Week 11: Matchings

November 11, 2020

1 Matching

Let $G = (V, E)$ be a graph.

- A **matching** of G is an edge subset M consisting of links that no two edges share a common vertex. The two endpoints of an edge in M are said to be **matched** under M. The vertices incident with edges of M are said to be **covered** by M.

- A matching M of G is **maximum** if $|M| \geq |M'|$ for all matchings M' of G, and is **perfect** if every vertex of G is covered by M. Clearly, perfect matchings are maximum matchings. The **matching number** of G is the cardinality of a maximum matching of G, denoted $\alpha'(G)$, i.e.,

  $$\alpha'(G) = \max\{|M| : M \text{ is a matching of } G\}.$$

  A graph is **matchable** if it admits a perfect matching.

- A **covering** of G is a vertex subset $K \subseteq V(G)$ that every edge has an endpoint in K. A covering of G is **minimal** if none of its subsets is a covering of G. A covering K^* of G is **minimum** if there is no covering K such that $|K| < |K^*|$. The **covering number** of G is the cardinality of a maximum covering of G, denoted $\beta(G)$, i.e.,

  $$\beta(G) = \min\{|K| : K \text{ is a covering of } G\}.$$

- Let M be a matching and K a covering of G. Since every edge of M is covering by a vertex of K, and distinct edges of M are covered by distinct vertices of K, we have $|M| \leq |K|$. Thus

  $$\alpha'(G) \leq \beta(G).$$

  It may be speculated that $\alpha'(G) = \beta(G)$. Unfortunately, this is not true in general. However, it holds whenever G is bipartite.

- Let M be a matching and K a covering of G. If $|M| = |K|$, then M is a maximum matching and K is a minimum covering.

- Given a matching M of G. An M**-alternating path** in G is a path whose edges alternate between M and M^c along its vertex-edge sequence. An M**-augmenting path** is an M-alternating path whose initial and terminal vertices are covered by M.

Lemma 1.1. *Let M be a matching of a graph G and P a path in G.*

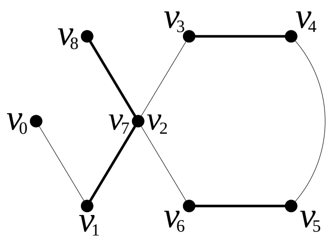
(a) *If P is M-alternating, then P has no self-intersect vertices.*

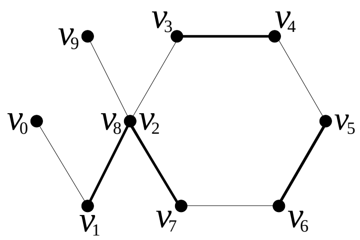
(b) *If P is an M-augmenting path, then the symmetric difference*

$$M' = M\Delta P := (M \smallsetminus P) \cup (P \smallsetminus M)$$

is a matching of G and $|M'| = |M| + 1$.

Proof. (a) Suppose that the path $P = v_0v_1 \ldots v_{2m+1}$ has self-intersection, i.e., two of the vertices $v_0, v_1, \ldots, v_{2m+1}$ are the same, say, $v_i = v_j$ with $i < j$. There are two possibilities: $j - i$ is odd and $j - i$ is even. In the former case, we see that either the edges $v_{i-1}v_i, v_jv_{j+1}$ belong to M or the edges $v_iv_{i+1}, v_{j-1}v_j$ belong to M. This is a contradiction since two edges of M share the common vertex $v_i(v_j)$. In the latter case, we see that either the edges $v_{i-1}v_i, v_{j-1}v_j$ belong to M or the edges v_iv_{i+1}, v_jv_{j+1} belong to M, so two edges of M share the common vertex $v_i(v_j)$.





(b) Since M is a matching, no two edges of $M \smallsetminus P$ share a common vertex. Since P has no self-intersection, no two edges of $P \smallsetminus M$ share a common vertex.

