Planar Graphs

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1 Planar Graphs

Theorem 1.1 (Jordan Curve Theorem). *A simple closed curve C in the plane $\mathbb{R}^2$ separates the plane into two disjoint open sets.*

- The Jordan Curve Theorem is not true for the Möbius band.
- A graph G is said to be **embeddable in a plane**, or **planar**, if it can be drawn in the plane i such a way that no two edges cross each other. Such a drawing is called a **planar embedding** of the graph.
- A planar graph G drawn in the plane is called a **plane graph**.
- A **subdivision** on an edge e with end-vertices u, v of a graph G is to add a new vertex w on e, i.e., by adding the new vertex w to G and replacing the edge e with two edges, one is incident with u, w and the other with v, w.
- A **subdivision** of a graph G is a graph obtained from G by a sequence of subdivisions on some edges of G.

Proposition 1.2. *A graph G is planar iff every subdivision of G is planar.*

Proposition 1.3. *A graph G is embeddable in the plane iff it is embeddable in the sphere.*

Let $G = (V, E)$ be a plane graph. We think ok G as the union $V \cup E$, which is considered to be a subspace of the plane $\mathbb{R}^2$ (or sphere S^2). The complement of G, $\mathbb{R}^2 \smallsetminus G$, is a collection disconnected open sets of $\mathbb{R}^2$ (or of S^2), each is called a **face** of G. Each plane graph has exactly one unbounded face, called the **outer face**. The numbers of vertices, edges, and faces are denoted by $v(G), e(G)$, and $f(G)$ respectively.

The **boundary** of a face σ is the topological boundary of the open set σ in $\mathbb{R}^2$ (or S^2). A face is **incident** with a vertex (an edge) if the vertex (edge) is contained in the boundary of the face. Two faces are said to be **adjacent** if their boundaries contain a common edge.

Proposition 1.4. *Let G be a planar graph. Let σ be a face of G in some planar embedding. Then G admits a planar embedding whose outer face has the same boundary as that of σ.*

The proposition above means that outer face is not special. Any face can be an outer face. For a plane graph G, we denote by $f(G)$ the number of faces of G, by $e(G)$ the number of edges, and $v(G)$ the number of vertices of G. For each face σ of G, let $|\sigma|$ denote the number of sides of the face σ. Then

$$\sum_{v\in V} \deg_G(v) = 2e(G) = \sum_{\sigma\in\mathcal{F}} |\sigma|,$$

Theorem 1.5 (Face Cycle Theorem)**.** *Let G be a nonseparable graph and is neither K_1 nor K_2. Then each face of G is bounded by a cycle.*

Proof. Consider an ear decomposition $G_0, G_1, \ldots, G_k$, where G_0 is a cycle, $G_k = G$, $G_i = G_{i-1} \cup P_i$ is nonseparable, P_i is a path, $i = 1, \ldots, k$. We apply induction on k. For $k = 0$, it is trivially true. Assume that it is true for G_{k-1} and let σ denote the outer face of G_{k-1}. Now for G_k, then σ is divided into one bounded face σ_1 and one unbounded face σ_2 of G_k. Let $P = v_0e_1v_1 \cdots e_lv_l$ be a closed path representing the boundary of σ, and let v_i, v_j be the initial and terminal vertices of P_k with $i < j$. Let P_1' denote the subpath of P from v_i to v_j, and P_2' the subpath from v_j to v_i in the direction of P. Then the closed paths $P_k^{-1}P_1'$ and PP_2 bound the faces σ_1 and σ_2 respectively. □

Corollary 1.6. *Let G be a loopless 3-connected plane graph. Then the neighbor of every vertex lie on a common cycle.*

Proof. Fix a vertex v, the graph $G \smallsetminus v$ is nonseparable. Let G be embedded in the plane, and let σ denote the face of $G \smallsetminus v$ in which the vertex v is located. Then by Theorem 1.5 the boundary of σ is a cycle, which contains all neighbors of v. □

2 Duality

Definition 2.1. Let G be a plane graph. For each face σ of G, we choose a point σ^* inside σ, and for each edge e bounding two faces σ, τ (the two faces may be the same), we draw a topological path e^* from σ^* to τ^* inside $\sigma \cup \tau \cup e$, crossing the edge e once. The new vertices σ^* and new edges e^* form a graph embedded in the plane, called the **dual graph** of G, denoted G^*.

Proposition 2.2. *The dual graph G^* of a plane graph G is always connected. Moreover,*

(a) Each cut edge of G corresponds to a loop of G^*.

(b) Each loop of G corresponds to a cut edge of G^*.

