i\sigma(i)} + b_{i\sigma(i)}) \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) \sum_{\substack{c_{i\sigma(i)} \in \{a_{i\sigma(i)}, b_{i\sigma(i)}\} \\ i=1,\ldots,n}} \prod_{i=1}^{n} c_{i\sigma(i)} \\
&= \sum_{\substack{\boldsymbol{c}_i \in \{\boldsymbol{a}_i, \boldsymbol{b}_i\} \\ i=1,\ldots,n}} \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} c_{i\sigma(i)} \\
&= \sum_{S \subseteq [n]} \det(A_S \cup B_{S^c}).
\end{aligned}$$

$\square$

Theorem 6.3. *The characteristic polynomial of the Laplacian* $\mathbf{L}$ *of a graph* G *with* n *vertices is*

$$\det(t\mathbf{I}-\mathbf{L}) = \sum_{k=0}^{n-1}(-1)^k c_k t^{n-k},$$

where c_k *is the number of rooted spanning forests of* G *with* k *edges. In particular, when* G *is connected,* c_{n-1} *is the* n *times of the number of spanning trees of* G*.*

Proof. Write $(t\mathbf{I}-\mathbf{L})$ as $(t\mathbf{I}+(-\mathbf{L}))$ and $\mathbf{L}=\mathbf{M}\mathbf{M}^T$, where $\mathbf{M}$ is the vertex-edge incidence matrix of G. Applying Lemma 6.2,

$$\begin{aligned}\det(t\mathbf{I}-\mathbf{L}) &= t^n + \sum_{k=1}^{n-1}(-1)^k t^{n-k} \sum_{S\subseteq V(G),\,|S|=k} \det\left(\mathbf{M}_S\mathbf{M}_S^T\right)\\ &= \sum_{k=0}^{n-1}(-1)^k c_k t^{n-k}.\end{aligned}$$

Since $\mathbf{M}$ is totally unimodular, applying Cauchy-Binet formula, we see that

$$\det\left(\mathbf{M}_S\mathbf{M}_S^T\right) = \#\{F\subseteq E : |F|=|S|,\, \det(\mathbf{M}_{S\times F})\neq 0\}.$$

Note that $\det(\mathbf{M}_{S\times F})\neq 0$ implies that $|S|=|F|$, $S\subseteq V(F)$, and the subgraph (S,F) (with possible half-edges) contains no cycle. Let (S,F) be decomposed into connected components (S_i,F_i). Then $\det(\mathbf{M}_{S\times F}) = \prod_i \det(\mathbf{M}_{S_i\times F_i})$, which implies $|S_i|=|F_i|$, $S_i\subseteq V(F_i)$, and $\det(\mathbf{M}_{S_i\times F_i})\neq 0$ for all i. Likewise, $\det(\mathbf{M}_{S_i\times F_i})\neq 0$ implies that (S_i,F_i) (with possible half-edges) contains no cycle. We claim that each graph $(V(F_i),F_i)$ is a tree. Suppose $(V(F_i),F_i)$ is not a tree, i.e., it contains a cycle. Then its number of independent cycles is

$$n(F_i) := |F_i| - |V(F_i)| + 1 \geq 1.$$

Consequently, $|V(F_i)|\leq |F_i| = |S_i|$. Since $S_i\subseteq V(F_i)$, we have $(S_i,F_i)=(V(F_i),F_i)$, which contains a cycle, a contradictory to that (S_i,F_i) contains no cycle.

Now each $(V(F_i),F_i)$ is a tree and $V(F_i)\smallsetminus S_i$ is a single vertex, which can be viewed as a root of the tree $(V(F_i),F_i)$. So each (S_i,F_i) may be considered as a rooted tree $(V(F_i),F_i)$ with the root v such that $\{v\}=V(F_i)\smallsetminus S_i$. Conversely, if $S_i\subseteq V(F_i)$, $|S_i|=|F_i|$, and $(V(F_i),F_i)$ is a tree, then it is clear that $\det(\mathbf{M}_{S_i\times F_i})\neq 0$ by expansion along its v-row with v a leaf. Thus we obtain

$$c_k = \#\{\text{acyclic}\,(S,F) : F\subset E,\, S\subseteq V(F),\, |S|=|F|=k\},$$

where each such (S,F) is identified as a rooted spanning forest F with k edges, i.e., each component of F is specified a root. In particular, c_{n-1} is the number of rooted spanning trees, which is n times of the number of spanning trees of G. □

Corollary 6.4. *Let* G *be a graph with* n *vertices. If the eigenvalues of* $\mathbf{L}(G)$ *are linearly ordered as* $0=\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n$ *(multiply listed with multiplicities). Then the number of spanning trees of* G *equals*

$$\frac{1}{n}\prod_{i=2}^{n}\lambda_i.$$

Proof. Let $\phi(t)$ denote the characteristic polynomial of $\mathbf{L}$, i.e., $\phi(t) = \det(t\mathbf{I} - \mathbf{L})$. Then $\phi(t) = t(t - \lambda_2) \cdots (t - \lambda_n)$. The coefficient of t in $\phi(t)$ is $(-1)^{n-1}\lambda_2 \cdots \lambda_n$, which is also the number of spanning trees of G times $(-1)^{n-1}n$ by Theorem 6.3. $\square$

Exercises

Ch10: 9.1.1; 9.1.8; 9.1.9; 9.2.1; 9.3.2; 9.3.7;