Week 3: Connected Subgraphs

September 23, 2020

1 Connected Graphs

Path, Distance:

- A path from a vertex x to a vertex y in a graph G is referred to an xy-path.
- Let $X, Y \subset V(G)$. An **(X,Y)-path** is an xy-path with $x \in X$ and $y \in Y$.
- The **distance** between two vertices x and y, denoted $d(x,y)$, is the minimal length of all xy-paths. If there is no path between x and y, we define $d(x,y) = \infty$.

Technique of Using Eigenvalues:

Theorem 1.1. *Let G be a simple graph with n vertices in which any two vertices have exactly one common neighbor. Then G has a vertex of degree $n-1$. Consequently, G must be obtained from a family of disjoint triangles by gluing selected vertices, one from each triangle, to a single vertex.*

Proof. Suppose it is not true, i.e., the maximal degree $\Delta(G) < n-1$. We first show that G is regular. Consider two non-adjacent vertices x and y. Let $f : N(x) \to N(y)$, where $f(v)$ is defined as the unique common neighbor of v and y. We claim that f is injective. In fact, if $f(u) = f(v)$ for distinct $u, v \in N(x)$, then $f(u)$ is a common neighbor of u, v, y; now u and v have two common neighbors x and $f(u)$, a contradiction. Thus $d(x) = |N(x)| \leq |N(y)| = d(y)$. Likewise, $d(y) \leq d(x)$. So $d(x) = d(y)$. This is equivalent to say that any two adjacent vertices in $\bar{G}$ (the complement simple graph of G) have the same degree. We claim that G is regular.

To this end, it suffices to show that $\bar{G}$ is connected. Note that $\bar{G}$ has no isolated vertices, since the minimal degree $\Delta(\bar{G}) = n - 1 - \Delta(G) > 0$. Suppose $\bar{G}$ has two or more connected components. Take two edges $e_i = u_i v_i$ from distinct components of $\bar{G}$, $i = 1, 2$. Then $u_1 u_2 v_1 v_2 u_1$ is a cycle of G. Thus u_1 and v_1 have at least two common neighbors u_2, v_2, a contradiction.

Let G be k-regular. Consider the number of paths of length 2 in G. Since any two vertices have exactly one common neighbor, there are $\binom{n}{2}$ paths of length 2. For each vertex v, there are $\binom{k}{2}$ paths with the middle vertex v. It follows that $\binom{n}{2} = n\binom{k}{2}$. So $n = k^2 - k + 1$.

Let $\mathbf{A}$ be the adjacency matrix of G. The (u,v)-entry of $\mathbf{A}^2$ is the number of (u,v)-walks of length 2. Then $\mathbf{A}^2$ has its diagonal entries k and other entries 1. So $\mathbf{A}^2 = (k-1)\mathbf{I} + \mathbf{J}$, where

$\mathbf{I}$ is the identity matrix and $\mathbf{J}$ is a matrix whose entries are 1. Note that $\mathbf{J}$ has eigenvalue 0 with multiplicity $n-1$ and simple eigenvalue n. Since $\mathbf{A}^2 - \lambda\mathbf{I} = (k-1-\lambda)\mathbf{I} + \mathbf{J}$ and $n = k^2 - k + 1$, we see that $\mathbf{A}^2$ has eigenvalue $k-1$ with multiplicity $n-1$ and a simple eigenvalue k^2 with eigenvector $(1, \ldots, 1)^T$. Since $\mathbf{A}^2 - \lambda^2\mathbf{I} = (\mathbf{A} - \lambda\mathbf{I})(\mathbf{A} + \lambda\mathbf{I})$, we see that $\mathbf{A}$ has the eigenvalues $\pm\sqrt{k-1}$ with multiplicity $n-1$ and a simple eigenvalue k.

Since the graph G is simple, we have $\mathrm{tr}(\mathbf{A}) = \mathbf{0}$ (the sum of its diagonal entries). Recall that the trace of $\mathbf{A}$ is the sum of its eigenvalues counted with multiplicities. We have $\pm(n-1)\sqrt{k-1} + k = 0$; it forces that $(n-1)\sqrt{k-1} = k$. The only possible choice is that $k = 2$ and $n = 3$, i.e., G is a triangle, where $\Delta(G) = 2$. This is contradict to that $\Delta(G) < n - 1$. □

Remark: The above proof is interesting, but the result is boring.