Note that the vertices of $M \cap P$ are internal vertices of P, and neither the initial vertex nor the terminal vertex of P is an endpoint of M. We see that the endpoints of $M \smallsetminus P$ are disjoint from P, of course, disjoint from $P \smallsetminus M$. Thus the symmetric difference $M' = M\Delta P$ is a matching. Clearly, $|M'| = |M| + 1$. $\square$

Theorem 1.2 (Berge, 1957)**.** *Let M be a matching of G. Then M is a maximum matching if and only if G contain no M-augmenting path.*

Proof. Suppose that M is not a maximum matching. Then there exists a matching M' such that $|M'| > |M|$. Consider the graph $G^* = (V, E^*)$, where

$$E^* = M\Delta M' := (M - M') \cup (M' - M).$$

Since $|M'| > |M|$, we have

$$|M' - M| > |M - M'|.$$

The graph G^* has the property that each vertex is incident with at most one edge in $M - M'$ and at most one edge in $M' - M$, i.e., at most two edges of E^*. This means that the degree of every vertex of G^* is at most two. Thus each component of G^* is either a simple path (without self-intersection) or a cycle. The paths and cycles must be M-alternating and can be classified into four types:

Type 1. A simple path whose first and last edges are in $M' - M$.

Type 2. A simple path whose first and last edges are in $M - M'$.

Type 3. A simple path whose first edge is in $M - M'$ and whose last edges is in $M' - M$, or whose first edge is in $M' - M$ and whose last edge is in $M - M'$.

Type 4. A cycle.

Note that a Type 1 path has more edges in M' than the edges in M. A Type 2 path has more edges in M than the edges in M'. A Type 3 path has equal number of edges in both M and M'. And a Type 4 cycle has the same number of edges in both M and M'. Since $|M' - M| > |M - M'|$, there exists at least one path $P = v_0 v_1 \ldots v_{2k+1}$ of Type 1, whose first and last edges are in $M' - M$. Since P is a component of G^*, the initial and terminal vertices v_0, v_{2k+1} are not incident with edges in $M - M'$. We claim that both v_0 and v_{2k+1} are *not* incident with edges of M.

Suppose that v_0 (v_{2k+1}) is incident with an edge e in M. The edge e cannot be in M' (since the vertex is already incident with an edge in $M' - M$). So e belongs to $M - M'$, i.e., v_0 (v_{2k+1}) is incident with at least two edges in G^*, which is a contradictory to the fact that v_0 (v_{2k+1}) is an initial (terminal) vertex of the path P, and that P is a component of G^*. Now we see that P is an M-augmenting path, since its initial and terminal vertices are not incident with edges of M. The symmetric difference $M \Delta P$ gives a larger matching of G, this is contradictory to that M is a maximum matching of G. $\square$

2 Matching in bipartite graphs

3 Perfect matchings

An **odd (even) component** of a graph $G = (V, E)$ is a connected component having odd (even) number of vertices. Let $o(G)$ denote the number of odd components of G.

Let M be a matching of G. Each odd component of G contains at least one vertex not covered by M. Let $U := U(M)$ denote the set of vertices not covered by M. Then obviously

$$|U| \geq o(G).$$

Let S be nonempty proper subset, i.e., $\varnothing \subsetneq S \subsetneq V$. Let H be an odd component of $G \smallsetminus S$. Then H cannot be covered by $M \cap H$. Assume that H is covered by M. Then the vertices of H not covered by $M \cap H$ must be matched some vertices in S by M. It is clear that there are at most $|S|$ odd components of $G \smallsetminus S$ covered by M. So the number of odd components of $G \smallsetminus S$ not covered by M is at least $o(G \smallsetminus S) - |S|$. Thus $|U| \geq o(G \smallsetminus S) - |S|$. We obtain the following proposition.

Proposition 3.1. *For all matching M of a graph G and all subsets $S \subseteq V(G)$,*

$$|U(M)| \geq o(G \smallsetminus S) - |S|. \tag{3.1}$$

In particular, if $|U(M)| = o(G \smallsetminus S) - |S|$ for a proper subset $S \subsetneq V(G)$, then M is a maximum matching.

