# Week 1-2: Graphs and Subgraphs

September 19, 2020

## 1 Graphs

Definition of Graphs:

- A **graph** $G$ is a system of ordered pair $(V, E)$ of two finite disjoint sets $V$ of **vertices** and $E$ of **edges**, such that each edge $e$ connects two (possibly identical) vertices $u, v$ (called the **endpoints** of $e$). We usually write $V = V(G)$ and $E = E(G)$. For convenience, we write each edge $e$ with endpoints $u, v$ as $e = uv$ or $\mathrm{End}_G(e) = \{u, v\}$. So $\mathrm{End}_G : E \to \mathcal{P}_{1,2}(V) = \{\{v\}, \{u, v\} : u, v \in V\}$.

- When an edge $e$ connects vertices $u$ and $v$, we also say that $e$ **joins** $u$ and $v$, or, $u$ and $v$ are **incident** with $e$, or, $u$ and $v$ are **adjacent** by $e$. We say that $e$ is a **link** if $u \neq v$ and a **loop** if $u = v$.

- Two edges are **parallel** if they have the same endpoints. Parallel edges are also called **multiple edges**.

Simple Graphs, Multigraphs, Complete Graphs, Bipartite Graphs:

- A graph is **simple** if it has no loops and parallel edges. A graph with possible loops and parallel edges is emphasized as a **multigraph**.

- The graph with empty vertex set (and hence empty edge set) is a **null graph**.

- A graph is **trivial** if it has only one vertex and no edges. All other graphs are **non-trivial**.

- An **empty graph** is a graph with possible vertices but no edges.

- A **complete graph** is a simple graph that every pair of vertices are adjacent. A complete graph with $n$ vertices is denoted by $K_n$.

- A graph $G$ is **bipartite** if its vertex set $V(G)$ can be partitioned into two disjoint nonempty subsets $X, Y$ such that every edge has one endpoint in $X$ and one endpoint in $Y$; such a partition $\{X, Y\}$ is called a **bipartition** of $G$, and such a bipartite graph is denoted by $G[X, Y]$.

- A **complete bipartite graph** is a bipartite graph $G[X,Y]$ that each vertex in $X$ is adjacent to every vertex in $Y$; we abbreviate $G[X,Y]$ to $K_{m,n}$ if $|X| = m$ and $|Y| = n$.

Neighbors, Degree:

- Two adjacent vertices are **neighbors** each other. The set of neighbors of a vertex $v$ in a graph $G$ is the set of all vertices adjacent with $v$, denoted $N_v(G)$ or $G[v]$.
- The **degree** of a vertex $v$ in a graph $G$, denoted $d_G(v)$, is the number of edges incident with the vertex, where each loop at $v$ is counted twice. A vertex is **isolated** if its degree is 0. For a simple graph $G$, $d_G(v) = |N_v(G)|$.
- A graph is **regular** if every vertex has the same degree. A graph is $k$**-regular** if every vertex has degree $k$.
- A **cycle** is a connected 2-regular graph. If a graph is connected and its degree is at least two at every vertex, then the graph contains a cycle.
- For each graph $G = (V, E)$,
$$2|E| = \sum_{v \in V} d_G(v).$$
- The number of odd-degree vertices of a graph is even.

**Proposition 1.1.** *Let $G[X,Y]$ be a bipartite graph without isolated vertices. If $d(x) \geq d(y)$ for all edge $xy$ with $x \in X$ and $y \in Y$, then $|X| \leq |Y|$, and the equality holds if and only if $d(x) = d(y)$ for all edges $xy$ with $x \in X$ and $y \in Y$.*

*Proof.* Since $d(x) \geq d(y)$ for all edges $xy$ with $x \in X$ and $y \in Y$, we have

$$|X| = \sum_{x \in X} \sum_{\substack{y \in Y \\ xy \in E}} \frac{1}{d(x)} = \sum_{\substack{x \in X, y \in Y \\ xy \in E}} \frac{1}{d(x)} \leq \sum_{\substack{x \in X, y \in Y \\ xy \in E}} \frac{1}{d(y)} = \sum_{y \in Y} \sum_{\substack{x \in X \\ xy \in E}} \frac{1}{d(y)} = |Y|.$$

It is clear that if $d(x) = d(y)$ for all edges $xy$ with $x \in X$ and $y \in Y$ then $|X| = |Y|$. Conversely, if $|X| = |Y|$, the above middle inequality must be equality. It forces that $d(x) = d(y)$ for all edges $xy$ with $x \in X$. □

Incidence Matrix, Adjacency Matrix:

- The **incidence matrix** of a graph $G$ is a matrix $\mathbf{M} = [m_{ve}]$, whose rows are indexed by vertices and whose columns are indexed by the edges of $G$, such that (i) the entry $m_{ve} = 0$ at $(v, e)$ if the vertex $v$ is not incident with the edge $e$, (ii) $m_{ve} = 1$ if $v$ is incident with $e$ once (i.e., $e$ is a link), and (iii) $m_{ve} = 2$ if $v$ is incident with $e$ twice (i.e., $e$ is a loop).
- The **adjacency matrix** of a graph $G$ is a quare matrix $\mathbf{A} = [a_{uv}]$, whose rows and columns are indexed by vertices of $G$, where $a_{uv}$ is the number of edges between the vertices $u$ and $v$, each loop is counted twice.

Walk, Trail, Path, Cycle, Connectedness:

- A **walk** from a vertex $u$ to a vertex $v$ in a graph $G$ is a sequence

$$W = v_0 e_1 v_1 e_2 \cdots v_{\ell-1} e_\ell v_\ell$$

  of vertices and edges with $v_0 = u$ and $v_\ell = v$, whose terms are alternate between vertices and edges of $G$, such that the edge $e_i$ is incident with the vertices $v_{i-1}$ and $v_i$, $1 \le i \le \ell$. The vertex $v_0$ is the **initial vertex**, $v_\ell$ is the **terminal vertex** of $G$, and the number $\ell$ is the **length** of $W$. A walk is **closed** if its initial and terminal vertices are identical.

- A walks is a **trail** if its edges are distinct.

- A walk is a **path** if its vertices are distinct (so are its edges), except possible identical initial and terminal vertices, for which it is referred to a **closed path**. If $P = v_0 e_1 v_1 \cdots v_{\ell-1} e_\ell v_\ell$ is a path, then $v_0, v_1, \ldots, v_\ell$ are distinct, or, $v_0 = v_\ell$, $v_1, v_2, \ldots, v_{\ell-1}$ are distinct; the vertices $v_1, v_2, \ldots, v_{\ell-1}$ are **internal vertices** of $P$.

- A graph is **connected** if there is a path between any two vertices of the graph.

- The underlying graph of a closed path is a cycle. The underlying graph of a closed trail is connected and has even degree everywhere.

- An **Euler trail** of a graph $G$ is a trail that uses every edge of $G$ exactly once. An **Euler tour** is a closed Euler trail. The underlying graph of an Euler tour is called an **Eulerian graph**.

- A graph is called an **even graph** if it has even degree everywhere.

- A graph is **Eulerian** if and only if it is a connected even graph.

- A **Hamilton path** of graph $G$ is a path that uses every vertex of $G$. A closed Hamilton path is called a **Hamilton cycle**.

**Theorem 1.2. (Fleury's Algorithm) Input:** *Graph $G = (V, E)$.*
**Output:** *Euler tour, or Euler trail, or no Euler trail.*

`STEP 1`: *If there are vertices of odd degree, choose such a vertex $v$; otherwise, choose any vertex $v$. Set* SEQ $= v$.

`STEP 2`: *If there is no edge remaining at the terminal vertex $v$ of* SEQ, *then* STOP. *(There is an Euler trail. If $v$ is the same as the initial vertex of* SEQ, *it is an Euler tour.)*

`STEP 3`: *If there is exactly one edge $e$ from $v$ to another vertex $w$, then remove $ve$ and go to* `STEP 5`.

`STEP 4`: *If there are more than one edges remaining at $v$, choose one of these edges, say an edge $e$ from $v$ to $w$, in such a way that the removal of $e$ will not disconnect the remaining graph, then remove $e$ and go to* `STEP 5`. *If such an edge can not be selected,* `STOP`. *(There is neither Euler tour nor Euler trail.)*

`STEP 5:` *Add $ew$ to the end of* SEQ*, replace $v$ by $w$, and return to* `STEP 2`*.*

Union, Intersection, Cartesian Product:

- Two graphs are said to be **disjoint** if they have no common vertices, and to be **edge-disjoint** if they have no common edges.
- The **union** of two graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G$ and $H$ are disjoint, we write their union as $G + H$.
- The **intersection** of two graphs $G$ and $H$ is the graph $G \cap H$ with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. If $G$ and $H$ are disjoint, then $G + H$ is the null graph.
- The **Cartesian product** of two simple graphs $G, H$ is the graph $G \Box H$, whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is

$$\big\{(u,x)(v,x) : uv \in E(G), x \in V(H)\big\} \bigcup \big\{(u,x)(u,y) : u \in V(G), xy \in E(H)\big\}.$$

Digraphs, Out-degree, In-degree:

- A **directed graph** (or **digraph** for short) is an ordered pair $D = (V, A)$ of disjoint set $V$ of **vertices** and a set $A$ of **arcs** (edges with a direction), such that each arc $a \in A$ is associated with an ordered pair $(u, v)$ of two (possible identical) vertices $u$ and $v$ of $V$; the arc $a$ **points** from $u$ to $v$, or, $e$ **points away from** $u$ and **points toward** $v$.
- A **tournament** is a directed complete graph.
- Let $v$ be a vertex in a digraph $D$. The **out-degree** of $v$ is the number of arcs of which $v$ is tail, denoted $d^+_D(v)$. The **in-degree** of $v$ is the number of arcs of which $v$ is a head, denoted $d^-_D(v)$.
- Let $a$ be an arc in a digraph $D$ from a vertex $u$ to a vertex $v$. We call $u$ an **in-neighbor** of $v$, and $v$ an **out-neighbor** of $u$. We denote by $N^+_D(v)$ the set of all out-neighbors of a vertex $v$, and by $N^-_D(v)$ the set of all in-neighbors of $v$.