2 Euler Tour

- A trail in a connected graph is called an **Euler tail** if it traverses every edge of the graph.
- Let G be a connected graph. A **tour** of G is a closed walk that traverses each edge at least once. An **Euler tour** is a tour that traverses each edge exactly once. A graph is said to be **Eulerian** if it admits an Euler tour.

Theorem 2.1. *A connected graph G has an Euler tour if, and only if, every vertex of G has even degree.*

Proof. "$\Rightarrow$" Let $W = v_0e_1v_1e_2 \ldots e_nv_n$ be an Euler tour of G. When one travels along the Euler tour and passes by a vertex v, the person must come towards v through one edge and depart from v through another edge. Then the number of times coming towards v equals the number times departing from v. Thus the degree of v must be even.

"$\Leftarrow$" Consider a longest trail $W = v_0e_1v_1e_2 \ldots e_nv_n$ in G. We show that W is an Euler tour.

(a) Claim $v_0 = v_n$. Suppose $v_0 \neq v_n$. Let v_n be appeared k times in the vertex sequence $(v_0, v_1, \ldots, v_{n-1})$, say, $v_{i_1} = \cdots = v_{i_k} = v_n$, $i_1 < \cdots < i_{i_k} < n$. Then the degree of v_n in W is $2k+1$. Since the degree of v_n in G is even, there is an edge e in G but not in W incident with v_n and another vertex v. Thus $W'WTev$ is a longer trail in G, a contradiction.

(b) Claim that $G - E(W)$ has no edges incident with a vertex in W. Suppose there is an edge $e \in E(G) - E(W)$ incident with a vertex $v_i \in V(W)$ and another vertex v. Then

$$W' = vev_1e_{i+1}v_{i+1} \cdots e_nv_n(v_0)e_1v_1 \cdots v_{i-1}e_iv_i$$

is a longer trail in G, a contradiction.

(c) Claim that W uses every edge of G. Suppose there exists an edge e not used in W. Let e be incident with vertices u, v. Then $u, v \notin V(W)$. Since G is connected, there is a shortest path $P = u_0x_1u_1 \cdots x_mu_m$ from W to u, where $u_0 = u_i$ and $u_m = u$. We claim that $x \not\in E(W)$. Suppose $x_1 \in E(W)$, then $u_1 \in V(W)$; thus $P' = u_1x_2u_2 \cdots x_mu_m$ is a shorter path from T to u, a contradiction. Now since $x_1 \notin E(W)$, we have a longer trail

$$W' = u_1x_1(u_0)v_ie_{i+1}v_{i+1} \cdots e_nv_n(v_0)e_1v_1e_2 \cdots v_{i-1}e_iv_i.$$

Again this is a contradiction.

Since W is a closed trail that uses every edge of G, we see that W is an Euler tour. $\square$

Corollary 2.2. *A connected graph G has a non-closed Euler trail if and only if G has exactly two vertices of odd degree.*

Proof. "$\Rightarrow$" Let $W := v_0e_1v_1 \cdots e_nv_n$ be an Euler trail of G with $v_0 \neq v_n$. Then $W' := We_0v_0$ is an Euler tour of the graph $G' := G \cup e_0$. Theorem 2.1 implies that G' is an even graph. It follows that G has exactly the two vertices v_0, v_n of odd degree.

"$\Leftarrow$" Let G have exactly two vertices u and v of odd degree. We add a new edge between u and v to G to obtain a new graph G'. Then G' is a connected even graph. Thus G' has an Euler tour by Theorem 2.1. Remove the edge e from the Euler tour for G', we obtain an Euler trail for G. $\square$

A **cut edge** of a graph G is an edge e such that $G \smallsetminus e$ has more connected components than G, i.e., $c(G - e) > c(G)$.

Lemma 2.3. *Let G be a connected graph with a specified vertex v. Assume that G is either an even graph (former case) or G has exactly two vertices u, v of odd degree (latter case).*

(a) If $d_G(v) = 1$ and $e = vw$ is a link at v, then G is the latter case. Moreover, $G \smallsetminus v$ is connected, either having all vertices of even degree (when $w = u$) or having exactly two vertices u, w of odd degree (when $w \neq u$).

having either $2(k-1)$ vertices of odd degree (when w is an odd-degree vertex) or $2k$ vertices of odd degree (when w is an even-degree vertex).