Proof. Let $|U(M)| = o(G \smallsetminus S) - |S|$ for a matching M and a proper subset $S \subsetneq V$. Suppose M is not a maximum matching, i.e., there exists a matching M' such that $|M| < |M'|$. Then $|U(M)| > |U(M')| \geq o(G \smallsetminus S) - |S|$, which is a contradiction. $\square$

A subset $B \subseteq V(G)$ is a **barrier** to a matching M of a graph G if

$$|U(M)| = o(G \smallsetminus B) - |B|.$$

Barriers are succinct certificates to check if a matching is maximum, and of course not unique for a graph. Recall that a graph is **matchable** if it admits a perfect matching. A graph G is **hypo-matchable** if its every single-vertex-deleted subgraph is matchable, i.e., $G \smallsetminus v$ admits a perfect matching for each vertex $v \in V(G)$.

Lemma 3.2. *(a) The empty set and all singletons are barriers of perfect matchings.*
(b) The empty set is a barrier for every hypo-matchable graph.

Proof. (a) Let M be a perfect matching of graph G. The empty set is obviously a barrier of M. For each $v \in V(G)$, the vertex v must be matched a vertex of an odd component of $G \smallsetminus v$, and each odd component of $G \smallsetminus v$ has a vertex matched v. So there is exactly one odd component of $G \smallsetminus v$. Thus $|U(M)| = 0 = o(G \smallsetminus v) - 1$. So $\{v\}$ is a barrier of M.

(b) Let M_v be a perfect matching of $G \smallsetminus v$ for each $v \in V(G)$. If the set $E_v(G)$ of edges at v is empty, then M_v is a perfect matching of G; clearly, $\varnothing$ is a barrier of G to M_v. If $E_v(G) \neq \varnothing$, then $|U(M_v)| = 1 = o(G)$, which means that $\varnothing$ is a barrier of G to M_v. □

A vertex v of a graph G is **essential** if every maximum matching of G covers v; otherwise **inessential**, i.e., there exists a maximum matching M of G such that v is not covered by M. Recall that the matching number $\alpha'(G)$ is the number of edges of a maximum matching of G. We claim: A vertex v is essential if and only if $\alpha'(G \smallsetminus v) = \alpha'(G) - 1$.

In fact, let v be an essential vertex of G. Given maximum matching M of G. Then v matches a vertex with an edge $e = uv \in M$. Thus $M \smallsetminus e$ is a matching of $G \smallsetminus v$. We have $\alpha'(G \smallsetminus v) \geq |M \smallsetminus e| = |M| - 1 = \alpha'(G) - 1$. Let M_v be a maximum matching of $G \smallsetminus v$. If $G \smallsetminus v$ has a vertex u not covered by M_v, then $M_v \cup e$ is a matching of G, where $e = uv$. Thus $\alpha'(G) \geq |M_v \cup e| = |M_v| + 1 = \alpha'(G \smallsetminus v) + 1$. Therefore $\alpha'(G \smallsetminus v) = \alpha'(G) - 1$.

Conversely, let $\alpha'(G \smallsetminus v) = \alpha'(G) - 1$ for a vertex $v \in V(G)$. Given a maximum matching M of G. Suppose v is not covered by M. Then M is a maximum matching of $G \smallsetminus v$. Thus $\alpha'(G \smallsetminus v) = |M| = \alpha'(G)$, a contradiction.

Lemma 3.3. *If v is an essential vertex of a graph G and B is a barrier of $G \smallsetminus v$, then $B \cup v$ is a barrier of G.*

Proof. Let M be a maximum matching of G. Then v is covered by an edge $e = uv \in M$. Then $|M - e| = |M| - 1 = \alpha'(G) - 1 = \alpha'(G - v)$. This means that $M - e$ is a maximum matching of $G - v$. Since B is a barrier of $G - v$, we have $|U(M - e)| = o(G - B \cup v) - |B|$. Thus $|U(M)| = |U(M - e) + 1 = o(G - B \cup v) - |B \cup e|$. This means that $B \cup e$ is a barrier of G to M. □

Lemma 3.4. *Let G be a graph whose every vertex is inessential. Then G is hypo-matchable.*

Proof. Inessential of every vertex implies that G has no perfect matching. We need to show that each vertex-deleted subgraph of G has a perfect matching. Suppose this is not true, i.e., there exists a vertex v such that $G - v$ has no perfect matching. Since v is not essential in G, there exists a maximum matching M of G such that v is not covered by M. Of course, M is a matching but not a perfect matching of $G - v$. So there exists a vertex u not covered by

M, where $u \neq v$. We see that there exist a pair of two vertices not covered by a maximum matching of G. We choose a pair of two vertices u, v among all pairs of two vertices not covered by a maximal matching of G such that the distance $d(u, v)$ is minimal.