Orientations and Oriented Incidence Matrix:

- We may think of each graph $G$ embedded in a Euclidean space $\mathbb{R}^d$, viewing each edge $e \in E(G)$ with endpoints $u, v$ as a simple path $e : [0,1] \to \mathbb{R}^d$ such that $e(0) = u$ and $e(1) = v$. If $e$ is a loop, then $e$ is a closed simple path with $e(0) = e(1)$.
- An **orientation** of an edge $e$ with endpoints $u, v$ is either the direction of the simple paths $e : [0,1] \to \mathbb{R}^d$ from $e(0)$ to $e(1)$ or its reverse direction. An **orientation** of a graph is an assignment that each edge of the graph is given an orientation. A graph $G$ is **oriented** (or **directed**) if $G$ is given an orientation $\omega$, denoted $(G, \omega)$.

- The **oriented incidence matrix** of an oriented graph $(G, \omega)$ is a $\{-1, 0, 1\}$-valued matrix $M$ indexed by $V(G) \times E(G)$, such that the entry $(v, e)$ has value 0 if $e$ is a loop or $e$ is not incident with $v$, and has value $+1$ $(-1)$ if $e$ is a link pointing toward (away from) $v$.

**Theorem 1.3.** *Every tournament has a directed Hamilton path.*

*Proof.* Let $D$ be a tournament with $n$ vertices. We proceed by induction on $n$. It is trivial for $n = 2, 3$ by inspection. Now remove one vertex $v$ from $D$ to obtain a digraph $D' = D \smallsetminus v$ with $n - 1$ vertices. By induction hypothesis, $D'$ has a directed Hamilton path $P = v_1 v_2 \ldots v_{n-1}$ from $v_1$ to $v_{n-1}$. The situation can be divided into the following cases.

CASE 1. *$(v, v_1)$ is a directed edge in $D$.* Then $P_1 = v v_1 v_2 \ldots v_{n-1}$ is a directed Hamilton path for $D$. Otherwise, $(v_1, v)$ is the directed edge.

CASE 2. *$(v_1, v)$ and $(v, v_2)$ are directed edges in $D$.* Then $P_2 = v_1 v v_2 \ldots v_{n-1}$ is a directed Hamilton path for $D$. Otherwise, $(v_2, v)$ is the directed edge.

CASE 3. *$(v_1, v)$, $(v_2, v)$, and $(v, v_3)$ are directed edges in $D$.* Then $P_3 = v_1 v_2 v v_3 \ldots v_{n-1}$ is a directed Hamilton path for $D$. Otherwise, $(v_3, v)$ is the directed edge.

CASE $k$. *$(v_1, v), (v_2, v), \ldots, (v, v_k)$ are directed edges in $D$.* Then $P_k = v_1 \ldots v_{k-1} v v_k \ldots v_{n-1}$ is a directed Hamilton path for $D$. Otherwise, $(v_k, v)$ is the directed edge.

CASE $n$. *$(v_1, v), (v_2, v), \ldots, (v, v_{n-1})$ are directed edges in $D$.* Then $P_n = v_1 v_2 \ldots v_{n-2} v v_{n-1}$ is a directed Hamilton path for $D$. Otherwise, $(v_{n-1}, v)$ is the directed edge.

CASE $n + 1$. $P_{n+1} = v_1 v_2 \cdots v_{n-1} v$ is a directed Hamilton path for $D$. □

Isomorphism, Automorphism, Homomorphism:

- Two graphs $G$ and $H$ are **equal** (or **identical**) if $V(G) = V(H)$ and $E(G) = E(H)$.

- A graph $G$ is **isomorphic** to a graph $H$ if there exist bijective mappings $f : V(G) \to V(H)$ and $g : E(G) \to E(H)$ such that $\mathrm{End}_G(e) = \{u, v\}$ if and only if $\mathrm{End}_H(g(e)) = \{f(u), f(v)\}$; such a pair $(f, g)$ of mappings is called an **isomorphism** from $G$ to $H$.

- An isomorphism from a graph $G$ to itself if called an **automorphism** of $G$. The set of all automorphisms of $G$ froms a group under the composition of mappings, called the **automorphism group** of $G$, denoted $\mathrm{Aut}(G)$.

- A **homomorphism** from a graph $G$ to a graph $H$ if there exist maps $f : V(G) \to V(H)$ and $g : E(G) \to E(H)$ such that if vertices $u, v$ are adjacent by an edge $e$ then the vertices $f(u), f(v)$ are adjacent by the edge $g(e)$. [The concept of homomorphism of graphs is not yet standardized. We rarely use the concept in our course.]

Labeled Graphs:

- Given a finite set $V$. A simple graph $G = (V, E)$ on $V$ can be considered as a subset of $\binom{V}{2}$, the set of all 2-element subsets of $V$. A simple graph whose vertices are labeled, but whose edges are not labeled, is referred to a **labeled simple graph**.

- Given a set $V$ of $n$ elements. There are $2^{\binom{n}{2}}$ labeled simple graphs with the vertex set $V$. We denote by $\mathcal{G}(V)$ the set of all labeled simple graphs with vertex set $V$.

- Let $G$ be an unlabeled graph with $n$ vertices. Then the number of labelings of $G$ is $\frac{n!}{\mathrm{Aut}(G)}$, where $\mathrm{Aut}(G)$ is the automorphism group of $G$ with any labeling. Then

$$\sum_{\substack{G \text{ unlabeled graph} \\ \text{with } n \text{ vertices}}} \frac{n!}{\mathrm{Aut}(G)} = 2^{\binom{n}{2}}.$$

- The number of unlabeled graphs with $n$ vertices is at least $\lceil 2^{\binom{n}{2}}/n! \rceil$.

Intersection Graphs, Interval Graphs, Polyhedral Graphs, Cayley Graphs:

- Let $\mathcal{F}$ be a family of subsets of set $V$. The **intersection graph** of $\mathcal{F}$ is a graph whose vertex set is $\mathcal{F}$, and two members of $\mathcal{F}$ are adjacent if their intersection is nonempty.
- Let $V = \mathbb{R}$ and $\mathcal{F}$ be a set of some closed intervals of $\mathbb{R}$. The intersection graph of $\mathcal{F}$ is called an **interval graph**.
- Given a polytope $P$ of $\mathbb{R}^3$. The vertices and edges of $P$ form a graph, called a **polyhedral graph**.
- Let $\Gamma$ be a group. Given a subset $S \subset \Gamma$ such that $S$ does not contain the identity element of $\Gamma$ and is closed under inverse operation. The **Cayley graph** of $\Gamma$ with respect to $S$ is a graph $G(\Gamma, S)$ with vertex set $\Gamma$ in which two vertices $x, y$ are adjacent if $xy^{-1} \in S$.
- Each Cayley graph is regular.

Networks, Big Graphs, Infinite Graphs:

# 2 Subgraphs

Definition of Subgraphs:

- A graph $H$ is called a **subgraph** of a graph $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\mathrm{End}_H : E(H) \to \mathcal{P}_{1,2}(V(H))$ is the restriction of $\mathrm{End}_G : E(G) \to \mathcal{P}_{1,2}(V(G))$ to $E(H)$. We then say that $G$ **contains** $H$ or $H$ is contained in $G$.
- A **copy** of a graph $H$ in a graph $G$ is a subgraph of $G$ which is isomorphic to $H$. Such a subgraph is also referred to as an $H$**-subgraph** of $G$.
- An **embedding** of graph $H$ in a graph $G$ is an isomorphism from $H$ to a subgraph of $G$. For each copy of $H$ in $G$. For each copy of $H$ in $G$, there are $|\mathrm{Aut}(H)|$ embeddings in $G$, whose image subgraph is fixed.
- A maximal connected subgraph of $G$ is called a **connected component** (or just **component**) of $G$. The number of connected components of $G$ is denoted by $c(G)$.

Deletion, Contraction:

- Let $v$ be a vertex in a graph $G$. We denote by $G \setminus v$ the graph obtained from $G$ by deleting the vertex $v$ and all edges incident with $v$. Such an operation is referred to as an **vertex deletion**, and $G \setminus v$ as a **vertex-deleted subgraph**.

