Network Flows

May 4, 2010

1 Transportation Networks

- A **network** $N = (D, x, y, c)$ is a digraph $D = (G, \varepsilon)$ with two distinguished vertices, a **source** x and a **sink** y, together with a nonnegative function $c : E(D) \to \mathbb{R}_{\geq 0}$, called the **capacity function** of N. For each edge $e \in E(D)$, the value $c(e)$ is called the **capacity** of e. Vertices other than x, y are called **intermediate vertices**.
- For any function $f : E(D) \to \mathbb{R}$ and a vertex subset $X \subseteq V(D)$, we define

$$f^+(X) := \sum_{e \in (X, X^c)} f(e), \quad f^-(X) := \sum_{e \in (X^c, X)} f(e).$$

- An (x, y)**-flow** of a network $N = (D, x, y)$ is a function $f : E(D) \to \mathbb{R}$ satisfying the **conservation condition**:

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e) \quad \text{i.e.,} \quad f^+(v) = f^-(v) \quad \text{for all } v \in V(D) - \{x, y\},$$

  where $E^+(v)$ is the set of edges whose tails are v, and $E^-(v)$ is the set of edges whose heads are v.

  Equivalently, an (x, y)-flow of N is just a real-valued function f on $E(D)$ such that for any $v \in V(D) - \{x, y\}$,

$$\sum_{e \in E(D)} \varepsilon(v, e) f(e) = 0.$$

- The **value** of an (x, y)-flow f of a network $N(x, y)$ is the flow value out of the source x, i.e.,

$$\text{val}(f) := f^+(x) = f^-(y).$$

  An (x, y)-flow f is called a **feasible flow** (or just a **flow**) of N if it satisfies the **capacity constraint**:

$$0 \leq f(e) \leq c(e) \quad \text{for all } e \in E(D).$$

  A flow is called a **maximum flow** if there is no flow of greater value.

Lemma 1.1. *Let f be a flow a network $N(x, y)$, and $X \subseteq V(N)$ be such that $x \in X, y \notin X$. Then*

$$\text{val}(f) = f^+(X) - f^-(X).$$

Proof. By definition of $f^+(X)$ and $f^-(X)$, and $\sum_{e \in E} \varepsilon(v, e) f(e) = 0$ for all $v \neq x, y$, we have

$$\begin{aligned} \text{val}(f) &= \sum_{v \in X} \sum_{e \in E} \varepsilon(v, e) f(e) \\ &= \sum_{e \in E} f(e) \sum_{v \in X} \varepsilon(v, e) \\ &= \left\{ \sum_{e \in [X, X]} + \sum_{e \in (X, X^c)} + \sum_{e \in (X^c, X)} \right\} f(e) \sum_{v \in X} \varepsilon(v, e) \\ &= f^+(X) - f^-(X). \end{aligned}$$

□

- An (x, y)**-cut** (or just a **cut**) of a network $N(x, y)$ is a cut $[X, X^c]$ **separating** x **from** y, i.e., $x \in X, y \notin X$. The **capacity** of such an cut $[X, X^c]$ is

$$c(X, X^c) := \sum_{e \in (X, X^c)} c(e).$$

- A cut $[X, X^c]$ of a network $N(x, y)$ is called a **minimum cut** if N has no cut of smaller capacity.

- Let f be a flow of a network $N(x, y)$. A cut $[X, X^c]$ is said to be f**-saturated at its edge** e if either (i) $e \in (X, X^c)$ and $f(e) = c(e)$, or (ii) $e \in (X^c, X)$ and $f(e) = 0$; otherwise it is said to be f**-unsaturated** at e, i.e., either (i) $e \in (X, X^c)$ and $f(e) < c(e)$, or (ii) $e \in (X^c, X)$ and $f(e) > 0$. If a cut $[X, X^c]$ is f-unsaturated at its edge e, we define

$$\iota(e) = \iota(e, f) := \begin{cases} c(e) - f(e) & \text{if } e \in (X, X^c), \\ f(e) & \text{if } e \in (X^c, X). \end{cases}$$

  If $[X, X^c]$ is f-unsaturated at an edge e, then $\iota(e) > 0$.

- A cut $[X, X^c]$ of a network N is said to be (i) f**-saturated** if it is f-saturated at its every edge, and (ii) f**-unsaturated** if it is f-unsaturated at one of its edges.

Proposition 1.2. *For any flow f of a network $N(x, y)$ and any cut $[X, X^c]$,*

$$\text{val}(f) \le c(X, X^c).$$

Moreover, the equality holds if and only if the cut $[X, X^c]$ is f-saturated.

