Week 3: Connected Subgraphs

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1 Connected Graphs

Path, Distance:

- A path from a vertex x to a vertex y in a graph G is referred to an xy-path.
- Let $X, Y \subset V(G)$. An (X, Y)**-path** is an xy-path with $x \in X$ and $y \in Y$.
- The **distance** between two vertices x and y, denoted $d(x, y)$, is the minimal length of all xy-paths. If there is no path between x and y, we define $d(x, y) = \infty$.

Technique of Using Eigenvalues:

Theorem 1.1. *Let G be a simple graph with n vertices in which any two vertices have exactly one common neighbor. Then G has a vertex of degree $n-1$. Consequently, G must be obtained from a family of disjoint triangles by gluing selected vertices, one from each triangle, to a single vertex.*

Proof. Suppose it is not true, i.e., the maximal degree $\Delta(G) < n - 1$. We first show that G is regular. Consider two non-adjacent vertices x and y. Let $f : N(x) \to N(y)$, where $f(v)$ is defined as the unique common neighbor of v and y. We claim that f is injective. In fact, if $f(u) = f(v)$ for distinct $u, v \in N(x)$, then $f(u)$ is a common neighbor of u, v, y; now u and v have two common neighbors x and $f(u)$, a contradiction. Thus $d(x) = |N(x)| \le |N(y)| = d(y)$. Likewise, $d(y) \le d(x)$. So $d(x) = d(y)$. This is equivalent to say that any two adjacent vertices in $\bar{G}$ (the complement simple graph of G) have the same degree. We claim that G is regular.

To this end, it suffices to show that $\bar{G}$ is connected. Note that $\bar{G}$ has no isolated vertices, since the minimal degree $\Delta(\bar{G}) = n - 1 - \Delta(G) > 0$. Suppose $\bar{G}$ has two or more connected components. Take two edges $e_i = u_i v_i$ from distinct components of $\bar{G}$, $i = 1, 2$. Then $u_1 u_2 v_1 v_2 u_1$ is a cycle of G. Thus u_1 and v_1 have at least two common neighbors u_2, v_2, a contradiction.

Let G be k-regular. Consider the number of paths of length 2 in G. Since any two vertices have exactly one common neighbor, there are $\binom{n}{2}$ paths of length 2. For each vertex v, there are $\binom{k}{2}$ paths with the middle vertex v. It follows that $\binom{n}{2} = n\binom{k}{2}$. So $n = k^2 - k + 1$.

Let $\mathbf{A}$ be the adjacency matrix of G. The (u, v)-entry of $\mathbf{A}^2$ is the number of (u, v)-walks of length 2. Then $\mathbf{A}^2$ has its diagonal entries k and other entries 1. So $\mathbf{A}^2 = (k-1)\mathbf{I} + \mathbf{J}$, where

$\mathbf{I}$ is the identity matrix and $\mathbf{J}$ is a matrix whose entries are 1. Note that $\mathbf{J}$ has eigenvalue 0 with multiplicity $n-1$ and simple eigenvalue n. Since $\mathbf{A}^2 - \lambda\mathbf{I} = (k-1-\lambda)\mathbf{I} + \mathbf{J}$ and $n = k^2 - k + 1$, we see that $\mathbf{A}^2$ has eigenvalue $k-1$ with multiplicity $n-1$ and a simple eigenvalue k^2 with eigenvector $(1,\ldots,1)^T$. Since $\mathbf{A}^2 - \lambda^2\mathbf{I} = (\mathbf{A} - \lambda\mathbf{I})(\mathbf{A} + \lambda\mathbf{I})$, we see that $\mathbf{A}$ has the eigenvalues $\pm\sqrt{k-1}$ with multiplicity $n-1$ and a simple eigenvalue k.

Since the graph G is simple, we have $\mathrm{tr}(\mathbf{A}) = \mathbf{0}$ (the sum of its diagonal entries). Recall that the trace of $\mathbf{A}$ is the sum of its eigenvalues counted with multiplicities. We have $\pm(n-1)\sqrt{k-1} + k = 0$; it forces that $(n-1)\sqrt{k-1} = k$. The only possible choice is that $k = 2$ and $n = 3$, i.e., G is a triangle, where $\Delta(G) = 2$. This is contradict to that $\Delta(G) < n-1$. $\square$

Remark: The above proof is interesting, but the result is boring.

2 Euler Tour

- A trail in a connected graph is called an **Euler tail** if it traverses every edge of the graph.
- Let G be a connected graph. A **tour** of G is a closed walk that traverses each edge at least once. An **Euler tour** is a tour that traverses each edge exactly once. A graph is said to be **Eulerian** if it admits an Euler tour.