- Let $e$ be an edge of graph $G$. We denote by $G \setminus e$ the graph obtained from $G$ by deleting the edge $e$ but leaving the endpoints of $e$. Such an operation is referred to as an **edge deletion**, and $G \setminus e$ as an **edge-deleted subgraph**. If $S \subseteq E(G)$, we denote by $G \setminus S$ the graph obtained from $G$ by deleting all edges of $S$.

- Let $e$ be an edge of a graph $G$. We denote by $G/e$ the graph obtained from $G$ by deleting the edge $e$ and gluing the endpoints of $e$ to become one vertex. Such an operation is called a **contraction**, and $G/e$ an **edge-contracted minor** of $G$. Note that there are edges (other than $e$) joining the endpoints of $e$, then those edges become loops in $G/e$. If $S \subseteq E(G)$, we denote by $G/S$ the graph obtained from $G$ by contracting all edges of $S$.

**Theorem 2.1.** *A graph $G$ whose every vertex has degree at least 2 contains a cycle.*

*Proof.* Let $P := v_0e_1v_1 \cdots e_\ell v_\ell$ be a longest path in $G$. Such a path does exist since $G$ is finite. Of course, $\ell \geq 1$. If $v_0 = v_\ell$, then the underlying graph of $P$ is already a cycle. If $v_0 \neq v_\ell$, then the degree of $v_\ell$ in $P$ must be 1. Since the degree of $v_\ell$ in $G$ is at least 2, there exists an edge $e_{\ell+1}$ (not in $P$) joining $v_\ell$ to another vertex $v_{\ell+1}$. If $v_{\ell+1} = v_i$ for some $i$ with $0 \leq i \leq \ell$, then the underlying graph of $P_i := v_ie_{i+1}v_{i+1}e_{i+1} \cdots e_\ell v_\ell e_{\ell+1}v_{\ell+1}$ is a cycle. Otherwise, $Q := Pe_{\ell+1}v_{\ell+1}$ is a longer path, a contradiction. □

**Corollary 2.2.** *A graph with some edges but no cycles has at least one vertex of degree 1; actually it has at leat two vertices of degree 1.*

*Proof.* The degree of the initial vertex and the terminal vertex of a longest path $P$ in $G$ have degree 1. □

Acyclic Graphs (=Forests):

- A graph is said to be **acyclic** it it contains no cycles. An acyclic graph is also called a **forest**. A **tree** is a connected forest.

- Each vertex of degree 1 in a tree is called a **leaf** of the tree.

- Each tree with edges contains at least **two** leaves.

- If $T = (V, E)$ is a tree, then $|E| = |V| - 1$. If $F = (V, E)$ is a forest, then

  $$|E| = |V| - c(F),$$

  where $c(F)$ is the number of connected components of $F$.

Spanning Subgraphs, Induced Subgraphs:

- A **spanning subgraph** $H$ of a graph $G$ is subgraph such that $V(H) = V(G)$.

- Let $X$ be a vertex subset of a graph $G$. An **induced subgraph** of $G$ by $X$ is a graph $G[X]$, whose vertex set is $X$ and whose edge set consists of the edges of $G$ having endpoints in $X$.
- Let $S$ be an edge subset of a graph $G$. An **induced subgraph** of $G$ by $S$ is a graph $G[S]$ whose edge set is $S$ and whose vertex set consists of the endpoints of edges in $S$. The **induced spanning subgraph** of $G$ by $S$ is the subgraph $(V, S)$.

Decomposition, Coverings:

- A **decomposition** of a graph $G$ is a family of edge-disjoint subgraphs of $G$ such that
$$E(G) = \bigcup_{H \in \mathcal{F}} E(H).$$
- A **covering** or **cover** of a graph $G$ is a family $\mathcal{F}$ of not necessarily edge-disjoint subgraphs of $G$ such that
$$E(G) = \bigcup_{H \in \mathcal{F}} E(H).$$
- A covering $\mathcal{F}$ of a graph $G$ is referred to a **path (cycle) covering** if all members of $\mathcal{F}$ are paths (cycles) of $G$.
- A covering of a graph $G$ is **uniform** if each edge of $G$ is covered the same number of times by the members of $\mathcal{F}$. When this number is $k$, the covering is called a $k$**-cover**. A 2-cover is usually called a **double cover**.

**Theorem 2.3.** *A graph admits a cycle decomposition if and only if it is an even graph.*

*Proof.* The necessity is trivial, for every cycle is 2-regular and the degree of each vertex in the graph is a sum of 2's. The sufficiency is as follows.

Let $G$ be an even graph. If $G$ contains some edges, then $G$ contains a cycle $C_1$ by Theorem 2.1. Remove the edge of $C_1$ from $G$ to obtain a graph $G_1$, which is still even. Then by Theorem 2.1 again there is a cycle $C_2$ in $G_1$. Remove the edges of $C_2$ from $G_1$ to obtain a graph $G_2$, which is still even. Continue this procedure, we obtain a family of edge-disjoint cycles $C_1, C_2, \ldots, C_k$ whose edge union is $E(G)$; the family forms a cycle decomposition of $G$. □

**Theorem 2.4.** *Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a family of complete bipartite graphs. If $\mathcal{F}$ is a decomposition of $K_n$, then $k \geq n - 1$.*

*Proof.* It is trivially true for $n = 1, 2$ (for $n = 1$, there is no edges so the family is empty; for $n = 2$, at least one bipartite graph is required). Suppose $n \geq 3$ is the smallest positive integer such that the statement is not true, i.e., $K_n = (V, E)$ can be partitioned into complete bipartite graphs $F_1, \ldots, F_k$ with $k < n-1$, where $F_i = [X_i, Y_i]$. Note that $E(K_n) = \bigsqcup [X_i, Y_i]$. Consider the system of linear equations:
$$\sum_{v \in V} x_v = 0, \quad \sum_{v \in X_i} x_v = 0, \quad i = 1, \ldots, k.$$

There are $n$ variables and $k+1$ equations with $k+1 < n$. The system has a nonzero solution $x_v = c_v$, $v \in V$ (not all zero). Since $E(K_n) = \bigsqcup [X_i, Y_i]$, we have

$$\sum_{uv \in E(K_n)} c_u c_v = \sum_{i=1}^{k} \Big( \sum_{u \in X_i} c_u \Big) \Big( \sum_{v \in Y_i} c_v \Big).$$

Thus

$$\begin{aligned} 0 = \Big( \sum_{v \in V} c_v \Big)^2 &= \sum_{v \in V} c_v^2 + 2 \sum_{uv \in E(K_n)} c_u c_v \\ &= \sum_{v \in V} c_v^2 + 2 \sum_{k=1}^{k} \Big( \sum_{u \in X_i} c_u \Big) \Big( \sum_{v \in Y_i} c_v \Big) \\ &= \sum_{v \in V} c_v^2 > 0, \end{aligned}$$

which is a contradiction. □

Cuts, Bonds, Even Graphs:

- Let $X$ and $Y$ be vertex subsets of a graph $G$ or digraph $D$. We introduce the edge subset and the arc subset of the form

  $$\begin{aligned} [X, Y] :&= \text{set of edges with one vertex in } X \text{ and the other vertex in } Y, \\ (X, Y) :&= \text{set of arcs having the head in } X \text{ and the tail in } Y. \end{aligned}$$

  We view each cut $[X, X^c]$ of $G$ as a spanning bipartite subgraph of $G$.

- An **edge cut** or just a **cut** of a graph $G$ is a nonempty edge subset of the form $[X, X^c]$, where $X$ is a vertex subset and $X^c$ is its complement in $V(G)$. We also write

  $$\delta X = \delta_G X := [X, X^c].$$

  If $\delta X$ is a cut, then $X, X^c$ must be proper subsets of $V(G)$.

- For each vertex subset $X$ of a graph $G$,

  $$\#[X, X^c] + 2\#[X, X] = \sum_{v \in X} d_G(v).$$

- A **bond** of a graph $G$ is a minimal cut, i.e., an edge cut none of whose proper subset is an edge cut.

- Deleting the edges of a cut increases the number of connected components. Deleting the edges of a bond increases exactly by one the number of connected components.

- An **even graph** is a graph whose every vertex has even degree.

- Every even graph can be decomposed into a family of edge-disjoint cycles. This why even graphs are also called **algebraic cycles**.