Proof. Note that $f^+(X) \le c(X, X^c)$ and $f^-(X) \ge 0$. Then

$$\text{val}(f) = f^+(X) - f^-(X) \le c(X, X^c).$$

As for the equality, the sufficiency is obvious. The necessity is as follows:

Suppose $[X, X^c]$ is f-unsaturated, i.e., $[X, X^c]$ has an f-unsaturated edge e. If $e \in (X, X^c)$, then $f(e) < c(e)$; thus

$$\text{val}(f) = f^+(X) - f^-(X) < c(X, X^c) - f^-(X) = c(X, X^c).$$

If $e \in (X^c, X)$, then $f(e) > 0$; thus

$$\text{val}(f) = f^+(X) - f^-(X) < f^+(X) \le c(X, X^c).$$

Both cases are contradictory to $\text{val}(f) = c(X, X^c)$. □

Corollary 1.3. *Let f be a flow and (X, X^c) a cut of a network $N(x, y)$. If* $\text{val}(f) = c(X, X^c)$*, then f is a maximum flow and $[X, X^c]$ is a minimum cut.*

Proof. Let f^* be a maximum flow and (X^*, X^{*c}) a minimum cut of N. Then by Proposition 1.2,

$$\text{val}(f) \le \text{val}(f^*) \le c(X^*, X^{*c}) \le c(X, X^c).$$

Since $\text{val}(f) = c(X, X^c)$, it follows that $\text{val}(f) = \text{val}(f^*)$ and $c(X^*, X^{*c}) = c(X, X^c)$. □

2 The Max-Flow Min-Cut Theorem

- Let $N(x, y)$ be a network, f a flow of $N(x, y)$, and P an x-path (not necessarily a directed path); the positive direction of P is denoted by ε_P. The f**-increment** of P is

$$\epsilon(P) = \epsilon(P, f) := \min\{\epsilon(e) \mid e \in E(P)\},$$

  where

$$\epsilon(e) = \epsilon(e, f) := \begin{cases} c(e) - f(e) & \text{if } \vec{e} \text{ is a forward arc in } P, \text{ i.e., } [\varepsilon, \varepsilon_P](e) = 1, \\ f(e) & \text{if } \vec{e} \text{ is a reverse arc in } P, \text{ i.e., } [\varepsilon, \varepsilon_P](e) = -1. \end{cases}$$

- Given a flow of a network N; an x-path P is said to be **f-saturated** if $\epsilon(P) = 0$ and **f-unsaturated** if $\epsilon(P) > 0$. An (x, y)-path is called an **f-incrementing path** if it is f-unsaturated.

Proposition 2.1. *Let f be a flow of a network $N(x, y)$ and P an (x, y)-path. Then $\epsilon(P) \geq 0$, $f' := f + \epsilon(P)[\varepsilon, \varepsilon_P]$ is a flow of N with $\mathrm{val}(f') = \mathrm{val}(f) + \epsilon(P)$, and f' is explicitly given by*

$$f'(e) := \begin{cases} f(e) + \epsilon(P) & \text{if } \vec{e} \text{ is a forward arc in } P, \\ f(e) - \epsilon(P) & \text{if } \vec{e} \text{ is a reverse arc in } P, \\ f(e) & \text{otherwise.} \end{cases}$$

Proof. We only need to verify that f' is a flow, for f' is clearly feasible. Since any linear combination of flows is also a flow, it is equivalent to check that $[\varepsilon, \varepsilon_P]$ is a flow. In fact,

$$\sum_{e \in E} \varepsilon(v, e)[\varepsilon, \varepsilon_P](e) = \sum_{e \in E} \varepsilon(v, e)\varepsilon(v, e)\varepsilon_P(v, e) = \sum_{e \in E} \varepsilon_P(v, e),$$

which is zero at each internal vertex v of P by definition of direction of a path. □

Proposition 2.2. *Let f be a flow of a network $N(x, y)$, and there is no f-incrementing path from x to y in N. Let X be the set of vertices reachable from x by f-unsaturated paths, including x itself. Then f is a maximum flow, $[X, X^c]$ is a minimum cut, and $\mathrm{val}(f) = c(X, X^c)$.*

Proof. It is clear that $[X, X^c]$ is a cut separating x from y. We claim that $[X, X^c]$ is f-saturated. In fact, suppose $[X, X^c]$ has an f-unsaturated edge e with end-vertices $u \in X, v \in X^c$. Let P_u be an f-unsaturated path from x to u. Then $P_v := P_u ev$ is an f-unsaturated path from x to v; this is a contradiction.