Theorem 2.1. *A connected graph G has an Euler tour iff G has even degree at every vertex.*

Proof. "$\Rightarrow$" Let $T = v_0e_1v_1e_2\ldots e_nv_n$ be an Euler tour of G. When one travels along the Euler tour and passes by a vertex v, the person must come towards v through one edge and depart from v through another edge. Then the number of times coming towards v equals the number times departing from v. Thus the degree of v must be even.

"$\Leftarrow$" Consider a longest trail $T = v_0e_1v_1e_2\ldots e_nv_n$ in G. We show that T is an Euler tour.

(a) Claim $v_0 = v_\ell$. Suppose $v_0 \neq v_\ell$. Let v_ℓ be appeared k times in the vertex sequence $(v_0, v_1, \ldots, v_{\ell-1})$, say, $v_{i_1} = \cdots = v_{i_k} = v_\ell$, $i_1 < \cdots < i_{i_k} < \ell$. Then the degree of v_ℓ in T is $2k+1$. Since the degree of v_ℓ in G is even, there is an edge e in G but not in T incident with v_ℓ and another vertex v. Thus $T' = Tev$ is a longer trail in G, a contradiction.

(b) Claim that $G - E(T)$ has no edges incident with a vertex in T. Suppose there is an edge $e \in E(G) - E(T)$ incident with a vertex $v_i \in V(T)$ and another vertex v. Then

$$T' = vev_1e_{i+1}v_{i+1}\cdots e_\ell v_\ell(v_0)e_1v_1\cdots v_{i-1}e_iv_i$$

is a longer trail in G, a contradiction.

(c) Claim that T uses every edge of G. Suppose there exists an edge e not used in T. Let e be incident with vertices u, v. Then $u, v \notin V(T)$. Since G is connected, there is a shortest path $P = u_0x_1u_1\cdots x_mu_m$ from T to u, where $u_0 = u_i$ and $u_m = u$. We claim that $x \not\in E(T)$. Suppose $x_1 \in E(T)$, then $u_1 \in V(T)$; thus $P' = u_1x_2u_2\cdots x_mu_m$ is a shorter path from T to u, a contradiction. Now since $x_1 \notin E(T)$, we have a longer trail

$$T' = u_1x_1(u_0)v_ie_{i+1}v_{i+1}\cdots e_\ell v_\ell(v_0)e_1v_1e_2\cdots v_{i-1}e_iv_i.$$

Again this is a contradiction.

Since T is a closed trail that uses every edge of G, we see that T is an Euler tour. $\square$

Corollary 2.2. *A connected graph G has an Euler trail iff G has even degree at all vertices or G has exactly two vertices of odd degree.*

Proof. "$\Rightarrow$" Let $T := v_0e_1v_1 \cdots e_\ell v_\ell$ be an Euler trail of G. If $v_0 = v_\ell$, then T is an Euler tour. By Theorem 2.1 G, G is an even graph. If $v_0 \neq v_\ell$, we add a new edge e_0 between v_0 and v_ℓ. Then $T' := Te_0v_0$ is an Euler tour of the graph $G' := G \cup e_0$. Again by Theorem 2.1, G' is an even graph. It follows that G has exactly the two vertices v_0, v_ℓ of odd degree.

"$\Leftarrow$" It is Theorem 2.1 when G is an even graph. Let G have exactly two vertices u and v of odd degree. We add a new edge between u and v to G to obtain a new graph G'. Then G' is a connected even graph. Thus G' has an Euler tour by Theorem 2.1. Remove the edge e from the Euler tour for G', we obtain an Euler trail for G. $\square$

A **cut edge** of a graph G is an edge e such that $G \setminus e$ has more connected components than G.

Lemma 2.3. *let G be a connected graph with a specified vertex v. Assume that G is either an even graph (former case) or G has exactly two vertices u and v of odd degree (latter case).*

(a) If $d_G(v) = 1$ and e is a link joining v to a vertex w, then G is the latter case. Moreover, $G \setminus v$ is connected, either having all vertices of even degree or having exactly two vertices v and w of odd degree.

(b) If $d_G(v) \geq 2$ and $d_G(v)$ is even, then G is the former case. Moreover, for each edge e joining v to a vertex w, $G \setminus e$ is connected, either having all vertices of even degree or having exactly two vertices v and w of odd degree.