**Theorem 2.5.** *A graph $G$ is even if and only if every cut of $G$ has even number of edges.*

*Proof.* Let $G$ be an even graph. For each proper subset $X \subset V(G)$, it is clear that $[X, X^c]$ contains even number of edges:

$$\#[X, X^c] = -2\#[X, X] + \sum_{v \in X} d_G(v).$$

Conversely, for each $v \in V(G)$, it is clear that $d_G(v) = \#[v, V \smallsetminus v] + 2\#[v, v]$ is even. $\square$

**Proposition 2.6** (Bond Charcaterization)**.** *Let $B$ be an edge subset of a connected graph $G$. Then $B$ is a bond if, and only if, there exist disjoint connected vertex subsets $X, Y$ such that $B = [X, Y] = [X, X^c]$.*

*Proof.* "$\Rightarrow$" Let $B$ be a minimal cut. Being a cut, there exists a proper vertex subset $X'$ such that $B = [X', X'^c]$. Let $X'$ be decomposed into connected vertex subsets $X_i$. Then each $[X_i, X'^c]$ is a cut and $B = [X', X'^c] = \bigsqcup [X_i, X'^c]$. Since $B$ is minimal, only one of $[X_i, X'^c]$ is nonempty, say $[X_1, X'^c]$. Set $X = X_1$, we have $B = [X, X'^c]$. Let $X'^c$ be decomposed into connected vertex subsets $Y_j$. Likewise, each $[X, Y_j]$ is a cut, and $B = \bigsqcup [X, Y_j]$. Since $B$ is minimal, only one of $[X, Y_j]$ is nonempty, say $[X, Y_1]$. Set $Y = Y_1$, we have $B = [X, Y] = [X, X^c]$.

"$\Leftarrow$" Let $B = [X, Y] = [X, X^c]$ be a cut, where $X, Y$ are connected vertex subsets. Note that $G[X \cup Y]$ is connected. Suppose $B$ is not minimal, i.e., there exists a proper subset $B' \subsetneq B$ such that $B'$ is a cut. Then $c(G \smallsetminus B') > c(G)$ as $B'$ is a cut. However, the edges of $B \smallsetminus B'$ are between $X$ and $Y$, and both $X, Y$ are connected vertex subsets. So $c(G \smallsetminus B') = c(G)$, contradictory to $c(G \smallsetminus B') > c(G)$. $\square$

**Proposition 2.7** (Bond Decomposition of Cut)**.** *Each cut of a graph $G$ is an edge-disjoint union of bonds of $G$.*

*Proof.* Given a cut $[X, X^c]$ of $G$. Let $X$ be decomposed into disjoint connected vertex subsets $X_i$. Then $[X, X^c]$ is decomposed into edge-disjoint (possibly empty) cuts $[X_i, X^c]$. Let $X^c$ be decomposed into disjoint connected vertex subsets $Y_j$. Each nonempty $[X_i, X^c]$ is decomposed into edge-disjoint (possibly empty) cuts $[X_i, Y_j] = [Y_j, X_i]$. Since $Y_j, X_i$ are connected vertex subsets, each nonempty $[Y_j, X_i] = [Y_j, Y_j^c]$ is a bond by Proposition 2.6. $\square$

**Definition 2.8.** The **symmetric difference** of two spanning subgraphs $G_i = (V, E_i)$ ($i = 1, 2$) of a graph $G = (V, E)$ is a spanning subgraph $G_1 \Delta G_2$ of $G$, whose edge set is

$$E_1 \Delta E_2 := E_1 \cup E_2 - E_1 \cap E_2.$$

We view each edge subset $E'$ of $G$ as a spanning subgraph with the edge set $E'$.

The class $\mathcal{P}(G)$ of all spanning subgraphs of $G$ forms an abelian group under the symmetric difference $\Delta$. We sometimes write

$$G_1 + G_2 := G_1 \Delta G_2 = (V, E_1 \Delta E_2).$$

The zero (or identity) element of $\mathcal{P}(G)$ is the spanning subgraph $(V, \varnothing)$ with the empty edge set. The negative (or inverse) of a subgraph $(V, E')$ is $(V, E')$ itself.

**Proposition 2.9.** *The symmetric difference of two cuts is a cut or the empty set of edges. For vertex subsets $X, Y$ of a graph $G$,*

$$[X, X^c]\Delta[Y, Y^c] = [X\Delta Y, (X\Delta Y)^c].$$

*Proof.* Note that $V(G)$ is partitioned into four disjoint parts $X \cap Y$, $X \cap Y^c$, $X^c \cap Y$ and $X^c \cap Y^c$. The identity follows clearly from Figure 1. The identity can be also verified logically

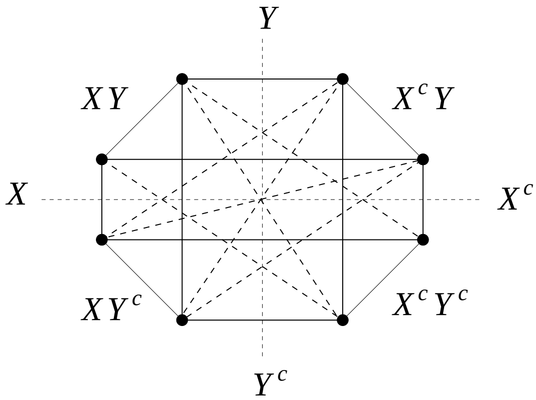


Figure 1: Symmetric difference of two cuts, where $XY = X \cap Y$.

as follows:

$$\begin{aligned}[X, X^c] &= [X \cap Y, X^c] \sqcup [X \cap Y^c, X^c] \\ &= [X \cap Y, X^c \cap Y] \sqcup [X \cap Y, X^c \cap Y^c] \sqcup \\ &\quad [X \cap Y^c, X^c \cap Y] \sqcup [X \cap Y^c, X^c \cap Y^c];\end{aligned}$$

$$\begin{aligned}[Y, Y^c] &= [X \cap Y, Y^c] \sqcup [X^c \cap Y, Y^c] \\ &= [X \cap Y, X \cap Y^c] \sqcup [X \cap Y, X^c \cap Y^c] \sqcup \\ &\quad [X^c \cap Y, X \cap Y^c] \sqcup [X^c \cap Y, X^c \cap Y^c].\end{aligned}$$

Note that $[X\cap Y^c, X^c\cap Y] = [X^c\cap Y, X\cap Y^c]$, which is canceled in $[X, X^c]\Delta[Y, Y^c]$. The cuts $[X\cap Y, X^c\cap Y]$, $[X\cap Y^c, X^c\cap Y^c]$ are disjoint from both $[X\cap Y, X\cap Y^c]$ and $[X^c\cap Y, X^c\cap Y^c]$. Thus $[X, X^c]\Delta[Y, Y^c]$ is the disjoint union

$$\begin{aligned}& [X \cap Y, X \cap Y^c] \cup [X \cap Y, X^c \cap Y] \cup [X \cap Y^c, X^c \cap Y^c] \cup [X^c \cap Y, X^c \cap Y^c] \\ = \; & [X \cap Y, (X \cap Y^c) \cup (X^c \cap Y)] \sqcup [(X \cap Y^c) \cup (X^c \cap Y), X^c \cap Y^c] \\ = \; & [(X \cap Y^c) \cup (X^c \cap Y), (X \cap Y) \cup (X^c \cap Y^c)].\end{aligned}$$

Since $X\Delta Y = (X \cap Y^c) \cup (X^c \cap Y)$ and

$$\begin{aligned}(X\Delta Y)^c &= (X^c \cup Y) \cap (X \cup Y^c) \\ &= [(X^c \cup Y) \cap X] \cup [(X^c \cup Y) \cap Y^c] \\ &= (X \cap Y) \cup (X^c \cap Y^c),\end{aligned}$$

we see that $[X, X^c]\Delta[Y, Y^c] = [X\Delta Y, (X\Delta Y)^c]$. □

**Proposition 2.10.** *For spanning subgraphs $G_i = (V, E_i)$ of a graph $G = (V, E)$, $i = 1, 2$, and each proper vertex subset $X \subset V$, we have*

$$\delta_{G_1 \Delta G_2} X = \delta_{G_1} X \Delta \partial_{G_2} X, \quad \text{i.e.,} \quad [X, X^c]_{G_1 \Delta G_2} = [X, X^c]_{G_1} \Delta [X, X^c]_{G_2}.$$

*Proof.* It follows that $\delta_{G_1 \Delta G_2} X = [X, X^c] \cap (E_1 \Delta E_2) = [X, X^c] \cap (E_1 \cup E_2 - E_1 \cap E_2)$ and

$$\begin{aligned} \delta_{G_1} X \Delta \partial_{G_2} X &= ([X, X^c] \cap E_1) \Delta ([X, X^c] \cap E_2) \\ &= ([X, X^c] \cap E_1) \cup ([X, X^c] \cap E_2) - [X, X^c] \cap E_1 \cap E_2 \\ &= [X, X^c] \cap (E_1 \cup E_2) - [X, X^c] \cap E_1 \cap E_2. \end{aligned}$$

□

**Theorem 2.11.** *The symmetric difference of two spanning even subgraphs of a graph $G$ is an even spanning subgraph of $G$.*

*Proof.* Let $G_i = (V, E_i)$ be spanning even subgraphs of $G = (V, E)$, $i = 1, 2$. Let $X$ be a proper vertex subset. By Proposition 2.10,

$$\#\delta_{G_1 \Delta G_2} X = \#(\delta_{G_1} X \Delta \delta_{G_2} X) = \#\delta_{G_1} X + \#\delta_{G_2} X - 2\#(\delta_{G_1} X \cap \delta_{G_2} X),$$

which is an even number. Then by Theorem 2.5, $G_1 \Delta G_2$ is a spanning even subgraph. □

**Corollary 2.12.** *The class $\mathcal{C}(G)$ of spanning even subgraphs of a graph $G$ is closed under the symmetric difference. So $\mathcal{C}(G)$ is a subgroup of $\mathcal{P}(G)$, called the* **cycle group** *of $G$.*

**Corollary 2.13.** *The class $\mathcal{B}(G)$ of cuts of a graph $G$ is closed under symmetric difference. So $\mathcal{B}(G)$ is a subgroup of $\mathcal{P}(G)$, called the* **bond group** *(or* **cut group***) of $G$.*

Vector Spaces Associated to Graphs:

- Let $S$ be a nonempty set and $\mathbb{F}$ a field. Let $\mathbb{F}^S$ denote the set of all functions from $S$ to $\mathbb{F}$. Then $\mathbb{F}^S$ becomes a vector space over $\mathbb{F}$ under the addition and the scalar multiplication of functions: For functions $f, g \in \mathbb{F}^S$ and a scalar $a \in \mathbb{F}$,

  $$(f + g)(s) = f(s) + g(s), \quad (af)(s) = af(s), \quad s \in S.$$

  If $|S| = n$ and $S = \{s_1, \ldots, s_n\}$, then $\mathbb{F}^S \cong \mathbb{F}^n$ and the isomorphism is given by $f \mapsto (f(s_1), \ldots, f(s_n))$. If you don't like an arbitrary field $\mathbb{F}$, just assume that $\mathbb{F}$ is the field $\mathbb{R}$ of real numbers.