If $d(u, v) = 1$, then $M \cup e$ with $e = uv$ is a matching of G, contradicting the maximality of M. So $d(u, v) \geq 2$. Let P_{uv} be a shortest path from u to v. Let w be an internal vertex of P_{uv}. Since $d(u, w) < d(u, v)$, the vertex w must be covered by M. Since w is not essential in G, there exists a maximum matching M' of G such that w is not covered by M'. Since $d(u, w) < d(u, v)$ and $d(w, v) < d(u, v)$, both u, v must be covered by M'.

Note that components of $M \Delta M'$ are vertex disjoint cycles and paths, whose edges alternate between M and M'. Clearly, each cycle of $M \Delta M'$ has even number of edges, and each endpoint of a path in $M \Delta M'$ is covered either by M or by M' but not by both. Note that w is covered by M but not by M', and u, v are covered by M' but not by M. Thus u, w, v must be endpoints of paths in $M \Delta M'$. Each path of $M \Delta M'$ also has even number of edges. In fact, if P is a path of $M \Delta M'$ with endpoints not covered by M (M'), then P is an M-augmenting (M'-augmenting) path. So $M \Delta P$ $(M' \Delta P)$ is a matching and $|M \Delta P| > |M|$ $(|M' \Delta P| > |M'|)$, contradicting the maximality of M (M').

Let P_u be the path of $M \Delta M'$ with endpoints u, x. Then x is covered by M but not covered by M'. If $x \neq w$, then $M \Delta P_u$ is a maximum matching and u, w are not covered by $M \Delta P_u$, but $d(u, w) < d(u, v)$; contradicting the minimality of $d(u, v)$. Thus $x = w$, i.e., P_u is a from u to w. Likewise, the path P_v of $M \Delta M'$ starting from v ends at w. This is contradictory to that w is an endpoint of a path in $M \Delta M'$. $\square$

Theorem 3.5 (Tutte-Berge Theorem)**.** *Every graph G has a barrier, i.e., there exists a matching M of G and a proper subset $S \subsetneq V(G)$ such that $|U(M)| = o(G \smallsetminus S) - |S|$. Moreover,*

$$\alpha'(G) = \frac{1}{2} \min\{v(G) - o(G \smallsetminus S) + |S| : S \subsetneq V(G)\}. \tag{3.2}$$

Proof. We proceed by induction on $|V(G)|$. For $|V| = 1$, choose $M = \varnothing$, then $S = \varnothing$ is a barrier to M, since $|U(M)| = 1 = o(G) = o(G \smallsetminus S) - |S|$. For $V = \{u, v\}$ with $E = \varnothing$, choose $M = \varnothing$, then $S = \varnothing$ is a barrier to M, since $|U(M)| = 2 = o(G) = o(G \smallsetminus S) - |S|$. For $V = \{u, v\}$ and $E = \{uv\}$, choose $M = E$, then $S = \varnothing$ is a barrier to M, since $|U(M)| = 0 = o(G) = o(G \smallsetminus S) - |S|$.

Given a graph $G = (V, E)$ with $|V| \geq 3$. If all vertices of G are inessential, then G is hypo-matchable. Thus the empty set is a barrier of G. If there exists an essential vertex $v \in V(G)$, then $G \smallsetminus v$ has a barrier S by induction. Thus $S \cup v$ is a barrier by Lemma 3.3.