(c) If $d_G(v) \geq 2$ and $d_G(v)$ is odd, then G is the latter case. Moreover, there exists an edge e joining v to a vertex w such that $G \setminus e$ is connected, either having all vertices of even degree or having exactly two vertices v and w of odd degree.

Proof. (a) It is clear the latter case. If $w = u$, i.e., $d_G(w)$ is odd, then $G \setminus v$ is a connected even graph. If $w \neq u$, i.e., $d_G(w)$ is even, then $G \setminus v$ has exactly two vertices u and w of odd degree.

(b) Since $d_G(v)$ is even, it turns out that G is the former case. If e is a loop at v, then $G \setminus e$ have the same property as G. If e is a link, then $G \setminus e$ has exactly two odd-degree vertices v and w. We still need to show that $G \setminus e$ is connected. Suppose $G \setminus e$ has two connected components G_1 and G_2 with $v \in V(G_1)$ and $w \in V(G_2)$. Then G_1 has exactly one vertex v of odd degree. This is impossible because the number of odd-degree vertices is always even.

(c) It is clear that G is the latter case. If G has no cut edge at v, then for any edge e at v joining to a vertex w, $G \setminus e$ is an even graph if $w = u$ or $G \setminus e$ has exactly two vertices u and w of odd degree.

Let G have a cut edge e' at v. Then $G \setminus e'$ has two connected components G_1 and G_2 with $v \in V(G_1)$, $d_{G_1}(v)$ is even, and $d_G(v) \geq 3$. If $u \in V(G_1)$, then G_1 has exactly one odd-degree vertex u; this is a contradiction. So $u \notin V(G_1)$, and G_1 is a connected even graph and $d_{G_1}(v) \geq 2$. Then by (b) for any edge $e \in E(G_1)$ at v joining to a vertex w, $G_1 \setminus e$

is connected, either having all vertices of even degree or having exactly two vertices v and w of odd degree. Consequently, $G \smallsetminus e$ is connected, either having all vertices of even degree or having exactly two vertices u and w of odd degree. $\square$

Theorem 2.4. (Fleury's Algorithm) Input: *a connected graph $G = (V, E)$.* **Output**: *an Euler tour, or an Euler trail, or no Euler trail for G.*

Step 1 *If there are vertices of odd degree, then start at one such vertex u. Otherwise, start at any vertex u. Set $T := u$ and $G' := G$.*

Step 2 *Let v be the terminal vertex of T. If there is no edge remaining at v in G',* **stop**. *(Now T is an Euler tour if $v = u$ and an Euler trail if $v \neq u$.)*

Step 3 *If there is exactly one edge e remaining at v in G', joining v to another vertex w, set $T := Tew$, $G' := G' \smallsetminus e$; and return to* Step 2.

Step 4 *If there are more than one edge remaining at v in G', choose one of these edges, say, an edge e with end-vertices v to w, in such a way that $G' \smallsetminus e$ is still connect; $T := Tew$, $G' := G' \smallsetminus e$, and return to* Step 2. *If such an edge can not be selected,* **stop**. *(There is no Euler trail.)*

Proof. Consider the pair (T, G'), called an **Eulerian pair** of G, where T is a trail of G, G' is a connected subgraph of G such that G' is either an even graph or has exactly two vertices of odd degree, the terminal vertex v of T is a vertex of G', and if G' is the case of having exactly two vertices of odd degree then v is one of the two odd-degree vertices. Such a pair is said to be **complementary** if $E(G) = E(T) \sqcup E(G')$ (disjoint union). The initial pair (T, G') in Fleury's algorithm is an Eulerian complementary pair.

Let (T, G') bean Eulerian complementary pair with the terminal vertex v of T in the process of Fleury's algorithm before entering Step 2. Now in Step 2, if there is no edge at v in G', then $E(G') = \varnothing$ (since G' is connected). It is clear that T is an Euler trail of G. In Step 3, we have $d_{G'}(v) = 1$; then by Lemma 2.3(c), $(Tew, G' \smallsetminus e)$ is an Eulerian complementary pair. In Step 4, we have $d_{G'}(v) \geq 2$; then by Lemma 2.3(b) and Lemma 2.3(c), $(Tew, G' \smallsetminus e)$ is an Eulerian complementary pair.