- Let $S$ be a nonempty set and $\mathbb{F}_2 = \{0, 1\} \cong \mathbb{Z}/2\mathbb{Z}$, the field of two elements, where $1 + 1 = 0$ (so $-1 = 1$). The power set $\mathcal{P}(S)$ is an abelian group under the symmetric difference, which is written as plus '+' now as follows: For subsets $A, B \subseteq S$, define the addition

  $$A + B := A \cup B - A \cap B.$$

  The zero element of $\mathcal{P}(S)$ is the empty set $\varnothing$; the negative element of $A \in \mathcal{P}(S)$ is $A$ itself. Moreover, for each $a \in \mathbb{F}_2$, we define the scalar multiplication

  $$aA = \begin{cases} A & \text{if } a = 1, \\ \varnothing & \text{if } a = 0. \end{cases}$$

  Then $\mathcal{P}(S)$ becomes a vector space over $\mathbb{F}_2$ under the addition and scalar multiplication.

- There is a bijection $\varphi : \mathcal{P}(S) \to \mathbb{F}_2^S$, defined by

$$\varphi(A) = 1_A, \quad \text{where} \quad 1_A(s) = \begin{cases} 1 & \text{if } s \in A, \\ 0 & \text{if } s \in S - A. \end{cases}$$

  The bijection $\varphi$ preserves the addition and scalar multiplication:

$$\begin{aligned} \varphi(A+B) &= 1_{A\cup B - A\cap B} = 1_{A-B} + 1_{B-A} \; (1_{A\cup B} - 1_{A\cap B}) \\ &= 1_A + 1_{A\cap B} + 1_B + 1_{A\cap B} \; (1_A + 1_B - 1_{A\cap B} - 1_{A\cap B}) \\ &= 1_A + 1_B = \varphi(A) + \varphi(B); \end{aligned}$$

$$\varphi(aA) = \begin{cases} 1_A & \text{if } a = 1 \\ 1_\varnothing = 0 & \text{if } a = 0 \end{cases} = \begin{cases} a\varphi(A) & \text{if } a = 1 \\ a\varphi(A) & \text{if } a = 0 \end{cases} = a\varphi(A).$$

  So $\varphi$ is a vector space isomorphism from $\mathcal{P}(S)$ to $\mathbb{F}_2$.

- A basis of $\mathcal{P}(S)$ is the set of singletons $\{s\} : s \in S$. A basis of $\mathbb{F}_2^S$ is the set of indicator functions of singletons $1_{\{s\}} : s \in S$. If $|S| = n$ and $S = \{s_1, \ldots, s_n\}$, the vector space $\mathbb{F}_2^S$ is isomorphic to $\mathbb{F}_2^n$ with $1_{\{s_i\}} \leftrightsquigarrow (0, \ldots, 1, \ldots, 0)$ (all coordinates are zero except 1 for the $i$th coordinate).

- For a graph $G = (V, E)$, the vector space $\mathbb{F}_2^V$ is called the **vertex space** of $G$, and $\mathbb{F}_2^E$ is called the **edge space** of $G$, whose dimension is $|E|$.

- The class of edge sets of even subgraphs of a graph $G$ is a vector subspace of its edge space, called the **cycle space** of $G$.

- The class of edge sets of cuts of a graph $G$ is a vector subspace of its edge space, called the **bond space** of $G$.

**Theorem 2.14.** *Let $T$ be a spanning tree of a connected graph $G$. Let $C(T, e)$ denote the unique cycle (called the* **fundamental cycle** *of $G$ with respect to $T$) contained in $T \cup e$ for each edge $e$ of the co-tree $T^c := E(G) \smallsetminus E(T)$. Then the cycle group $\mathcal{C}(G)$ is generated by the cycles $C(T, e)$ with $e \in T^c$. Moreover, if $\mathcal{C}(G)$ is viewed a vector space over $\mathbb{F}_2$, then $\{C(T, e) : e \in T^c\}$ is a basis of $\mathcal{C}(G)$.*

*Proof.* The fundamental cycles of $G$ with respect to $T$ are linearly independent: Assume

$$\sum_{e\in T^c} x_e C(T, e) = 0 \, (= \varnothing) \quad \text{with} \quad x_e \in \mathbb{F}_2.$$

For each $e \in T^c$, we have $e \in C(T, e)$ and $C(T, e) \smallsetminus e \subset T$; we see that $e$ cannot be canceled in the LHS if $x_e = 1$. So $x_e = 0$ for all $e \in T^c$. This means that $C(T, e), e \in T^c$ are linearly independent over $\mathbb{F}_2$.

Let $C$ be an even spanning subgraph of $G$. Then $C$ is an addition of edge-disjoint cycles. Consider the following even spanning subgraph

$$C' := C + \sum_{e\in C\cap T^c} C(T, e).$$

We claim that $C' = 0\,(=\varnothing)$. Note that $C'$ is contained in $T$ (the edges in $T^c$ are canceled by definition). Suppose $C'$ is nonempty. Then $C'$ is an edge disjoint union of cycles $C_i$ contained in $T$, which is a contradiction. Thus $C' = 0$ and $C = \sum_{e\in C\cap T^c} C(T,e)$. We have shown that $\{C(T,e) : e \in T^c\}$ is a basis of $\mathcal{C}(G)$. □

**Theorem 2.15.** *Let $T$ be a spanning tree of a connected graph with $G$. Let $B(T^c, e)$ denote the unique bond (called a* **fundamental bond** *of $G$ with respect to $T$) contained in $T^c \cup e$ for each edge $e \in T$. Then the bond group $\mathcal{B}(G)$ is generated by the bonds $B(T^c, e)$, where $e \in T$. Moreover, if $\mathcal{B}(G)$ is viewed as a vector space over $\mathbb{F}_2$, then $\{B(T^c, e) : e \in T\}$ is a basis of $\mathcal{B}(G)$.*

*Proof.* The fundamental bonds of $G$ with respect to $T$ are linear independent: Assume

$$\sum_{e\in T} x_e B(T^c, e) = 0\,(=\varnothing) \quad \text{with} \quad x_e \in \mathbb{F}_2.$$

For each $e \in T$, we have $e \in B(T^c, e)$ and $B(T^c, e) \smallsetminus e \subset T^c$; we see that $e$ cannot be canceled in the LHS if $x_e = 1$. So $x_e = 0$ for all $e \in T$. This means that $B(T^c, e), e \in T$ are linearly independent over $\mathbb{F}_2$.

Let $U$ be a cut of $G$. Consider the following additions of cuts

$$U' := U + \sum_{e\in U\cap T} B(T^c, e).$$

We claim that $U' = 0\,(=\varnothing)$. Note that $U'$ is contained in $T^c$ (the edges in $T$ are canceled in the RHS). Suppose $U' \neq 0$, i.e., $U'$ is a nonempty cut contained $T^c$, which is a contradiction, since each cut contains at least one edge of $T$. Thus $G' = 0$ and $U = \sum_{e\in U\cap T} B(T^c, e)$. We have shown that $\{B(T^c, e) : e \in T\}$ is a basis of $\mathcal{B}(G)$. □

**Corollary 2.16.** *For each even subgraph $H$ and each cut $U$ of a graph $G$, we have*

$$|H \cap U| = |E(H) \cap E(U)| = \text{even}.$$

*Proof.* Let $H$ be decomposed into edge-disjoint cycles $C$. Then $|H \cap U| = \sum_C |C \cap U|$. It suffices to show that $|C \cap U|$ is even. Let $U = [X, X^c]$ and $C$ be arranged as a closed path $W = v_0 e_1 v_1 \dots e_n v_n$ with $v_n = v_0$. Then $W$ goes through $U$ between $X$ and $X^c$ even number of times. So $|C \cap U|$ is even. □

**Question**: Is the group $\mathcal{P}(G)$ a direct sum of the cycle group $\mathcal{C}(G)$ and the bond group $\mathcal{B}(G)$, i.e., $\mathcal{P}(E) = \mathcal{C}(P) \oplus \mathcal{B}(G)$?

