Planar Graphs and Regular Polyhedra

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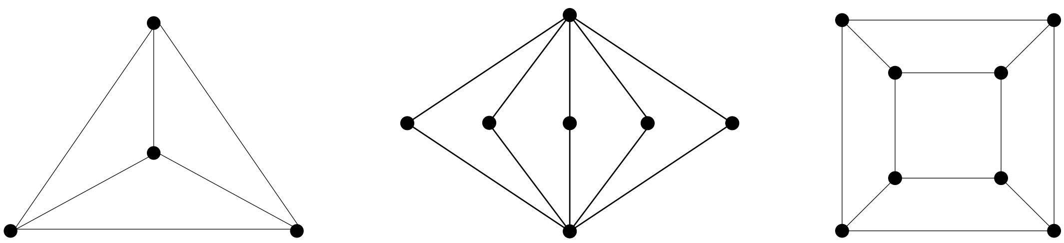
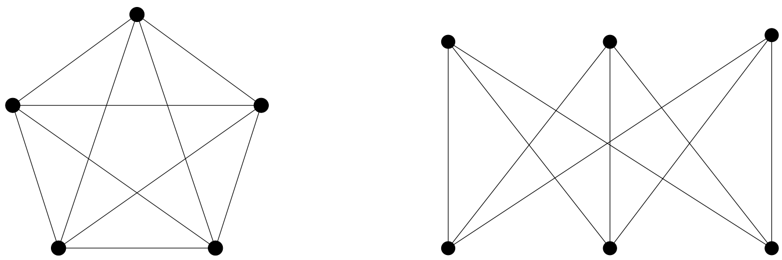
1 Planar Graphs

- A graph G is said to be **embeddable** in a plane, or **planar**, if it can be drawn in the plane in such a way that no two edges cross each other. Such a drawing is called a **planar embedding** of the graph.
- Let G be a planar graph and be embedded in a plane. The plane is divided by G into disjoint **regions**, also called **faces** of G. We denote by $v(G)$, $e(G)$, and $f(G)$ the number of vertices, edges, and faces of G respectively.
- Strictly speaking, the number $f(G)$ may depend on the ways of drawing G on the plane. Nevertheless, we shall see that $f(G)$ is actually independent of the ways of drawing G on any plane. The relation between $f(G)$ and the number of vertices and the number of edges of G are related by the well-known Euler Formula in the following theorem.
- The complete graph K_4, the bipartite complete graph $K_{2,n}$, and the cube graph Q_3 are planar. They can be drawn on a plane without crossing edges; see Figure 5. However, by try-and-error, it seems that the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not planar.



(a) Tetrahedron, K_4 (b) $K_{2,5}$ (c) Cube, Q_3

Figure 1: Planar graphs

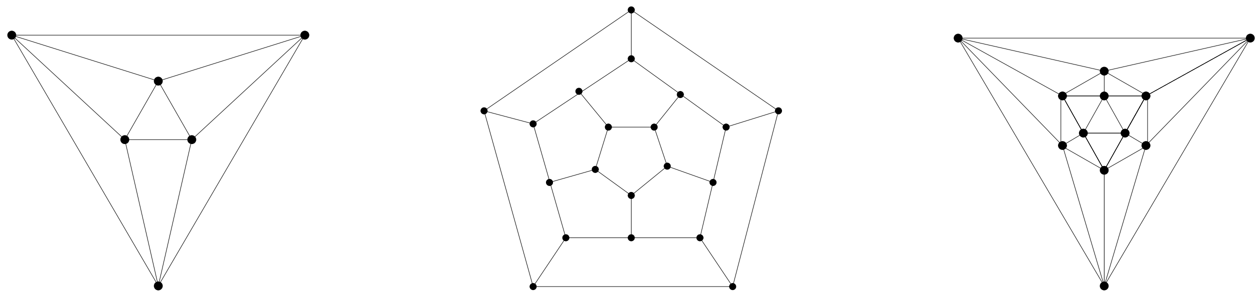


(a) K_5 (b) $K_{3,3}$

Figure 2: Nonplanar graphs

Theorem 1.1. (Euler Formula) *Let G be a connected planar graph with v vertices, e edges, and f faces. Then*

$$v - e + f = 2. \tag{1}$$



(a) Octahedron (b) Dodecahedron (c) Icosahedron

Figure 3: Regular polyhedra

Proof. We prove it by induction on the number of edges. When $e = 0$, the graph G must be a single vertex, and in this case the number of faces is one. Clearly, $v - e + f = 2$.