(b) If $d_G(v) \geq 2$ and $d_G(v)$ is even, then G is the former case. Moreover, for each edge $e = vw$ at v, $G \smallsetminus e$ is connected, either having all vertices of even degree (when e is a loop) or having exactly two vertices v, w of odd degree (when e is a link).

having $2k$ vertices of odd degree (when e is a loop) or $2k$ vertices of odd degree (when w is an odd vertex in G) or $2(k+1)$ vertices of odd degree (when w is an even vertex in G).

(c) If $d_G(v) \geq 2$ and $d_G(v)$ is odd, then G is the latter case. Moreover, there exists an edge $e = vw$ at v such that $G \smallsetminus e$ is connected, either having all vertices of even degree (when $w = u$) or having exactly two vertices u, w of odd degree (when $w \neq u$).

having $2k$ vertices of odd degree (when e is a loop) or $2(k-1)$ vertices of odd degree (when w is an odd vertex in G) or $2k$ vertices of odd degree (when w is an even vertex in G).

Proof. (a) It is clearly the latter case. If $w = u$, i.e., $d_G(w)$ is odd, then $G \smallsetminus v$ is a connected even graph. If $w \neq u$, i.e., $d_G(w)$ is even, then $G \smallsetminus v$ has exactly two vertices u, w of odd degree.

(b) Since $d_G(v)$ is even, it turns out that G is the former case. If e is a loop at v, i.e., $w = v$, then $G \smallsetminus e$ have the same property as G. If e is a link, then $G \smallsetminus e$ has exactly two vertices v, w of odd degree. We still need to show that $G \smallsetminus e$ is connected. Suppose $G \smallsetminus e$ has two connected components G_1, G_2 with $v \in V(G_1), w \in V(G_2)$. Then G_1 has exactly one vertex v of odd degree. This is impossible because the number of vertices of odd degree is always even.

(c) It is clear that G is the latter case. If G has no cut edge at v, then for any edge $e = vw$ at v, the graph $G \smallsetminus e$ is an even graph (when $w = u$) or $G \smallsetminus e$ has exactly two vertices u, w of odd degree (when $w \neq u$).

Let G have a cut edge $e' = vw'$ at v. Then $G \smallsetminus e'$ has two connected components G_1, G_2 with $v \in V(G_1)$, $w' \in V(G_2)$, $d_{G_1}(v)$ is even, and $d_G(v) \geq 3$. Suppose $u \in V(G_1)$; then G_1 has the only vertex u of odd degree; this is a contradiction. So we must have $u \in V(G_2)$, G_1 is a connected even graph, and $d_{G_1}(v) \geq 2$. Then by (b), for each edge $e = vw$ at v in G_1, the graph $G_1 \smallsetminus e$ is connected, either having all vertices of even degree (when e is a loop) or having exactly two vertices v, w of odd degree (when e is a link). Consequently, the graph $G \smallsetminus e$ is connected, having exactly two vertices u, w of odd degree for either case of $G_1 \smallsetminus e$. $\square$

Theorem 2.4. (Fleury's Algorithm) Input: *a connected graph* $G = (V, E)$. **Output**: *an Euler tour, or an Euler trail, or none of the previews two for* G.

Step 1 *If there are vertices of odd degree, then start at one such vertex* u. *Otherwise, start at any vertex* u. *Set* $W := u$ *and* $G' := G$.

Step 2 *Let* v *be the terminal vertex of* W. *If there is no edge remaining at* v *in* G', **stop**. *(Now* W *is an Euler tour if* $v = u$ *and an Euler trail if* $v \neq u$.*)*

Step 3 *If there is exactly one edge* $e = vw$ *remaining at* v *in* G', *set* $W := Wew$, $G' := G' \smallsetminus e$ *when* e *is a loop,* $G' := G' \smallsetminus v$ *when* e *is a link, and return to* Step 2.