# 3 Flow space and tension space

Let $G = (V, E)$ be a graph and let $A$ an abelian group.

- Recall that **orientation** of an edge $e = uv$ with endpoints $u, v$ is one of the two arcs (directed edges) $\overrightarrow{uv}$ and $\overleftarrow{uv}$ $(= \overrightarrow{vu})$. If an orientation of $e$ is denoted by $\vec{e}$, say, $\vec{e} = \overrightarrow{uv}$, then the other (opposite) orientation of $e$ is denoted by $-\vec{e}$, i.e., $-\vec{e} = \overleftarrow{uv}$. Let $\vec{E}(G)$ denote the set of all oriented edges from the edge set $E(G)$. Then

$$|\vec{E}(G)| = 2|E(G)|.$$

- Recall that an **orientation** $\omega$ of $G$ is an assignment that each edge of $G$ is given one of its two orientations. We may view $\omega$ as an arc subset $\omega \subset \vec{E}(G)$ such that

$$\omega \cap (-\omega) = \varnothing, \quad \omega \cup (-\omega) = \vec{E}(G).$$

  A graph $G$ with an orientation $\omega$ is called a **directed graph**, denoted $(G, \omega)$.

- A **flow** valued in $A$ or $A$**-flow** of a digraph $(G, \omega)$ is a function $f \in A^E$ such that for each vertex $v \in V$,

$$\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0, \tag{3.1}$$

  where $E^+(v)$ is the set of arcs of $\omega$ pointing to $v$, and $E^-(v)$ the set of arcs of $\omega$ pointing away from $v$. The $A$**-flow group** of $G$ with respect to $\omega$, denoted $F(G, \omega; A)$, is the group of all $A$-flows of $(G, \omega)$.

- A **sink** (**source**) of a digraph $(G, \omega)$ is a vertex $v$ such that all arcs of $\omega$ with endpoint $v$ point to (away from) $v$.

- A **circuit** of a graph $G$ is a minimal spanning even subgraph. Circuit is just another name for cycle in graphs. A **direction** of a circuit $C$ is an orientation $\omega_C$ on $C$ such that the directed subgraph $(C, \omega)$ has neither a sink nor a source. Each circuit has exactly two directions.

- Given an orientation $\omega$ of $G$. For each directed circuit $(C, \omega_C)$ of $G$, the function $[\omega, \omega_C] : E \to \mathbb{Z}$, defined by

$$[\omega, \omega_C](e) = \begin{cases} 1 & \text{if } e \in C \text{ and } \omega, \omega_C \text{ have the same orientation on } e, \\ -1 & \text{if } e \in C \text{ and } \omega, \omega_C \text{ have opposite orientations on } e, \\ 0 & \text{if } e \notin C, \end{cases}$$

  is an integer-valued flow of the digraph $(G, \omega)$, called a **circuit flow** generated by $C$.

- A **tension** valued in $A$ or $A$**-tension** of a digraph $(G, \omega)$ is a function $g \in A^E$ such that for each directed circuit $(C, \omega_C)$,

$$\langle [\omega, \omega_C], g \rangle := \sum_{e \in C} [\omega, \omega_C](e) g(e) = 0.$$

  The $A$**-tension group** of $G$ with respect to $\omega$, denoted $T(G, \omega; A)$, is the group of all $A$-tensions of $(G, \omega)$.

- Let $U = [X, X^c]$ be a cut of $G$. A **direction** of $U$ is an orientation $\omega_U$ on $B$ such that the arcs of $\omega_U$ have heads all in $X$ or all in $X^c$. Each cut has exactly two directions.

- Given an orientation $\omega$ of $G$. For each directed cut $(U, \omega_U)$ of $G$, the function $[\omega, \omega_U] : E \to \mathbb{Z}$, defined by

$$[\omega, \omega_U](e) = \begin{cases} 1 & \text{if } e \in U \text{ and } \omega, \omega_U \text{ have the same orientation on } e, \\ -1 & \text{if } e \in U \text{ and } \omega, \omega_U \text{ have opposite orientations on } e, \\ 0 & \text{if } e \notin U, \end{cases}$$

  is an integral tension of the digraph $(G, \omega)$, called a **cut tension** generated by $U$. The cut tension is called a **bond tension** if the cut is a bond.

- A function $f \in A^E$ is a flow of $(G,\omega)$ if, and only if, for each directed bond $(B,\omega_B)$,
$$\langle [\omega,\omega_B], f\rangle = \sum_{e\in\omega_B} [\omega,\omega_B](e) f(e) = 0.$$
In particular, a digraph $(G,\omega)$ is **even** (i.e., in-degree equals out-degree at every vertex) if, and only if, for each directed bond $(B,\omega_B)$,
$$\sum_{e\in\omega_B} [\omega,\omega_B](e) = 0.$$
- A **local direction** of a cut $U = [X, X^c]$ of a graph $G$ is an orientation $\omega_U$ such that $(U,\omega_U)$ can be decomposed into an edge-disjoint directed bonds. Local directions of a cut are not necessarily unique up to sign.
- Let $U$ be a nonempty edge subset of $G$, and let $\omega_U$ be an orientation on $U$. If the function $[\omega,\omega_U] : E \to \mathbb{Z}$, defined by
$$[\omega,\omega_U](e) = \begin{cases} 1 & \text{if } e\in U \text{ and } \omega,\omega_U \text{ have the same orientation on } e, \\ -1 & \text{if } e \in U \text{ and } \omega,\omega_U \text{ have opposite orientations on } e, \\ 0 & \text{if } e \notin U, \end{cases}$$
is an integral tension of $G,\omega)$, then $U$ is a cut, i.e., $U = [X, X^c]$ for a vertex subset $X$ of $G$, and $\omega_U$ is a local direction of $U$.

The flow groups $F(G,\omega;\mathbb{Z})$, $F(G,\omega;\mathbb{R})$ are called **flow lattice**, **flow space** of $(G,\omega)$ respectively. Likewise, the tension groups $T(G,\omega;\mathbb{Z})$, $T(G,\omega;\mathbb{R})$ are called **tension lattice**, **tension space** of $(G,\omega)$ respectively. Sometimes we abbreviate $F(G,\omega;\mathbb{R})$ to $F(G,\omega)$, and $T(G,\omega;\mathbb{R})$ to $T(G,\omega)$

**Theorem 3.1.** *Let $T$ be a spanning tree of a connected graph $G$ with an orientation $\omega$.*

(a) *For each $e \in T^c$, let $C(T,e)$ denote the unique circuit contained in $T \cup e$ with a direction $\omega_e$ such that $\omega,\omega_e$ have the same orientation on $e$. Then $\{[\omega,\omega_e] : e \in T^c\}$ is a basis of $F(G,\omega;\mathbb{Z})$, and for each $f \in F(G,\omega;\mathbb{Z})$,*
$$f = \sum_{e\in T^c} f(e)[\omega,\omega_e].$$

(b) *For each $e \in T$, let $B(T^c,e)$ denote the unique bond contained in $T^c \cup e$ with a direction $\omega_e$ such that $\omega,\omega_e$ have the same orientation on $e$. Then $\{[\omega,\omega_e] : e \in T\}$ is basis of $T(G,\omega;\mathbb{Z})$, and for each $g \in T(G,\omega;\mathbb{Z})$,*
$$g = \sum_{e\in T} g(e)[\omega,\omega_e].$$

(c) *Vector spaces $F(G,\omega;\mathbb{R})$ and $T(G,\omega;\mathbb{R})$ are orthogonal complement each other in $\mathbb{R}^E$; in particular,*
$$T(G,\omega;\mathbb{R}) \oplus F(G,\omega;\mathbb{R}) = \mathbb{R}^E.$$
*However, $T(G,\omega;\mathbb{Z}) \oplus F(G,\omega;\mathbb{Z}) \subseteq \mathbb{Z}^E$ and*
$$|\mathbb{Z}^E/(T(G,\omega;\mathbb{Z}) \oplus F(G,\omega;\mathbb{Z}))| = \#\{\text{spanning forests of } G\}.$$

*Proof.* (a) Consider the equation $\sum_{e\in T^c} x_e[\omega,\omega_e] = 0$. Note that $[\omega,\omega_e]$ is supported on $T\cup e$. For each $e_0 \in T^c$, we have

$$0 = \Big(\sum_{e\in T^c} x_e[\omega,\omega_e]\Big)(e_0) = \sum_{e\in T^c} x_e[\omega,\omega_e](e_0) = x_{e_0}.$$

We see that $[\omega,\omega_e] : e \in T^c$ are linearly independent.