Suppose it is true for planar graphs with k edges, $k \geq 0$. We consider a connected planar graph G with $k+1$ edges. The graph G may or may not have cycles. If G has no cycles, i.e., G is a tree, then $e = v - 1$ (every tree with v vertices has $v - 1$ edges), $f = 1$; so $v - e + f = 2$. If G has a cycle C, choose an edge x on C and remove x from G to obtain a new planar graph G'. Since x is on a cycle, G' is still connected and have the same vertices of G, but have k edges. By induction we have $v' - e' + f' = 2$. Note that x bounds two regions, one inside C and the other is outside C. So $v' = v$, $e' = e - 1$, $f' = f - 1$. It is clear that $v - e + f = 2$. □

Planar graphs can be equivalently described as graphs drawn on a sphere with no edges crossing each other. Note that the boundary of any polyhedron is homeomorphic to a sphere. Then the graph consists of the vertices and edges of a polyhedron is planar. The graphs for regular polyhedra are the first and third graphs in Figure 5 and the three graphs in Figure 3.

- For a connected planar graph G drawn on a plane, let $D_1, D_2, \ldots, D_f$ denote the faces of G, and let $e(D_i)$ denote the number of edges on the boundary of D_i, $1 \leq i \leq r$. If one counts the edges along the boundary of each region of G, every edge of G is counted exactly twice; we then have

$$\sum_{i=1}^{r} e(D_i) = 2e(G). \tag{2}$$

- If G is a simple graph (no loops and no multiple edges joining any two vertices), then $e(D_i) \geq 3$. Thus

$$3\, f(G) \leq 2\, e(G).$$

Corollary 1.2. *If G is a connected, simple, planar graph with v vertices and e edges, then*

$$e \leq 3v - 6. \tag{3}$$

Example 1.1. The complete graph K_5 is not planar.

Proof. Notice that K_5 has 5 vertices and $\binom{5}{2}$ $(= 10)$ edges. Suppose K_5 is planar. Then by Corollary 1.2, we have $10 \leq 3 \cdot 5 - 6 = 9$, a contradiction. □

Example 1.2. The complete bipartite graph $K_{3,3}$ is not planar.

Proof. Note that every region of a graph bounds a cycle. Since $K_{3,3}$ is a bipartite graph, it has only cycles of even length. Since there is no cycles of length 2, every cycle of $K_{3,3}$ has length of at least 4. Suppose $K_{3,3}$ is a planar graph. Then every region of $K_{3,3}$ bounds at least 4 edges. By the equation (2), we have

$$4f \leq 2e.$$

Combine this inequality and the Euler formula (1); we obtain

$$e \leq 2v - 4.$$

Note that $K_{3,3}$ has 6 vertices and 9 edges. Thus $9 \leq 2 \cdot 6 - 4 = 8$ is a contradiction. $\square$

Example 1.3. The **Petersen graph** P_5 is not planar; see Figure 4.

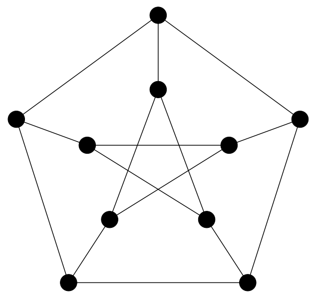


Figure 4: Petersen graph P_5

Proof. Note that each cycle of the Petersen graph has at least 5 edges. So if it is planar, then $5f \leq 2e$. It follows from the Euler formula that $3e \leq 5v - 10$. However, the graph has 10 vertices and 15 edges. Thus $45 = 3 \cdot 15 \leq 5 \cdot 10 - 10 = 40$, a contradiction. $\square$

2 Classification of Regular Polyhedra

A convex polygon is said to be **regular** if all its sides are of equal length and all its internal angles are equal. A polyhedron is said to be **regular** if (i) all its faces (convex polygons) are regular and have the same number of sides; (ii) all vertices have the same number of edges joining them. The **Platonic solids** are the five regular polyhedra: *tetrahedron, cube, octahedron, dodecahedron, and icosahedron.*

Theorem 2.1. *The only regular convex polyhedra are the five Platonic solids.*

Proof. Let P be a regular polyhedron with v vertices, e edges, and f faces. Let n be the number of sides of a face, and d the number of edges joining a vertex. Then

$$2e = nf,$$

[It follows from the counting of the number of ordered pairs (ε, σ), where ε is an edge, σ is a face, and ε bounds σ.]

$$2e = dv.$$

[It follows from the counting of the number of ordered pairs (ν, ε), where ν is a vertex, ε is an edge, and ε joins ν.] Thus

$$f = \frac{2e}{n}, \quad v = \frac{2e}{d}.$$

Recall the Euler formula $v - e + f = 2$; we have $\frac{2e}{d} - e + \frac{2e}{n} = 2$. Dividing both sides by $2e$, we have

$$\frac{1}{d} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2}. \tag{4}$$

Note that $n \geq 3$, as a convex polygon must have at least 3 sides; likewise $d \geq 3$, since it is geometrically clear that in a polyhedron a vertex must belong to at least 3 edges. Since the right hand side of (4) is at least $\frac{1}{2}$, it follows that we cannot have both $d \geq 4$ and $n \geq 4$. So we have either $d \leq 3$ or $n \leq 3$, and subsequently either $d = 3$ or $n = 3$.