Thus $\mathrm{val}(f) = c(X, X^c)$. By Corollary 1.3, f is a maximum flow and (X, X^c) is a minimum cut. □

Theorem 2.3 (Max-Flow Min-Cut Theorem)**.** *The value of a maximum flow in a network is equal to the capacity of a minimum cut.*

Proof. Let f be a maximum flow. By Proposition 2.1, there is no f-incrementing path by Proposition 2.1. Then by Proposition 2.2, $\mathrm{val}(f)$ is equal to the capacity of a cut of N. □

Theorem 2.4 (Ford-Fulkerson Algorithm)**.** INPUT: *a network $N = (D, x, y)$ with a capacity function $c : E \to \mathbb{R}$; a feasible flow f of N.*

OUTPUT: *a maximum flow f and a minimum cut $[T, D - T]$.*

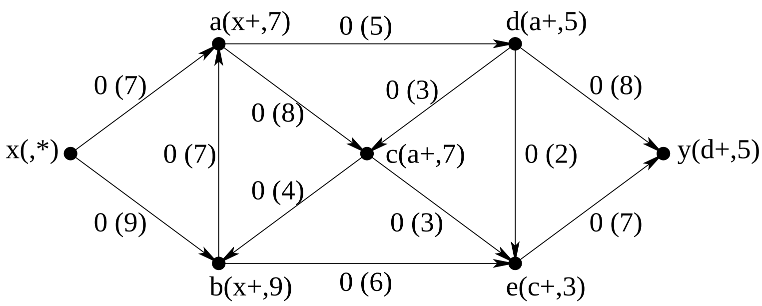
STEP 1: *Initialize a tree $T := \{x\}$, set $\iota(x) = \infty$, then go to* STEP 2.

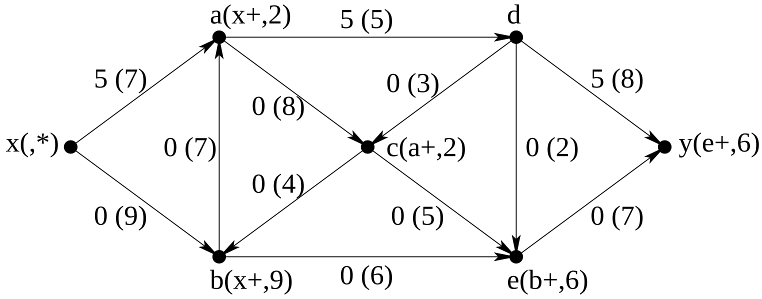
STEP 2: *If $y \in T$, set $f := f + \iota(y)[\varepsilon, \varepsilon_P]$ with P the unique path from x to y in T, then go to* STEP 1.
If $y \notin T$, go to STEP 3.

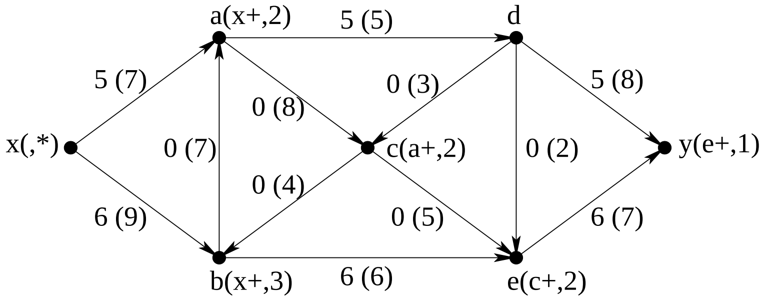
STEP 3: *If $[T, D - T]$ is f-saturated,* STOP*;* f is a maximum flow, $[T, D - T]$ is a minimum cut.
If $[T, D-T]$ is f-unsaturated, select an f-unsaturated edge $e \in [T, D-T]$ with end-vertices $u \in T$ and $v \notin T$, add e and v to T, set $\iota(v) := \min\{\iota(u), \iota(e)\}$, then go to STEP 2.

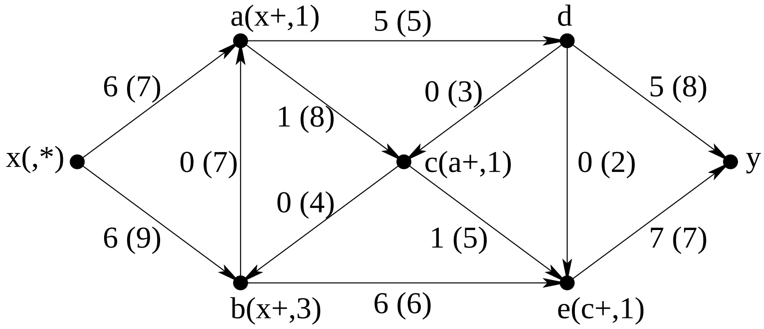
Proof. Trivial with previous preparation. □

Example 2.1. Consider the following network with capacity function specified on the edges.