(c) If G is connected, then G^{**} is isomorphic to G.

Proof. For two vertices σ^*, τ^* of G^*, where σ, τ are two faces of G, there exists a topological path from σ^* to τ^*, crossing finite number of edges $e_1, \ldots, e_k$ of G, where each e_i bounds two faces σ_{i-1}, σ_i, $1 \le i \le k$. Then $P = \sigma_0^*e_1^*\sigma_1^* \cdots e_k^*\sigma_k^*$ is a path from σ^* to τ^*.

(a) Each cut edge e of G is on the boundary of one face σ of G, i.e., the outer face of G. The corresponding edge e^* is incident with the vertex σ^* twice in G^*. So e^* is a loop.

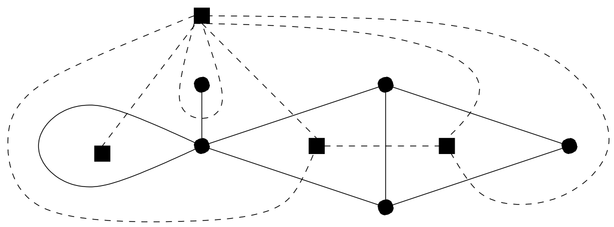

Figure 1: The dual graph of a plane graph

(b) Each loop e of G encloses a face σ of G. The corresponding edge e^* connects the part of G^* inside the loop e and the part of G^* outside the loop e. So e^* is a cut edge of G^*.

(c) For each vertex v of G, let $e_j = vv_j$ be the links and f_k the loops of G at v, where $1 \leq j \leq m$ and $1 \leq k \leq n$. Let σ_0 denote the unbounded face of G.

Case 1: G has no edges at v. Then the conclusion is trivially true.

Case 2: G has only loops at v, i.e., $G \smallsetminus v = \varnothing$. Then the face σ_0 of G is enclosed by the closed walk

$$W = \sigma_0^* f_1^* \sigma_1^* f_1^* \sigma_0^* f_2^* \sigma_2^* \cdots \sigma_0^* f_n^* \sigma_n^* f_n^* \sigma_0^*.$$

in G^* and contains the only vertex v.

Case 3: G has only links at v. Let G_i be the components of $G \smallsetminus v$, where $1 \leq i \leq l$. We may assume that the edges of G at v are ordered clockwise as $e_1, \ldots, e_m$ so that

$$[v, G_1] = \{e_1, \ldots, e_{m_1}\}, \ \ [v, G_2] = [e_{m_1+1}, \ldots, e_{m_2}], \ \ \ldots, \ \ [v, G_l] = \{e_{m_{l-1}+1}, \ldots, e_{m_l}\},$$

where $m_0 = 0$ and $m_l = m$. Let σ_j denote the face of G between the edges e_j and e_{j+1}, where $m_{i-1} < j < m_i$ and $1 \leq i \leq l$. Then

$$P_i = \sigma_0^* e_{m_{i-1}+1}^* \sigma_{m_{i-1}+1}^* e_{m_{i-1}+2}^* \sigma_{m_{i-1}+2}^* \cdots e_{m_i-1}^* \sigma_{m_i-1}^* e_{m_i}^* \sigma_0^*$$

is a closed path (i.e. cycle) in G^*, where $1 \leq i \leq l$ and $m_0 = 0$. In particular, when $m_i = m_{i-1} + 1$, the closed path P_i is a loop. The face σ_0 is enclosed by the closed walk $W' = P_1 P_2 \cdots P_l$ in G^* and contains the only vertex v. In particular, if $n = 1$, then W' is a closed path and the face σ_0 of G is enclosed by a cycle in G^*.

Case 4: G has both links and loops at v. Then σ_0 is enclosed by the closed walk $W'' = W'W$ and contains the only vertex v.

For each face F of G^*, take an edge e^* of F in G^*, there exists an edge e crossing e^*, then F contains an end-vertex of the edge e in G. We have obtained a one-to-one correspondence between the vertices of G^{**} and the vertices of G.

By definition of plane dual graph, for each face F of G^*, we may choose the vertex F^* of G^{**} to be the only vertex of G inside F. Then G and G^{**} have the same vertex set. Now two vertices F_1^*, F_2^* of G^{**}, which correspond to two faces F_1, F_2 of G^*, are adjacent in G^{**} iff there exists an edge e^* which corresponds an edge e of G with end vertices v_1, v_2.