CASE $d = 3$. Then (4) becomes

$$\frac{1}{n} = \frac{1}{e} + \frac{1}{6}.$$

Since e is positive, it follows that $3 \leq n \leq 5$. So $(n, e) = (3, 6), (4, 12), (5, 30)$; i.e., $(v, e, f) = (4, 6, 4), (8, 12, 6), (20, 30, 12)$.

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We thus have five regular polyhedra: tetrahedron $(4, 6, 4)$; cube $(8, 12, 6)$; octahedron $(6, 12, 8)$; dodecahedron $(20, 30, 12)$; icosahedron $(12, 30, 20)$. □

Example. The surface graph on a football is known as the **football graph**, denoted C_{60}. It is easy to see that C_{60} has only faces of pentagons (5-sided polygons) and hexagons (6-sided polygons), each vertex is joined by three edges, each pentagon is surrounded by five hexagons, and each hexagon is surrounded by three pentagons and three hexagons. Find the number of vertices, edges, and faces of the football graph.

Let v, e, f_5, f_6 be the number of vertices, edges, pentagons, hexagons of G respectively. Counting the number of pairs (u, x), where u is a vertex incident with an edge x; the number of pairs (P, X), where P is a pentagon, X is a hexagon, and P, X share a common edge; and the number of pairs (x, F), where x is an edge bounding a face F; we obtain the following equations:

$$3v = 2e, \quad 5f_5 = 3f_6, \quad 2e = 5f_5 + 6f_6.$$

The Euler formula, $v - e + f_5 + f_6 = 2$, is another equation. Substitute $v = 2 + e - f_5 - f_6$ into $3v = 2e$; we obtain

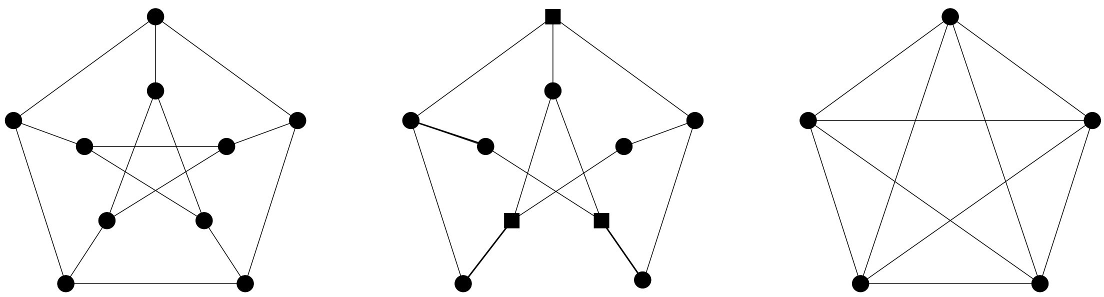
$$3(2 + e - f_5 - f_6) = 2e; \quad \text{i.e.,} \quad e = 3f_5 + 3f_6 - 6.$$

Substitute $e = 3f_5 + 3f_6 - 6$ into $2e = 5f_5 + 6f_6$; we obtain $f_5 = 12$. Thus $f_6 = 20$, $e = 90$, $v = 60$. We have

$$(v, e, f_5, f_6) = (60, 90, 12, 20).$$

3 Kuratowski's Characterization of Planar Graphs

- A **subdivision** of an edge e in a graph G is to delete the edge e and introduce a new vertex x and joins x to the end-vertices of x.
- A **subdivision** of a graph G is a graph obtained from G by a sequence of edge subdivisions.
- A **minor** of a graph G is any graph obtained from G by means of a sequence of vertex and edge deletions and edge contractions.
- The Peterson graph P_5 has both K_5 and $K_{3,3}$ as minors. However, P_5 contains a subdivision of $K_{3,3}$; but P_5 does not contain any subdivision of K_5.



(a) Petersen graph P_5 (b) Subdivision of $K_{3,3}$ (c) Complete graph K_5

Figure 5: Nonplanar graphs

Theorem 3.1 (Kuratowski Theorem). *A graph G is planar if and only if G contains no subdivision of K_5 and $K_{3,3}$.*

Proof. To be continued. □

The Kuratowski Theorem can be stated equivalently as follows.

Theorem 3.2 (Kuratowski). *A graph G is planar if and only if G has no minor of K_5 and $K_{3,3}$.*