Let $f \in F(G,\omega;R)$. Consider the flow

$$f' := f - \sum_{e\in T^c} f(e)[\omega,\omega_e].$$

For each $e_0 \in T^c$, note that $f'(e_0) = f(e_0) - \sum_{e\in T^c} f(e)[\omega,\omega_e](e_0) = f(e_0) - f(e_0) = 0$. Then $f'|_{T^c} = 0$. Note that $T$ is a tree and $f'$ is a flow of $(G,\omega)$. For each edge $e' \in T$ at a leaf $v$ of $T$, the net flow of $f'$ at $v$ is $\pm f'(e')$, which must be zero by definition of flow. Then $f$ is zero on edges of $T \smallsetminus e'$ at its leaves. Continue this procedure, we see that $f'|_T = 0$. We have show that $f = \sum_{e\in T^c} f(e)[\omega,\omega_e]$. Thus $\{[\omega,\omega_e] : e \in T^c\}$ is basis of $F(G,\omega;\mathbb{Z})$.

(b) Likewise, consider the equation $\sum_{e\in T} x_e[\omega,\omega_e] = 0$. Note that $[\omega,\omega_e]$ is supported on $T^c \cup e$. For each $e_0 \in T$, we have

$$0 = \Big(\sum_{e\in T} x_e[\omega,\omega_e]\Big)(e_0) = \sum_{e\in T^c} x_e[\omega,\omega_e](e_0) = x_{e_0}.$$

We see that $[\omega,\omega_e] : e \in T$ are linearly independent.

Let $g \in T(G,\omega;R)$. Consider the tension

$$g' := g - \sum_{e\in T} g(e)[\omega,\omega_e].$$

For each $e_0 \in T$, note that $g'(e_0) = f(e_0) - \sum_{e\in T} g(e)[\omega,\omega_e](e_0) = g(e_0) - g(e_0) = 0$. Then $g'|_T = 0$. For each $e' \in T^c$, consider the direction $\omega_{e'}$ of $C(T,e')$ in the former part of the theorem. By definition of tension, we have

$$g(e') = \langle [\omega,\omega_{e'}], g\rangle = 0.$$

We see that $g'|_{T^c} = 0$. We have shown that $g = \sum_{e\in T} g(e)[\omega,\omega_e]$. Thus $\{[\omega,\omega_e] : e \in T\}$ is basis of $T(G,\omega;\mathbb{Z})$.

(c) Since tensions are orthogonal to circuit flows, and flows are spanned by circuit flows, it follows that all tensions are orthogonal to flows by (a). Since $\dim F(G,\omega;\mathbb{R}) = |T^c|$ and $\dim T(G,\omega;\mathbb{R}) = |T|$, we see that

$$|E| = \dim F(G,\omega;\mathbb{R}) + \dim T(G,\omega;\mathbb{R}).$$

It follows that $F(G,\omega;\mathbb{R})$ and $T(G,\omega;\mathbb{R})$ are orthogonal complement each other in $\mathbb{R}^E$. Thus $\mathbb{R}^E$ is a direct sum of $F(G,\omega;\mathbb{R})$ and $T(G,\omega;\mathbb{R})$. $\square$

The **incidence matrix** of a digraph $(G,\omega)$ is the $V\times E$ matrix

$$\mathbf{M} = \mathbf{M}(G,\omega) = [m_{ve}], \quad m_{ve} = \begin{cases} 1 & \text{if } \vec{e} = \overrightarrow{uv} \in \omega,\ u \neq v, \\ -1 & \text{if } \vec{e} = \overrightarrow{vw} \in \omega,\ v \neq w, \\ 0 & \text{otherwise.} \end{cases}$$

For each $v \in V(G)$, we have the flow equation:

$$\sum_{e\in E^+(v)} x_e - \sum_{e\in E^-(v)} x_e = 0 \quad \Leftrightarrow \quad \sum_{e\in E} m_{ve}x_e = 0.$$

Let $\boldsymbol{x} = (x_e : e \in E) \in \mathbb{R}^E$. Then the flow space $F(G, \omega; \mathbb{R})$ is the solution set of the matrix equation

$$\mathbf{M}\boldsymbol{x} = \mathbf{0}.$$

Since the the row space of $\mathbf{M}$ is the orthogonal complement of $\ker \mathbf{M}$, we have

$$F(G, \omega; \mathbb{R}) = \ker \mathbf{M}, \quad T(G, \omega; \mathbb{R}) = \mathrm{Row}\mathbf{M}.$$

# 4 Chain group

Given an abelian $A$. The flow groups $F(G, \omega; A)$ depend on the chosen orientations $\omega$, though all of them are isomorphic. Likewise, the tension groups $T(G, \omega; A)$ also depend on the chosen orientations $\omega$, and all of them are isomorphic. It is desired to obtain a unique flow group of $G$ in some intrinsic way, so that it is independent of the chosen orientations. For this purpose we need to introduce so-called chains groups.

Boundary Operator and Co-boundary Operator:

- Each vertex of a graph $G$ is referred to a 0-**cell**, and each edge of $G$ is referred to a 1-**cell**. So a graph $G$ can be viewed as a 1-dimensional **cell complex**.

- A 0-**chain** of $G$ valued in $A$ is a function $p : V \to A$, also called a **potential function**. A 1-**chain** or just **chain** of $G$ valued in $A$ is a function $f : \vec{E}(G) \to A$ satisfying

  $$f(-e) = -f(e) \quad \forall\, e \in \vec{E}(G).$$

  This means that the two orientations of each edge are coupled by opposite values. Let $C_i(G, A)$ denote the group of $i$-chains of $G$, $i = 0, 1$, called the $i$**th chain group** of $G$.

- The **support** of a chain $f$ is the edge set $\mathrm{supp} f = \{e \in E(G) : f(\vec{e}) \neq 0\}$. We usually write a chain $f$ as

  $$f = \sum_{e\in \mathrm{supp} f} f(\vec{e})\vec{e}.$$

  Here we do not care which oriented edge $\vec{e}$ is selected: since $f(-\vec{e}) = -f(\vec{e})$, we have

  $$\sum_{e\in \mathrm{supp} f} f(-\vec{e})(-\vec{e}) = \sum_{e\in \mathrm{supp} f} (-1)(-1)f(\vec{e})\vec{e} = \sum_{e\in \mathrm{supp} f} f(\vec{e})\vec{e}.$$

  The chain group $C_1(G, A)$ is generated by the arcs $\vec{e} \in \vec{E}(G)$. Each member of $C_1(G, A)$ is of the form $\sum_{e\in E} a_{\vec{e}}\, \vec{e}_i$ with $\vec{e} \in \vec{E}(G)$ and $a_{\vec{e}} \in A$.

- Choose an orientation $\omega$ of $G$. The chain group $C_1(G, A)$ can be identified to the group $A^E$ of functions from $E$ to $A$ as follows: $C_1(G, A) \cong A^E$, each $f : \vec{E} \to A$ is identified as a function $f : E \to A$ given by

  $$f(e) = f(\vec{e}) \quad \text{for} \quad e \in E(G), \text{ where } \vec{e} \in \omega.$$

- There is a canonical **pairing** $\langle , \rangle : C_1(G, \mathbb{Z}) \times C_1(G, A) \to A$, defined by

$$\langle f, g \rangle = \sum_{e \in E(G)} f(\vec{e}) g(\vec{e}).$$

  Again, here it does not matter which oriented edge $\vec{e}$ is selected in the sum, since

$$f(-\vec{e}) g(-\vec{e}) = (-f(\vec{e}))(-g(\vec{e})) = f(\vec{e}) g(\vec{e}).$$

- The **boundary operator** of $G$ is the group homomorphism

$$\partial : C_1(G, A) \to C_0(G, A), \quad \partial \vec{e} = v - u, \quad \vec{e} = \overrightarrow{uv},$$

  extended by linearity (homomorphism is uniquely determined by its values on generators). More specifically, for each $f \in C_1(G, A)$,

$$(\partial f)(v) = \sum_{\vec{e} = \overrightarrow{uv}} f(\vec{e}) = - \sum_{\vec{e} = \overrightarrow{vw}} f(\vec{e}) = \sum_{\vec{e} \in \omega \cap \vec{E}(v)} f(\vec{e}),$$

  where $\omega$ is an orientation of $G$ and $\vec{E}(v)$ is the set of arcs having $v$ as an endpoint.

  The **flow group** $F(G, A)$ of $G$ with coefficients in $A$ is $\ker \partial$. Each member of $F(G, A)$ is called a **flow** of $G$ valued in $A$ or $A$**-flow**.

- The **co-boundary operator** of $G$ is the group homomorphism

$$\delta : C_0(G, A) \to C_1(G, A), \quad (\delta p)(\vec{e}) = p(u) - p(v), \quad \vec{e} = \overrightarrow{uv}.$$

Circuit Chains, Cut Chains, Bond Chains:

- A **circuit chain** of a graph $G$, associated with a directed circuit $(C, \omega_C)$, is a chain

$$\mathrm{I}_{\omega_C} : \vec{E} \to \mathbb{Z}, \quad \mathrm{I}_{\omega_C}(e) = \begin{cases} 1 & \text{if } e \in \omega_C, \\ 0 & \text{if } e \notin \omega_C \cup (-\omega_C). \end{cases}$$

  Sometimes we simply write $\mathrm{I}_{\omega_C}$ as $\omega_C$, since

$$\mathrm{I}_{\omega_C} = \sum_{e \in E} \mathrm{I}_{\omega_C}(\vec{e}) \cdot \vec{e} = \sum_{\vec{e} \in \omega_C} 1 \cdot \vec{e} = \sum_{\vec{e} \in \omega_C} \vec{e}.$$

- A **direction** of a cut $U = [X, X^c]$ of a graph $G$ is an orientation $\omega_U$ on $U$ such that its oriented edges have tails in $X$ and heads in $X^c$. A cut $U$ with a direction $\omega_U$ is called a **directed cut**, denoted $(U, \omega_U)$.