3 Arc Disjoint Paths

Proposition 3.1. *Let f be a flow in a digraph $D = (V, A) = (G, \varepsilon)$.*

(a) If f is nonzero, then the support of f contains a cycle.

(b) If f is nonnegative and nonzero, then the support of f contains a directed cycle.

Proposition 3.2. *(a) Every nonnegative flow f in a digraph D is a nonnegative linear combination of flows associated with its directed cycles.*

(b) If the flow f is integer-valued, the coefficients in the linear combination may be chosen to be integers.

Corollary 3.3. *let $N = (D, x, y)$ be a network in which each arc has unit capacity 1. Then N has an (x, y)-flow of value k if and only if the digraph $D(x, y)$ has k arc-disjoint directed (x, y)-paths.*

Theorem 3.4 (Menger's Theorem)**.** *(a) In any digraph $D(x, y)$, the maximum number of arc-disjoint directed (x, y)-paths is equal to the minimum number of forward arcs in an (x, y)-cut.*

(b) In any graph $G(x, y)$, the maximum number of edge-disjoint (x, y)-paths is equal to the minimum number of edges in an (x, y)-cut.

4 Matchings in Bipartite Graphs

Let $G = (V, E)$ be a bipartite graph with vertex set $V = X \cup Y$, each edge is between a vertex of X and a vertex of Y.

- A **matching** in G is a subset of E such that no two edges share a common vertex in X and Y.

- A **complete matching** of X into Y is a matching in G such that every vertex $x \in X$ is an end-vertex of an edge.

Theorem 4.1. *Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \cup Y$. Then there exists a complete matching of X into Y if and only if for each subset $A \subseteq X$,*

$$|A| \leq |R(A)|,$$

where $R(A) \subseteq Y$ is set of vertices adjacent to at least one vertex in A.

Proof. The necessity is trivial. We only need to prove sufficiency.

Let $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$. Let N be a network with a source a, a sink z, and intermediate vertex set V, where (a, x_i) and (y_j, z) have capacity 1, $(x_i, y_j) \in E$ has capacity M, and $M \geq m$. It is clear that there exists a matching in G if and only if there is a maximum flow in N that uses all arcs (a, x_i), $1 \leq i \leq m$. Then the value of such a maximum flow is $m = |X|$. To show that a maximum flow in N uses all arcs (a, x_i), it suffices to show that for any cut $[P, P^c]$,

$$c(P, P^c) \geq |X|.$$

Fix a cut $[P, P^c]$; let $A := P \cap X$ and $B := P \cap Y$. Then $P^c = (X - A) \cup (Y - B) \cup \{z\}$. We may relabel the vertices of X so that $A = \{x_1, \ldots, x_i\}$, $0 \leq i \leq m$. (When $i = 0$, $A = \emptyset$.) note that

$$(P, P^c) = (a, X - A) \cup (A, Y - B) \cup (B, z).$$

If $[A, Y - B] \neq \emptyset$, then $c(P, P^c) \geq M \geq |X|$. If $[A, Y - B] = \emptyset$, then $(P, P^c) = [a, X - A] \cup [B, z]$. Thus

$$c(P, P^c) = |X - A| + |B|.$$

Since $[A, Y - B] = \emptyset$, then $R(A) \subseteq B$. Hence $c(P, P^c) \geq |X - A| + |R(A)| \geq |X - A| + |A| = |X|$.

Conversely, suppose there is a subset $A \subseteq X$ such that $|A| > |R(A)|$. Consider the cut $[P, P^c]$ with

$$P := \{a\} \cup A \cup R(A), \quad P^c := (X - A) \cup (Y - R(A)) \cup \{z\}.$$

Then $(P, P^c) = [a, X - A) \cup [A, Y - R(A)] \cup [R(A), z]$. Thus

$$c(P, P^c) = |X - A| + |R(A)| < |X - A| + |A| = |X|.$$

The cut $[P, P^c]$ has capacity smaller than $|X|$. □

5 Matchings

- A **matching** in a graph G is a set of non-loop edges, having end-vertices in common. If M is a matching, the two end-vertices of each edge of M are said to be **matched** under M, and each vertex incident with an edge of M is said to be **covered** by M.