Step 4 *If there are more than one edge remaining at* v *in* G', *choose one of these edges, say* $e = vw$, *in such a way that* $G' \smallsetminus e$ *is still connect; set* $W := Wew$, $G' := G' \smallsetminus e$, *and return to* Step 2. *If such an edge cannot be selected,* **stop**. *(There is neither Euler tour nor Euler trail.)*

Proof. At each step until stop in the algorithm, we construct a pair (W, G') dynamically, where W is a trail in G, G' is a connected subgraph of G, and W, G' have no common edges and $E(G) = E(W) \cup E(G')$, said to be **complementary**.

In Step 4, in the case that an edge e cannot be selected at v so that $G' \smallsetminus e$ is still connected, i.e., every edge at v in G' is a cut edge, then all edges at v in G' are cut edges of G, since W and G' are connected. Let $e_i = vv_i$ be edges at v in G', where $i = 1, \ldots, k$ and $k \geq 2$. Then $G \smallsetminus \{e_1, \ldots, e_k\}$ has components $G_0, G_1, \ldots, G_k$ with $v_i \in V(G_i)$ and $v_0 = v$. It is clear that it is impossible to have a trail in G from v to visit both G_1 and G_2. So Euler tour and Euler trail is impossible for the graph G.

A pair (W, G') above is said to be **Eulerian** if G' is either an even graph or has exactly two vertices of odd degree, the terminal vertex v of W is a vertex of G', and if G' is the case of having exactly two vertices of odd degree then v is one of the two odd-degree vertices. Whenever G is an even graph or have exactly two vertices of odd degree, the initial pair (W, G') in Fleury's algorithm is an Eulerian complementary pair.

Let (W, G') be an Eulerian complementary pair with the terminal vertex v of W in the process of Fleury's algorithm before entering Step 2. Now in Step 2, if there is no edge at v in G', then $E(G') = \varnothing$ (since G' is connected). It is clear that W is an Euler trail for G.

In Step 3, if e is a loop, it is clear that $(Wew, G' \smallsetminus e)$ is an Eulerian complementary pair; if e is a link then by Lemma 2.3(a), $(Wew, G' \smallsetminus v)$ is an Eulerian complementary pair. In Step 4, we have $d_{G'}(v) \geq 2$; then by Lemma 2.3(b) and Lemma 2.3(c), $(Wew, G' \smallsetminus e)$ is an Eulerian complementary pair.

Since all (W, G') constructed in Fleury's algorithm are Eulerian complementary pairs, and edges of G' are reducing when iterates, Fleury's algorithm stops at $(W, \varnothing)$ with W an Euler trail for G after finite number of iterates. $\square$

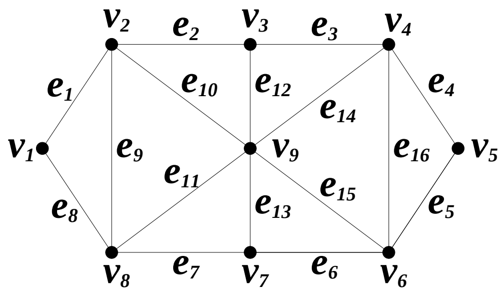

Figure 1: A graph with an Euler trail.

Example 2.1. An Euler trail for the graph in Figure 1 is given as

$$v_3e_3v_4e_4v_5e_5v_6e_6v_7e_7v_8e_8v_1e_1v_2e_2v_3e_{12}v_9e_{14}v_4e_{16}v_6e_{15}v_9e_{10}v_2e_9v_8e_{11}v_9e_{13}v_7.$$

3 Connection in Digraphs

- A **directed walk** in a digraph D is an alternating sequence of vertices and arcs

  $$W := v_0a_1v_1 \dots a_\ell v_\ell$$

  such that the arc a_i has the tail v_{i-1} and head v_i, $i = 1, \dots, \ell$. We call v_0 the **initial vertex** and v_ℓ the **terminal vertex** of W. Such a walk is referred to a **directed** (v_0, v_ℓ)**-walk**; the subwalk of W from a vertex v_i to a vertex v_j is referred to a (v_i, v_j)**-segment** of W.

- A **directed trail** is a directed walk with distinct arcs. A **directed path** is a directed walk with distinct arcs and distinct vertices, except the possible case that the initial vertex equals the terminal vertex.