Let $B \subset V(G)$ be a barrier to a matching M of G, i.e., $|U(M)| = o(G \smallsetminus B) - |B|$. Then M is a maximum matching and

$$\alpha'(G) = |M| = \frac{1}{2}|V(G) - U(M)| = \frac{1}{2}\Big(v(G) - o(G \smallsetminus B) + |B|\Big).$$

Since $|U(M)| \geq o(G \smallsetminus S) - |S|$ for all $S \subseteq V(G)$, we have $v(G) - |U(M)| \leq v(G) - o(G \smallsetminus S) + |S|$ for all $S \subseteq V(G)$. The Tutte-Berge formula follows immediately. $\square$

Theorem 3.6 (Tutte Theorem)**.** *A graph G has a perfect matching if and only if for each $S \subseteq V$,*

$$o(G \smallsetminus S) \leq |S|.$$

Proof. Let M be a perfect matching of G. Then $0 = |U(M)| \geq o(G \smallsetminus S) - |S|$, namely, $o(G \smallsetminus S) \leq |S|$ for all $S \subseteq V(G)$. Conversely, assume $o(G \smallsetminus S) \leq |S|$ for all $S \subseteq V(G)$. Let B be a barrier of G, i.e., there exists a maximum matching M such that $|U(M)| = o(G \smallsetminus B) - |B|$. Then $|U(M)| = o(G \smallsetminus B) - |B| \leq 0$. This means that M is a perfect matching. □

Corollary 3.7 (Petersen's Theorem)**.** *Every* 3*-regular simple graph* G *without cut edges has a perfect matching.*

Proof. We may assume that G is connected. For each subset $S \subsetneq V(G)$, let $S_1, \ldots, S_k$ denote the vertex sets of odd components of $G \smallsetminus S$. Note that

$$3|S| = \sum_{v \in S} \deg(v) = \#[S, S^c] + 2\#E(G[S]),$$

$$\text{odd} = 3|S_i| = \sum_{v \in S_i} \deg(v) = \#[S_i, S] + 2\#E(G[S_i]), \quad i = 1, \ldots, k.$$

Then $\#[S, S^c] \leq 3|S|$ and

$$\#[S_i, S] = \sum_{v \in S_i} \deg(v) - 2\#E(G[S_i]) = \text{odd}, \quad i = 1, \ldots, k.$$

Since G is connected and has no cut edge, we have $\#[S_i, S] > 0$ and $\#[S_i, S] \neq 1$. So $\#[S_i, S] \geq 3$. Thus

$$o(G - S) = k \leq \frac{1}{3} \sum_{i=1}^{k} \#[S_i, S] = \frac{1}{3} \#[S, S^c] \leq |S|.$$

By Tutte's theorem, G has a perfect matching. □

4 Matching algorithm

Given a matching M of a graph G. Recall that if P is an M-augmenting path, then $M \Delta P$ is a matching and $|M \Delta P| = |M| + 1$. We shall describe a polynomial-time algorithm, which either finds an M-augmenting path (subsequently, the matching M is improved), or a certificate that such M-augmenting path does not exist. Let u be a vertex not covered by M. A u-rooted tree T of G is an **M-alternating tree** if the unique path in T from u to each vertex v of T is an M-alternating path. A u-rooted M-alternating tree is **M-covered** if all vertices of T other than u are covered by $M \cap T$. Each u-rooted M-covered tree has a bipartition $T[R(T), B(T)]$, where $R(T), B(T)$ are the vertex bipartition of $V(T)$, consisting of vertices v having even, odd distances respectively from u to v in T.

Algorithm 4.1 (Augmenting Path Search (APS))**.** Input: a graph G with a matching M and a vertex u uncovered by M. Output: a matching $\hat{M}$ with one more edge than the input matching M, or a u-rooted maximal M-covered tree T (APS-tree).

1. **Set** a tree T with $V(T) = \{u\}$ and $E(T) = \varnothing$, $R(T) = \{u\}$.

2. **If** $[R(T), V(T)^c] = \varnothing$, **stop**, a maximal u-rooted M-tree is found.

3. **While** $\exists e = xy \in [R(T), V(T)^c]$, **do** $V(T) := V(T) \cup y$, $E(T) := E(T) \cup e$.