Given two faces F_1, F_2 of G^*. If F_1^*, F_2^* are adjacent in G^{**}, then F_1 and F_2 bound a common edge e^* of G^*, since G^{**} is the dual of G^*. Thus there exists an edge e of G crossing e^*, since G^* is the dual of G. It follows that the end-vertices v_1, v_2 of e must be located in the faces F_1, F_2 respectively, say, $v_i \in F_i$, $i = 1, 2$. Then $F_i^* = v_i$ by the choices of F_i^* inside F_i, $i = 1, 2$. We have seen that F_1^*, F_2^* are already adjacent in G.

Conversely, if F_1^*, F_2^* are adjacent in G by an edge $e = v_1v_2$ with $F_i^* = v_i$, $i = 1, 2$. Then there exists an edge e^* of G^* crossing the edge e of G. Since e is adjacent with F_1^* and F_2^* in G, it means that e^* bounds the faces F_1, F_2 of G^*. Then by definition of dual the vertices F_1^*, F_2^* are adjacent in G^{**}. □

Proposition 2.3. *A simple plane graph G is a triangulation iff its dual graph G^* is 3-regular.*

Proposition 2.4. *Let G be a connected plane graph with an edge e. If e is not a cut edge, then $(G \setminus e)^* \simeq G^*/e^*$. If e is a link, then $(G/e)^* \simeq G^* \setminus e^*$.*

Proof. Since e is not a cut edge, it must be on the boundary of two faces σ, τ of G. Deleting the edge e results the faces σ, τ amalgamated into one face ρ. Let $e_1, \ldots, e_k$ be the edges bounding σ, and $f_1, \ldots, f_l$ the edges bounding τ in G. Then in G^*, the dual edges $e_1^*, \ldots, e_k^*$ are incident with the dual vertex σ^*, and $f_1^*, \ldots, f_l^*$ are incident with the dual vertex τ^*. However in $(G \setminus e)^*$, the dual edges $e_1^*, \ldots, e_k^*$ and $f_1^*, \ldots, f_l^*$ are incident with the dual vertex ρ^*, which is the result by contracting the edge e^* in G^*. Hence $(G \setminus e)^* \simeq G^*/e^*$.

Since $G^{**} \simeq G$, we have $(G^* \setminus e^*)^* \simeq G^{**}/e^{**} \simeq G/e$. $G^* \setminus e^* \simeq (G/e)^*$. □

Proposition 2.5. *The dual graph G^* of a nonseparable plane graph G is nonseparable.*

Proof. It is trivial to verified it when G has only one or two vertices. We may assume that G has at least three vertices. Then G has neither loops nor cut edges, so is G^*. Actually, both G and G^* are 2-connected. We apply induction on the number of edges of G. It is trivial when G has no edges, i.e., G is a single vertex. When G has some edges, take an edge e. Then either $G \setminus e$ or G/e is nonseparable. Consequently, either $(G \setminus e)^*$ or $(G/e)^*$ is nonseparable by induction, i.e., either G^*/e^* or $G^* \setminus e^*$ is nonseparable. In the latter case, G^* is clearly nonseparable by adding the edge e^*.

In the former case, i.e., G^*/e^* is nonseparable, we claim that G^* is nonseparable. Suppose G^* is separable with the separating vertex v^*. If v^* is not incident with e^*, then G^*/e^* is separable, a contradiction. If v^* is incident with e^*, let H denote the maximal nonseparable component of G^* that contains the edge e^*. Then H cannot be just the edge e^*, otherwise, e^* is a link of G^*. Thus H/e^* is nontrivial and is a nonseparable component of G^*/e^*, a contradiction. □

Proposition 2.6. *Let G be a plane graph and G^* its dual. Let C be a cycle of G. Let X^* be the vertices of G^* that lie inside C. Then $G^*[X^*]$ is connected.*

Theorem 2.7. *Let G be a plane graph and G^* its dual graph.*

(a) If C is a cycle of G, then C^* is a bond of G^*.

(b) If B is a bond of G, then B^* is a cycle of G^*.

Proof. (a) Let X^* be the set of vertices lie inside C and Y^* the set of vertices outside C. Then both $G^*[X^*]$ and $G^*[Y^*]$ are connected. So $C^* = [X^*, Y^*]$ is a bond of G^*.

(b) Let the edges of B be listed clockwise as $e_1, e_2, \ldots, e_k$, where e_i is bounds the faces σ_{i-1} and σ_i, $i = 1, \ldots, k$ and $\sigma_k = \sigma_0$. Then $\sigma_0^* e_1^* \sigma_1^* e_2^* \sigma_2^* \cdots e_k^* \sigma_k^*$ form a cycle of G^*. □

Corollary 2.8. *The cycle space of a plane graph G is isomorphic to the bond space of its dual graph G^*.*

Let $D = (G, \omega)$ be a plane digraph. Choose an orientation of the plane $\vec{n}$. The **directed plane dual** of D is the digraph $D^* = (G^*, \omega^*)$, where ω^* is the orientation on G^* such that $(a, a^*, \vec{n})$ is in right system.