- A **cut chain** of $G$, associated with a directed cut $(U, \omega_U)$, is a chain

$$\mathrm{I}_{\omega_U} : \vec{E} \to \mathbb{Z}, \quad \mathrm{I}_{\omega_U}(e) = \begin{cases} 1 & \text{if } e \in \omega_U, \\ 0 & \text{if } e \notin \omega_C \cup (-\omega_U). \end{cases}$$

  Likewise, we sometimes simply write $\mathrm{I}_{\omega_U}$ as $\omega_U$. A cut chain is called a **bond chain** if the cut is a bond.

**Definition 4.1.** A **tension** of a graph $G$ with values in an abelian group $A$ is a chain $g \in C_1(G, A)$ such that for each directed circuit $(C, \omega_C)$,

$$\sum_{e \in \omega_C} g(e) = 0, \quad \text{i.e.,} \quad \langle \mathrm{I}_{\omega_C}, g \rangle = 0.$$

The **tension group** $T(G, A)$ of $G$ with coefficients in $A$ is the group of all tensions of $G$ with values in $A$.

**Proposition 4.2.** *For each potential $p \in C_0(G, A)$, $\delta p$ is a tension of $G$.*

*Proof.* Let $C$ be a circuit with a direction $\omega_C$. We may arrange the vertices and directed edge of $(C, \omega_C)$ as a directed walk $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n$, where $v_n = v_0$ and the direction of $e_i$ is pointing from $v_{i-1}$ to $v_i$, $i = 1, 2, \ldots, n-1$. Then

$$\langle \delta p, \mathrm{I}_{\omega_C} \rangle = \sum_{i=1}^{n} \delta p(e_i) = \sum_{i=1}^{n} \big[ p(v_i) - p(v_{i-1}) \big] = \sum_{i=1}^{n} p(v_i) - \sum_{i=0}^{n-1} p(v_i) = 0.$$

□

**Theorem 4.3.** *Let $F$ be a spanning forest of a graph $G = (V, E)$. Let $C(F, e)$ denote the unique circuit contained in $F \cup e$ with a direction $\omega(F, e)$, where $e \in F^c$. Let $B(F^c, e)$ denote the unique bond contained in $F^c \cup e$ with a direction $\omega(F^c, e)$, where $e \in F$. Then*

$$F(G, A) = \bigoplus_{e \in F^c} A\, \mathrm{I}_{\omega(F,e)} \cong A^{F^c} \quad \textit{and} \quad f = \sum_{e \in F^c} f(e) \mathrm{I}_{\omega(F,e)};$$

$$T(G, A) = \bigoplus_{e \in F} A\, \mathrm{I}_{\omega(F^c,e)} \cong A^{F} \quad \textit{and} \quad g = \sum_{e \in F} g(e) \mathrm{I}_{\omega(F^c,e)};$$

$$F(G, A) + T(G, A) \subseteq C_1(G, A) \cong A^E.$$

*However, $F(G, A) + T(G, A)$ is* not *necessarily a direct sum. The vector spaces $F(G, \mathbb{R}), T(G, \mathbb{R})$ are orthogonal complement each other in $C_1(G, \mathbb{R})$, and*

$$F(G, \mathbb{Z}) \oplus T(G, \mathbb{Z}) \subseteq C_1(G, \mathbb{Z}), \quad F(G, \mathbb{R}) \oplus T(G, \mathbb{R}) = C_1(G, \mathbb{R}).$$

*Proof.* Consider the chain $f' = f - \sum_{e \in F^c} f(\vec{e}) \mathrm{I}_{\omega(F,e)}$ with $\vec{e} \in \omega(F, e)$, which is a member of $F(G, A)$. It is clear by definition that $f' = 0$ on $\vec{F^c}$. So $f'$ is only possibly nonzero on $\vec{F}$. Since $F$ is a forest, each component $T$ of $F$ is a tree. For each nontrivial tree $T$ of $F$, let $e' = uv$ be an edge at a leaf $v$ of $T$ and let $\vec{e}' = \overrightarrow{uv}$. Since $\partial f' = 0$, we have $0 = (\partial f')(v) = f'(\vec{e}')$. Continue this procedure by selecting leaves of $T \smallsetminus e$, we conclude that $f' = 0$ on $\vec{T}$. Subsequently, $f' = 0$ on $\vec{F}$. Thus $f = \sum_{e \in F^c} f(\vec{e}) \mathrm{I}_{\omega(F,e)}$, where $\vec{e} \in \omega(F, e)$.

Likewise, set $g' = g - \sum_{e \in F} g(e) \mathrm{I}_{\omega(F^c,e)}$ with $\vec{e} \in \omega(F^c, e)$, which is a member of $T(G, A)$. It is clear by definition that $g' = 0$ on $\vec{F}$. For each $e' \in F^c$, we have $g'(\vec{e}') = \langle g', \mathrm{I}_{\omega(F,e')} \rangle = 0$. Thus $g' = 0$ on $\vec{F^c}$. Therefore $g = \sum_{e \in F} g(\vec{e}) \mathrm{I}_{\omega(F^c,e)}$, where $\vec{e} \in \omega(F^c, e)$.

Let $G$ be the graph with two vertices connected by two parallel edges. Then $C_1(G, \mathbb{Z}_2) \cong \mathbb{Z}_2^2$ and $F(G, \mathbb{Z}_2) = T(G, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Of course $F(G, \mathbb{Z}_2) + T(G, \mathbb{Z}_2)$ cannot be a direct sum, and $F(G, \mathbb{Z}_2) + T(G, \mathbb{Z}_2) \neq C_1(G, \mathbb{Z}_2)$. More complicated examples can be constructed in a similar fashion. □

**Theorem 4.4.** *Let $c(G)$ denote the number of connected components of a graph $G$. Then for each abelian group $A$,*

$$C_0(G,A) \cong T(G,A) \oplus A^{c(G)}.$$

*Proof.* Given a tension $g \in T(G)$. Fix a vertex $v_0$ in a component of $G$, and let $p(v_0)$ be any member of $A$. For each $v \in V(G_0)$, let $P = v_0 e_1 v_1 \cdots e_m v_m(v)$ be a directed path from $v_0$ to $v$. Define

$$p(v) = p(v_m) = p(v_0) + \sum_{i=1}^{m} g(e_i), \quad e_i = \overrightarrow{v_{i-1}v_i}.$$

Let $P' = v_0 e_1' v_1' \ldots e_n' v_n(v)$ be a another directed path from $v_0$ to $v$. Let $P'^{-1}$ denote the reversal of $P'$, having $e_n'$ pointing away from $v$. Then $PP'^{-1}$ is a directed closed path. Thus

$$\langle g, \mathrm{I}_{PP'^{-1}} \rangle = \sum_{i=1}^{m} g(e_i) - \sum_{j=1}^{n} g(e_j') = 0.$$

This means that $p(v)$ is well-defined. For a directed edge $e := e_m = \overrightarrow{v_{m-1}v_m}$. We have

$$\begin{aligned} (\delta p)(e) &= p(v_m) - p(v_{m-1}) \\ &= \left(p(v_0) + \sum_{i=1}^{m} g(e_i)\right) - \left(p(v_0) + \sum_{i=1}^{m-1} g(e_i)\right) \\ &= g(e_m) = g(e). \end{aligned}$$

□

**Proposition 4.5.** *Let $X$ be a proper vertex subset of a graph $G = (V,E)$. Let $\omega(X,X^c)$ denote the direction of the cut $[X,X^c]$ such that all arcs of $\omega(X,X^c)$ have heads in $X$ and tails in $X^c$. Then*

$$\delta 1_X = \mathrm{I}_{\omega(X,X^c)}.$$

*This explains again why we adopt the notation $\delta X = [X,X^c]$.*

*Proof.* For each $\vec{e} = \overrightarrow{uv} \in \omega(X,X^c)$, we have $v \in X$ and $u \in X^c$; then $(\delta 1_X)(\vec{e}) = 1_X(v) - 1_X(u) = 1$. Clearly, $(\delta 1_X)(\vec{e}) = 0$ for $\vec{e} = \overrightarrow{uv} \in [X^c,X^c]$ and $(\delta 1_X)(\vec{e}) = 1 - 1 = 0$ for $\vec{e} = \overrightarrow{uv} \in [X,X]$. □

**Exercises**

Ch1: 1.1.21; 1.1.22; 1.2.8; 1.4.2; 1.5.6; 1.5.7; 1.5,12.
Ch2: 2.1.2; 2.1,11; 2.2.12; 2.4.1; 2.4.2; 2.4.9; 2.5.2; 2.5.4; 2.5.7; 2.6.2; 2.6.4.