- Given a digraph D. A vertex y is **reachable** from a vertex x in D if $x = y$ or there is a directed (x, y)-path from x to y in D. Two vertices x and y of D are **strongly connected** if each of them is reachable from the other in D. Strongly connectedness is an equivalence relation on the vertex set $V(D)$. A sub-digraph of D induced by an equivalence class of the strong connectedness is called a **strongly connected component** or **strong component** of D.

- A **directed Euler trail** in a digraph D is a directed trail that uses every arc of D. A closed directed Euler trail is called a **directed Euler tour**. A digraph is said to be **Eulerian** if it has a directed Euler tour.

Theorem 3.1. *Let x, y be two vertices of a digraph D. Then y is reachable from x in D if and only if $(X, X^c) \neq \varnothing$ for every subset $X \subset V(D)$ such that $x \in X$ and $y \notin X$.*

Proof. For necessity, let P be a directed path from x to y. For each proper subset $X \subset V(D)$ with $x \in X$ and $y \in X^c$, the path P passes between X and X^c. The first arc of P from X to X^c; so $(X, X^c) \neq \varnothing$.

Conversely, for sufficiency, suppose that y is not reachable from x. Let X be the set of vertices reachable from x. Then $y \in X^c$. Since every vertex of X^c is not reachable from x, there is no arc from X to X^c. So $(X, X^c) = \varnothing$, a contradiction. □

Theorem 3.2. *A connected digraph D is Eulerian if, and only if, the in-degree equals the out-degree at each vertex of D.*

Proof. It is analogous to the proof of the existence of Euler tour. □

4 Cycle Double Cover

Cycle Cover, Cycle Double Cover

- A **cycle cover** of a graph G is a family $\mathcal{F}$ of subgraphs of G such that $E(G) = \bigcup_{H \in \mathcal{F}} E(H)$ and each member of $\mathcal{F}$ is a cycle.
- A **cycle double cover** of a graph G is a cycle cover such that each edge of G belongs to exactly two members of $\mathcal{F}$, i.e., each edge of G is covered exactly twice by $\mathcal{F}$.

Proposition 4.1. *Let G be a graph having a cycle covering $\mathcal{C}$ that each edge of G is covered at most twice. Then G has a cycle double cover.*

Proof. Let $E_1 \subset E(G)$ be the edge subset whose edges are covered exactly one by $\mathcal{C}$. Since $\mathcal{C}$ is a covering and each edge of G is covered at most twice by members of $\mathcal{C}$, we see that $G[E_1]$ is an Eulerian graph (i.e. even graph). So $G[E_1]$ is a union of edge disjoint cycles, i.e., $G[E_1]$ has a covering $\mathcal{C}_1$ that each edge of $G[E_1]$ is covered exactly once. Thus $\mathcal{C}_2 = \mathcal{C} \cup \mathcal{C}_1$ is a cycle double covering of G. □

Cycle Double Cover Conjecture

Conjecture 4.2. *Every 2-connected graph (i.e. having no cut edge) has a cycle double cover.*

5 Chinese Postman Problem

A postman wishes to walk minimal distance to pass every street at least once in order to deliver mails in worst case. The problem is known as the **Chinese postman problem**; it was first formulated by a Chinese mathematician Meigu Guan (Mei-Ku Kuan) in 1962 as finding a closed walk of minimal distance on a connected graph that passes every edge at least once.

Pairing odd vertices

Let $G = (V, E)$ be a connected graph. It is well known that G contains even number of vertices of odd degree. Let $W = v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n$ be a closed walk of minimal distance that uses every edge at least once.

Lemma 5.1. *Let W be a closed walk on a connected graph G that uses every edge of G at least once. Let u be an odd-degree vertex. Then*

(a) There exists an edge e at u such that e appears in W more than once.

(b) There exists a path P from u to another odd vertex v of G such that every edge of P appears at least twice in W.

(c) There exist paths pairing odd vertices of G. If G has $2m$ odd-degree vertices, then there exist m paths $P_1, \ldots, P_m$ pairing the $2m$ odd-degree vertices. Moreover, for each edge e appeared exactly in k paths of $P_1, \ldots, P_m$, the edge e appears at least $k+1$ times in W, and

$$|W| \geq |E(G)| + \sum_{i=1}^{\infty} |P_i|.$$

Proof. (a) When one follows the order of W and crosses the vertex u, the number of times of moving toward u equals the number of times departing away from u. Then the number of edges in W at u (counted with multiplicities) is even and positive. Since the number of edges at u in G is odd, we see that one of the edges at u in G must appear more than once in W.