- A **perfect matching** in a graph G is a matching that covers every vertex of the graph. A **maximum matching** is a matching which cover as many vertices as possible; the number of edges of such a matching is called the **matching number** of the graph, denoted $\alpha'(G)$. A graph is said to be matchable if it has a perfect matching.

Let M be a matching in a graph G. An M**-alternating path (cycle)** in G is a path (cycle) whose edges are alternating between M and $E - M$. An M-alternating path may not start or end with a vertex incident with an edge of M. An M**-augmenting path** is an M-alternating path of which neither its initial vertex nor its terminal vertex is covered by M.

Theorem 5.1 (Berge's Theorem)**.** *A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.*

Proof. "$\Rightarrow$": Suppose G contains an M-augmenting path P. Then M has more edges of in $E - M$ than of M, and the initial and terminal vertices are not covered by M. Set $M' := M \Delta E(P)$. Then M' is a matching in G with $|M'| > |M|$. So M is not a maximum matching, a contradiction.

"$\Leftarrow$": Suppose M is not a maximum matching. Given a maximum matching M^*; set $H := G(M\Delta M^*)$. Then H is a graph whose vertices have degree either 1 or 2. Thus H is a vertex-disjoint union of paths and cycles, alternating between M and M^*. Since $|M^*| > |M|$, the subgraph H contain more edge of M^* than of M. Then H has at least one path component P, whose initial and terminal vertices are not covered by M^*, i.e., P is an M-augmenting path in G, a contradiction. □

6 Matching in Arbitrary Graphs

- Let $o(G)$ denote the number of odd components in a graph G. An **odd component** is a connected component having odd number of vertices.

- For matching M of a graph G, let U denote the set of vertices that are not covered by M. Then

  $$|U| \geq o(G).$$

  [Each odd component must have a vertex uncovered by M.]

- Let M be a matching in a graph G, and let U be the set of vertices uncovered by M. Then for any proper subset $S \subsetneq V$,

  $$|U| \geq o(G - S) - |S|.$$

  [Let $O(G - S)$ be the set of odd components of $G - S$. For each $H \in O(G - S)$, if $V(H)$ is covered by M, then at least one vertex of H must be matched by an edge of M with a vertex in S. Let S_H denote the set of vertices in S that are matched to the vertices of H by M. Note that $\{S_H \mid H \in O(G - S)\}$ is a collection of disjoint subsets of S. There are at most $|S|$ odd components of $G - S$ that are covered by M. Thus there are at least $o(G - S) - |S|$ odd components of $G - S$ that are not covered by M. So $|U| \geq o(G - S) - |S|$.]

- Let U be the set of vertices uncovered by a matching M in a graph G. Then $|U| = |V(G)| - 2|M|$. If there is a proper subset $B \subsetneq V$ such that

  $$|V(G)| - 2|M| = o(G - B) - |B|,$$

  the matching M is necessarily to be maximal. Such a vertex set B is a called a **barrier** of G with respect to the maximum matching M.

Theorem 6.1 (Tutte-Berge Theorem)**.** *Every graph has a barrier.* □

Theorem 6.2 (Tutte's Theorem)**.** *A graph G has a perfect matching M if and only if for each subset $S \subseteq V$,*

$$o(G - S) \leq |S|.$$

Proof. "$\Rightarrow$": Fix a nonempty proper subset $S \subsetneq V$. Let $O(G - S)$ be the set of odd components of $G - S$. For each $H \in O(G - S)$, let S_H be the set of vertices in S that are matched to the vertices of H by M. Then $\{S_H \mid H \in O(G - S)\}$ is a collection of disjoint subsets of S. Of course,

$$o(G - S) = \#\{S_H \mid H \in O(G - S)\} \leq |S|.$$

"$\Leftarrow$": Too long. □

Theorem 6.3 (Petersen's Theorem)**.** *Every 3-regular graph without cut edges has a perfect matching.* □

7 Matchings and Coverings

- A **covering** of a graph G is a subset S of $V(G)$ such that every edge of G has at least one end-vertex in S.
- A covering S^* of G is called a **minimum covering** if G has no covering S such that $|S| < |S^*|$. The **covering number** of G is the number of vertices of a minimum covering of G.
- A covering is said to be **minimal** if none of its proper subsets is a covering.
- If M is a matching and S a covering of G, then

$$|M| \leq |S|.$$

Proposition 7.1. *Let M be a matching and S a covering of a graph G. If $|M| = |S|$, then is a maximum matching and S is a minimum covering of G.* □