Theorem 2.9. *Let D be a plane digraph and D^* its directed plane dual digraph.*
(a) If C is a directed cycle of D, then C^* is a directed bond of D^*.
(b) If B is a directed bond of D, then B^* is a directed cycle of D^*.

3 Euler Formula

Proposition 3.1. *Let G be a plane graph. Let $F(G)$ denote the set of faces of G. Then*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)| = \sum_{\sigma \in F(G)} |\sigma|,$$

where $|\sigma|$ is the number of sides of the face σ.

Proof. The first equality follows from the fact that each edge two in the sum of vertex degrees. The second equality follows from the fact that each edge contributes two sides either to one face or to two faces, each for one face. □

Theorem 3.2 (Euler's Formula)**.** *For each connected plane graph G,*

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

Proof. We apply induction on the number of edges of G. When $|E(G)| = 0$, the graph G must be a single vertex, and in this case the number of faces is one. Clearly,

$$|V(G)| - |E(G)| + |F(G)| = 1 - 0 + 1 = 2.$$

Assume that it is true for planar graphs with m edges, where $m \geq 1$. We consider a connected planar graph G with $m+1$ edges. The graph G may or may not have cycles. If G has no cycles, i.e., G is a tree, then $|E(G)| = |V(G| - 1$ by the tree formula, which is the Euler Formula since $|F(G)| = 1$. If G has a cycle C, choose an edge e on C and remove e from G to obtain a new planar graph G'. Since e is on a cycle, G' is still connected, but have less number of edges. By induction we have $|V(G')| - |E(G')| + |F(G')| = 2$. Note that e bounds two faces, one is inside the cycle C and the other is outside C. So $|V(G')| = |V(G)|$, $|E(G')| = |E(G)| - 1$, and $|F(G')| = |F(G)| - 1$. We thus obtain $|V(G)| - |E(G)| + |F(G)| = 2$. □

All planar embeddings of a planar graph G have the same umber of faces.

4 Kuratowski's Theorem

- A **subdivision** of an edge e with end-vertices u, v of a graph G is to add a new vertex w on e, i.e., by adding the new vertex w to G and replacing the edge e with two edges, one is incident with u, w and the other with v, w.

- A **subdivision** of a graph G is a graph obtained from G by a sequence of subdivisions on some edges of G.

- A minor of a graph G is any graph obtained from G by means of a sequence of vertex deletions, edge deletions, and edge contractions.

  Let $V(G)$ be partitioned into nonempty sets $V_0, V_1, \ldots, V_k$. Let H be the graph obtained from G by deleting V_0 and shrinking $G[V_i]$ for $1 \leq i \leq k$. Then any spanning subgraph of H is a minor of G.

- The Peterson graph P_5 has both K_5 and $K_{3,3}$ as its minors. However, P_5 does contain a subdivision of $K_{3,3}$, but does not contain any subdivision of K_5.

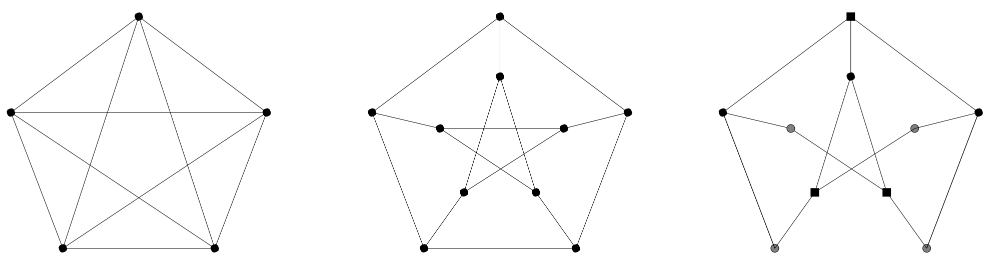

Figure 2: K_5, P_5, and subdivision of $K_{3,3}$ embedded in P_5

Theorem 4.1 (Kuratowski)**.** *A graph G is planar iff if G contains neither subdivision of K_5 nor subdivision of $K_{3,3}$.*

Proof. Too long. □

Exercises

Ch10: 10.1.3; 10.2.1; 10.2.2; 10.2.4; 10.2.9; 10.3.5.