(b) Without loss of generality we may assume that G is loopless. We perform the following algorithm to construct a walk P from u to another odd-degree vertex.

Step 1 Initially, set $P' = u$, $G' = G$, $W' = W$, $v = u$.

Step 2 If $\deg_{G'}(v) =$ odd, select an edge $e = vw$ in G' that appears at least twice in W'; set $P' := P'ew$, $G' := G' \cup e'$ (with e' an edge having the endpoints as e), $W' := (W' \setminus e) \cup e'$ (replace one copy e by e'), $v := w$; and **return** to Step 2.

If $\deg_{G'}(v) =$ even (i.e., $\deg_G(v) =$ odd), **stop**. (P' is a walk from the odd vertex u to an odd vertex v in G.)

Note that $\deg_{G'}(v)$ is even if and only if $\deg_G(v)$ is odd. If an even-degree vertex v of G' (odd-degree vertex of G) is met in an iteration of the algorithm, a walk P' is found from the odd-degree vertex u to an odd-degree vertex v in G. Of course, a path P from u to v can be easily constructed along the walk P'. Each iteration reduce the multiplicity of a multiple edge of W' by 1. Eventually, W' contains no multiple edges after finite number of iterations, i.e., an even-degree vertex v is met eventually in G'. So the algorithm stops.

(c) When a walk P_1' is found from one odd vertex to another odd vertex in G, if there are still odd-degree vertices in G', perform the algorithm again to find a walk P_2' from one odd-degree vertex to another odd-degree vertex in G'. Continue this procedure by the above algorithm, we obtain walks $P_1', \ldots, P_k'$ pairing the odd-degree vertices of G. Of course, the paths $P_1, \ldots, P_k$ pairing the odd-degree vertices of G are easily constructed along the walks. Since the construction, the multiset $W - \bigcup_{i=1}^{k} P_i'$ is a covering of $E(G)$. Thus

$$|W| \geq |E(G)| + \sum_{i=1}^{k} |P_i'| \geq |E(G)| + \sum_{i=1}^{k} |P_i|.$$

□

Lemma 5.2. *Let G be a connected graph G with $2k > 0$ odd-degree vertices. Let $P_1, \ldots, P_k$ be paths paring the odd-degree vertices. Then there exists a closed walk W in G such that each P_i is a part of W and every edge of G appears at least once in W.*

Proof. Let u_i and v_i be the initial and terminal vertices of P_i, $i = 1, \ldots, k$. Let Q_i be a copy of the path P_i, having the same vertices as P_i but edges distinct from that of G. Then the graph $G' := G \cup P_1 \cup \cdots \cup P_k$ is an even graph. Thus G' has an Euler tour W'. Clearly, each Q_i is a subwalk of W'. Replacing each Q_i by P_i in W', we obtain a closed walk W that uses every edge of G at least once. □

Theorem 5.3 (Chinese Postman Problem Algorithm)**. Input***: connected graph G.* **Output***: close walk W of minimum length that uses every edge of G.*

Step 1 *Pairing the odd-degree vertices by minimum distance paths P_i.*

Step 2 *Minimize the sum $\sum |P_i|$ among all pairings.*

Step 3 *Whenever $\{P_i\}$ is a minimum paring of odd-degree vertices, construct a new graph G' by adding a copy of each P_i into G so that G' is Eulerian.*

Step 4 *Find an Euler tour W' for G'. Replace the copy of each P_i in W' by P_i to obtain a closed walk W of minimum length that uses every edge of G.*

Proof. It is clear that the walk found in the theorem is a closed walk that uses every edge of G. Lemma (4) shows that the length of every closed walk that uses all edge of G is larger than or equal to the length of the walk found by the theorem. Thus the walk found in the theorem is a closed walk of minimum length that uses every edge of G. □

Exercises

Ch3: 3.1.2; 3.1.5; 3.2.1; 3.2.2; 3.4.1; 3.4.4; 3.4.5; 3.4.11.