Since all pair (T, G') constructed in Fleury's algorithm are Eulerian complementary pairs, and edges of G' are reducing when iterates, Fleury's algorithm stops at $(T, \varnothing)$ with T an Euler train of G after finite number of iterates. $\square$

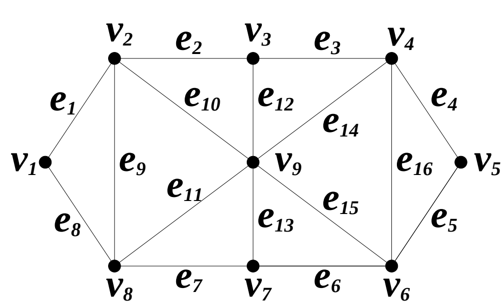

Figure 1: A graph with an Euler trail.

Example 2.1. An Euler trail for the graph in Figure 1 is given as

$$v_3e_3v_4e_4v_5e_5v_6e_6v_7e_7v_8e_8v_1e_1v_2e_2v_3e_{12}v_9e_{14}v_4e_{16}v_6e_{15}v_9e_{10}v_2e_9v_8e_{11}v_9e_{13}v_7.$$

3 Connection in Digraphs

- A **directed walk** in a digraph D is an alternating sequence of vertices and arcs $W := v_0a_1v_1 \ldots a_\ell v_\ell$ such that the arc a_i has the tail v_{i-1} and head v_i, $i = 1, \ldots, \ell$. We call v_0 the **initial vertex** and v_ℓ the **terminal vertex** of W. Such a walk is referred to a **directed** (v_0, v_ℓ)**-walk**; the subwalk of W from a vertex v_i to a vertex v_j is referred to a (v_i, v_j)**-segment** of W.

- A **directed trail** is a directed walk with distinct arcs. A **directed path** is a directed walk with distinct arcs and distinct vertices, except the possible case that the initial vertex equals the terminal vertex.

- Let D be a digraph. A vertex y is said to be **reachable** from a vertex x in D if there is a directed (x, y)-path from x to y in D. Two vertices x and y of D are said to be **strongly connected** if y is reachable from x and x is reachable from y in D. Strongly connectedness is an equivalence relation on $V(D)$. A sub-digraph of D induced by an equivalence class of the strong connectedness is called a **strongly connected component** or **strong component** of D.

- A **directed Euler trail** in a digraph D is a directed trail that uses every arc of D. A closed directed Euler trail is called a **directed Euler tour**. A digraph is said to be **Eulerian** if it has a directed Euler tour.

Theorem 3.1. *Let x and y be two vertices of a digraph D. Then y is reachable from x in D iff $(X, X^c) \neq \varnothing$ for every subset $X \subset V(D)$ such that $x \in X$ and $y \notin X$.*

Proof. For necessity, let P be a directed path from x to y. For each proper subset $X \subset V(D)$ with $x \in X$ and $y \in X^c$, the path P passes between X and X^c. The first arc of P from X to X^c; so $(X, X^c) \neq \varnothing$.

Conversely, for sufficiency, suppose that y is not reachable from x. Let X be the set of vertices reachable from x. Then $y \in X^c$. Since every vertex of X^c is not reachable from x, there is no arc from X to X^c. So $(X, X^c) = varnothing$, a contradiction. □

Theorem 3.2. *A connected digraph D is Eulerian iff the in-degree equals the out-degree at every vertex of D.*

Proof. □

4 Cycle Double Cover

Cycle Cover, Cycle Double Cover

- A **cycle cover** of a graph G is a family $\mathcal{F}$ of subgraphs of G such that $E(G) = \bigcup_{H \in \mathcal{F}} E(H)$ and each member of $\mathcal{F}$ is a cycle.

- A **cycle double cover** of a graph G is a cycle cover such that each edge of G belongs to exactly two members of $\mathcal{F}$, i.e., each edge of G is covered exactly twice by $\mathcal{F}$.

Proposition 4.1. *Let G be a graph having a cycle covering $\mathcal{C}$ that each edge of G is covered at most twice. Then G has a cycle double cover.*

Proof. Let $E_1 \subset E(G)$ be the edge subset whose edges are covered exactly one by $\mathcal{C}$. Since $\mathcal{C}$ is a covering and each edge of G is covered at most twice by $\mathcal{C}$, we see that $G[E_1]$ is an Eulerian graph (i.e. even graph). So $G[E_1]$ is a union of edge disjoint cycles, i.e., $G[E_1]$ has a covering $\mathcal{C}_1$ that each edge of $G[E_1]$ is covered exactly once. Thus $\mathcal{C}_2 = \mathcal{C} \cup \mathcal{C}_1$ is a cycle double covering. of G. $\square$

Cycle Double Cover Conjecture

Conjecture 4.2. *Every graph (i.e. having no cut edge) has a cycle double cover.*