   If y is not covered by M, **stop**, a required matching $\hat{M} := M \Delta P$ is found, where P is the unique path from u to y in T.

   If y is covered by M, **choose** an edge $e = yz \in M$, **do** $V(T) := V(T) \cup z$, $E(T) := E(T) \cup e$, $R(T) := R(T) \cup z$; **return** to Step 2.

The APS algorithm ends up with either a matching $\hat{M}$ with $|\hat{M}| = |M| + 1$ (see the right of Figure 1 with an M-augmented path), or a u-rooted M-covered tree T (see the middle of Figure 1), we have

$$|B(T)| = |R(T)| - 1, \quad B(T) \subseteq N_G[R(T)] \subseteq V(T),$$

where $N_G[R(T)]$ is the set of vertices of G adjacent to some vertices of $R(T)$. Note that whenever a u-rooted M-covered tree is the case, it does not mean that there is no M-augmented path in G. For instance, the right of Figure 1 demonstrates an M-augmented path starting from u that is not tested by the APS algorithm.

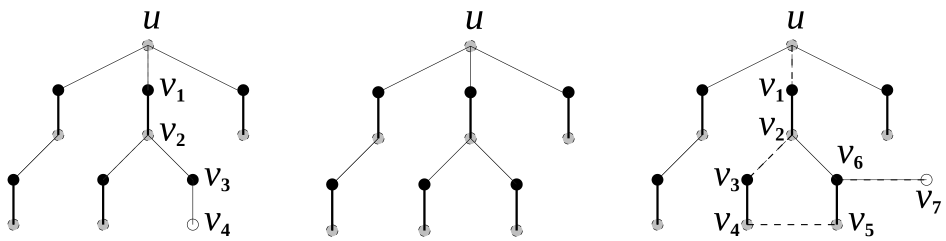


Figure 1: An APS-tree and an M-augmented path.

Proposition 4.1. *Let T be an APS-tree returned by the APS Algorithm. If no two vertices of $R(T)$ are adjacent in G, then no M-augmenting path in G include any vertex of T, in other words, each M-augmenting path in G is disjoint from T.*

Assume the algorithm is end up with a u-rooted maximal M-covered tree T. If $V \smallsetminus V(T)$ is covered by M, then M is a maximum matching. If $V \smallsetminus V(T)$ is uncovered by M, choose a vertex $v \in V \smallsetminus V(T)$ uncovered by M and repeat the APS algorithm starting from v.

Algorithm 4.2 (Hungarian or Egerváry's Algorithm)**.** Input: a bipartite graph $G[X, Y]$ with a matching M. Output: a matching $\hat{M}$ of G such that $|\hat{M}| > |M|$.

1. **Set** a tree T with $V(T) = \{u\}$ and $E(T) = \varnothing$, $R(T) = \{u\}$.

2. **If** $[R(T), V(T)^c] = \varnothing$, **stop**, a maximal u-rooted M-tree is found.

3. **While** $\exists e = xy \in [R(T), V(T)^c]$, **do** $V(T) := V(T) \cup y$, $E(T) := E(T) \cup e$.

   If y is not covered by M, **stop**, a required matching $\hat{M} := M \Delta P$ is found, where P is the unique path from u to y in T.

If y is covered by M, **choose** an edge $e = yz \in M$, **do** $V(T) := V(T) \cup z$, $E(T) := E(T) \cup e$, $R(T) := R(T) \cup z$; **return** to Step 2.

Repeating the APS algorithm, we have

- A set $\mathcal{T}$ of pairwise disjoint APS-trees.
- A set $R := \bigcup_{T \in \mathcal{T}} R(T)$ of red vertices.
- A set $B := \bigcup_{T \in \mathcal{T}} B(T)$ of blue vertices.
- A subgraph $F := G \setminus (R \cup B)$ with perfect matching $M(F)$.
- A matching $M^* := M(F) \cup \bigcup M(T)$ of G.
- A set $U := \{u(T) : T \in \mathcal{T}\}$ of vertices not covered by M^*.

Theorem 4.2. *The matching M^* returned above is a maximum matching.*

Exercises

Ch11: