Week 4: Trees

September 30, 2020

1 Forests and Trees

- A graph is **acyclic** if it contains no cycles. **Forest** is just another name for acyclic graph. A **tree** is a connected acyclic graph. A tree is **trivial** if it contains exactly one vertex.

- Every nontrivial tree has at least two leaves (= vertices of degree 1).

- Any two vertices in a tree are connected by a unique path.

- If T is tree, then $|E(T)| = |V(T)| - 1$. More generally, if F is a forest, then

$$|E(F)| = |V(F)| - c(F),$$

  where $c(F)$ is the number of connected components of F.

2 Rooted Trees

- A tree with a specified vertex x is called a **rooted tree** with the **root** x, denoted $T(x)$. A tree with a root x is also referred to an x**-tree**.

- A **branching** is a rooted tree with an orientation such that every vertex but the root has in-degree 1. A branching with a root x is referred to an x**-branching**.

- Let D be a digraph and x a vertex of D. Let X be the set of vertices reachable from x in D. Then there exists an x-branching $T(x)$ in D with $V(T) = X$.

3 Spanning Trees

- A **spanning subgraph** H of a graph G is a subgraph such that $V(H) = V(G)$. A spanning tree T of a connected graph G is a tree and is also a spanning subgraph. We identify each edge subset $S \subseteq E(G)$ as the spanning subgraph of G with edge set S.

- Every connected graph has a spanning tree.

- Let T be a spanning tree of a graph G. Let T^c denote the spanning subgraph whose edge set is $E(G) - E(T)$. For each edge $e \in T^c$, there exists a unique cycle $C(T, e)$ contained in $T \cup e$ (the graph having e added to T), known a **fundamental cycle** of G with respect to T. Each edge $e \in T^c$ must be contained in $C(T, e)$.

  Proof. Let u, v denote endpoints of the edge $e \in T^c$. There are two internal-vertex disjoint paths from u and v in $T \cup e$: the path uev and the unique path from u to v in T. The two paths form a cycle $C(T, e)$ in $T \cup e$. Suppose C' is a cycle other than $C(T, e)$ in $T \cup e$. Then $C' \Delta C(T, e)$ a nontrivial even graph contained in T, which is an edge disjoint cycles, contradicting the acyclicity of T. □

- Let T be a tree of a graph G. For each edge $e \in T$, there exists a unique bond $B(T, e)$ of G contained in $T^c \cup e$, known as a **fundamental bond** of G with respect to T. Each edge $e \in T$ must be contained in $B(T, e)$.

  Proof. The spanning subgraph $T \smallsetminus e$ (with the edge $e \in T$ removed) is decomposed into two disjoint trees T_1 and T_2. Then the edge set between the vertices of T_1 and the vertices of T_2 is a bond $B(T, e)$ of G, and clearly, $e \in B(T, e)$. Suppose B' is a bond contained in $T^c \cup e$ but other than $B(T, e)$. Then $B' \Delta B(T, e)$ is an edge cut of G contained in T^c. Taking edge set complement of T^c, we see that T is contained in the spanning subgraph of G with the sedge set $E(G) \backslash B' \Delta B(T, e)$, which is disconnected as $B' \Delta B(T, e)$ is an edge cut of G. This is a contradiction since T is connected. □

- Let F be a forest of a graph G. For each edge $e \in F^c$, there exists a unique cycle $C(F, e)$ contained in $F \cup e$, known as a **fundamental cycle** of G with respect to F. For each edge $e \in F$, there exists a unique bond $B(F, e)$ of G contained in $F^c \cup e$, known as a **fundamental bond** of G with respect to F.

4 Cycle Space and Bond Space

- The **edge space** of a graph $G = (V, E)$ over the field $\mathbb{F}_2 = \{0, 1\}$ is the vector space of all functions from E to $\mathbb{F}_2$, denote $\mathbb{F}_2^E$.

- The set $C(G)$ of indicator functions of edge sets of even subgraphs of a graph G forms a subspace of $\mathbb{F}_2^E$, called the **cycle space** of G. Let $\mathcal{C}(G)$ denote the set of even spanning subgraphs of G, which can be made into a vector space over $\mathbb{F}_2$ by the one-to-one correspondence $\mathcal{C}(G) \to C(G)$, $H \mapsto 1_{E(H)}$.

- The set $B(G)$ of indicator functions of edge sets of cuts of a graph G forms a subspace of $\mathbb{F}_2^E$, called the **bond space** of G. Let $\mathcal{B}(G)$ denote the set of cuts of G, which can be made into a vector space over $\mathbb{F}_2$ by the one-to-one correspondence $\mathcal{B}(G) \to B(G)$, $U = [X, X^c] \mapsto 1_U$.

Theorem 4.1. *The collection of fundamental cycles of a connected graph G with respect to a spanning tree T forms a basis of the cycle space of G. So the cycle space has dimension* $|E(T^c)| = |E(G)| - |V(G)| + 1$.

Proof. We claim that the collection $\{1_{C(T,e)} : e \in E(T^c)\}$ is a basis of the cycle space. We first show that $\{1_{C_e} : e \in E(G-T)\}$ is linearly independent over $\mathbb{F}_2 = \{0,1\}$.

Assume

$$\sum_{e\in E(G-T)} a_e 1_{C_e} = 0 \tag{4.1}$$

for some coefficients a_e in $\mathbb{F}_2$. For an particular edge $e_0 \in E(G-T)$, we have $e_0 \in C_{e_0}$ and $e_0 \notin C_e$ for all $e \in E(G-T)$ but $e \neq e_0$. So the value of left-hand side of (4.1) at e_0 is a_0, i.e., $a_0 = 0$. This proves the linear independence.

To show that $\mathbb{F}_2 = \{0,1\}$ spans the cycle space of G, given a cycle C of G. We claim that $1_C = \sum_{e\in E(C-T)} 1_{C_e}$, i.e.,

$$1_C + \sum_{e\in E(C\cap T^c)} 1_{C_e} = 0. \tag{4.2}$$

It is clear that the left-hand side of (4.2) is zero on $E(T^c - C)$. The left-hand side of (4.2) also cancels to zero on $E(C \cap T^c)$. Thus the left-hand side of (4.2) is the indicator function of an even graph on T. Since T does not contain cycle, it follows that the left-hand side of (4.2) is zero on $E(G)$. □

Theorem 4.2. *The collection of fundamental bonds of a connected graph G with respect to a spanning tree T forms a basis of the bond space of G. So the bond space has dimension $|T(G)| = |V(G)| - 1$.*

Proof. Let $\{B_e : e \in E(T)\}$ be the collection of fundamental bonds of G with respect to T. We first show that $\{1_{B_e} : e \in E(T)\}$ is linearly independent over $\mathbb{F}_2$.

Assume

$$\sum_{e\in E(T)} a_e 1_{B_e} = 0 \tag{4.3}$$

for some coefficients a_e in $\mathbb{F}_2$. For an particular edge $e_0 \in E(T)$, we have $e_0 \in B_{e_0}$ and $e_0 \notin B_e$ for all $e \in E(T)$ but $e \neq e_0$. So the value of left-hand side of (4.3) at e_0 is a_0, i.e., $a_0 = 0$. This proves the linear independence.

To show that $\mathbb{F}_2 = \{0,1\}$ spans the bond space of G, given a bond B of G. We claim that $1_B = \sum_{e\in E(T)} 1_{B_e}$, i.e.,

$$1_B + \sum_{e\in E(B\cap T)} 1_{B_e} = 0. \tag{4.4}$$

It is clear that the left-hand side of (4.4) is zero on $E(T - B)$. The left-hand side of (4.4) also cancels to zero on $E(B \cap T)$. Thus the left-hand side of (4.4) is the indicator function of an edge cut on T^c, i.e., $U \subset T^c$. Since $T = G - T^c$ is connected, thus $G - U$ is also connected. □

Corollary 4.3. *Let G be a graph with connected components $G_1, \ldots, G_k$. Then*

$$C(G) = \bigoplus_{i=1}^{k} C(G_i), \quad B(G) = \bigoplus_{i=1}^{k} B(G_i),$$

and $\dim C(G) = |E(G)| - |V(G)| + c(G)$, $\dim B(G) = |V(G)| - c(G)$. *However, $C(G) + B(G)$ is not a direct sum over $\mathbb{F}_2$.*

Proof. The first part is trivial. For the graph G with two vertices u and v and two edges between u and v. Then the edge space is isomorphic to $\mathbb{F}_2^2$, $C(G) = B(G) \cong \mathbb{F}_2$. □

5 Flow group and flow space

Let $G = (V, E)$ be a graph, where each edge e is viewed as a path $e : [0,1] \to \mathbb{R}^d$ with its endpoints $e(0)$ and $e(1)$ glued to two (possibly identical) vertices in V. So we view $e(0)$ and $e(1)$ as vertices of G. There are two directions on $[0,1]$, one from 0 to 1 and the other from 1 to 0, they are represented by the identity map $\sigma : [0,1] \to [0,1]$ and the reverse direction map $\sigma' : [0,1] \to [0,1]$ given by $\sigma'(t) = 1 - t$. So the images of the two maps $e\sigma, e\sigma'$ are the same; both $e\sigma, e\sigma'$ are called **orientations** of the edge e. For each edge e we put an arrow on e, namely $\vec{e}$, to denote an orientation of e, the other one is denoted by $-\vec{e}$. Let $\vec{E}(G) = \{\vec{e}, -\vec{e} : e \in E(G)\}$ denote the set of all oriented edges of G.

An **orientation** ω of G is an assignment $\varnothing$ that each edge of G is given an orientation of that edge. So an orientation of G can be viewed as a subset $\omega \subset \vec{E}(G)$ such that

$$\omega \cap (-\omega) = \varnothing, \quad \omega \cup (-\omega) = \vec{E}(G).$$

There exists a unique function $[,] : V \times \vec{E} \to \mathbb{Z}$, called the **incidence function** of G, satisfying the following properties:

(1) $[v, e] = 1$ if e is a directed link pointing to v,

(2) $[v, e] = 0$ if e is a directed loop or v is not incident with e,

(3) $[v, -e] = -[v, e]$, $\forall v \in V$, $e \in \vec{E}$.

Given an abelian group A. A 1-**chain** of G over A is a function $c : \vec{E} \to A$ such that $c(-e) = -c(e)$ for $e \in \vec{E}$. Let $C_1(G, A)$ denote the group of all 1-chains of G. If A is a field $\mathbb{F}$, then $C_1(G, \mathbb{F})$ is a vector space over $\mathbb{F}$, and its dimension is $|E(G)|$.

A 0-**chain** of G over A is a function from V to A, the set of all 0-chains is denoted $C_0(G, A)$. There is a **boundary operator** $\partial : C_1(G, A) \to C_0(G, A)$ defined by

$$\partial c = \sum_{v \in V} \sum_{e \in E} c(\vec{e})[v, \vec{e}]v, \quad c = \sum_{e \in E} c(\vec{e})\vec{e}.$$

A **flow** of G is a 1-chain f such that $\partial f = 0$. If C is a cycle and ω_C is a direction of C, then the indicator chain I_{ω_C} is a flow. Since $\mathrm{I}_{\omega_C} = \sum_{e \in \omega_C} e$, so for simply we can identify I_{ω_C} as ω_C itself, and write $\omega_C = \sum_{e \in \omega_C} e$ as the formal sum.

$$\mathrm{I}_{\omega_C}(e) = \begin{cases} \pm 1 & \text{if } \pm e \in \omega_C, \\ 0 & \text{otherwise.} \end{cases}$$

Given an orientation ω on G to have a digraph (i.e. oriented graph) $D = (G, \omega)$. Then ω forms a basis of $C_1(G, A)$. Viewing V as basis of $C_0(G, A)$, the matrix of the boundary operator ∂ relative to the basis ω and V is the **incidence matrix** $\mathbf{M} = [m_{ve}]$, whose rows are indexed by vertices v and columns by edges e, defined by $m_{ve} = [v, \vec{e}]$, where $\vec{e} \in \omega$.

Proposition 5.1. *Let* $\mathbf{M}$ *be the incidence matrix of a digraph* $D = (G, \omega)$. *Let* $S \subseteq E(G)$ *be an edge subset, and* $\mathbf{M}_S$ *the submatrix of* $\mathbf{M}$ *whose columns are indexed by the members of* S. *Then the columns of* $\mathbf{M}_S$ *are linearly dependent if and only if the subgraph* $G[S]$ *contains a cycle.*

Proof. "$\Rightarrow$" Let the columns of $\mathbf{M}_S$ be linearly dependent. If S contains some loops, clearly, these loops are cycles of $G[S]$. We assume that S contains no loops. Let $S' \subset S$ be a minimal subset such that the columns of $\mathbf{M}_{S'}$ are linearly dependent. We may assume $S = S'$.

Let v be a vertex of $V(G[S])$. There exist some edges of S connecting v, and all such edges are links; so the v-row of $\mathbf{M}_S$ is nonzero. Suppose the v-row of $\mathbf{M}_S$ has only one nonzero entry, say, $m_{ve_0} \neq 0$; then any solution of $\mathbf{M}_S\mathbf{x} = \mathbf{0}$ must have $x_{e_0} = 0$; thus $\mathbf{M}_{S \smallsetminus e_0}\mathbf{x} = \mathbf{0}$ has nonzero solutions, i.e., the columns of $\mathbf{M}_{S \smallsetminus e_0}$ are linearly dependent, contradicting the minimality of S. It then follows that the v-row of $\mathbf{M}_S$ has at least two nonzero entries. This means that the degree of v in $G[S]$ is at least 2. So $G[S]$ has degree at least 2 everywhere. We see that $G[S]$ contains a cycle.

"$\Leftarrow$" Let C be a cycle contained in $G[S]$. It suffices to show that the columns of $\mathbf{M}_C$ are linearly dependent. In fact, let $P = v_0e_1v_1e_2v_2 \cdots e_nv_n$ with $v_0 = v_n$ the close path on C and choose a direction of C to follow W, then the matrix $\mathbf{M}_C$ is

$$\mathbf{M}_C = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For the case $n = 1$, the cycle C is a loop and $\mathbf{M}_C$ is a zero column. For $n \geq 2$, it is clear that all columns of $\mathbf{M}_C$ add up to zero, so the columns are linearly dependent. $\square$

If one doesn't like chain group, one can simply choose an orientation to have a digraph $D = (G, \omega)$, and define a real-valued **flow** of D to be a function $f : E(G) \to \mathbb{R}$ such that the inflow equals the outflow at every vertex $v \in V(G)$, i.e., satisfying the **circulation law**

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e), \tag{5.1}$$

where $E^+(v)$ ($E^-(v)$) is the set of directed edges of D whose arrows point to (away from) v. If we view each $f : E(G) \to \mathbb{R}$ as a vector whose coordinates are indexed by the members of $E(G)$, the circulation law (5.1) can be stated as the matrix equation

$$\mathbf{Mx} = 0, \quad \mathbf{x} \in \mathbb{R}^E. \tag{5.2}$$

The set of all real-valued flows of (G, ω) forms a subspace of $\mathbb{R}^E$, called the **flow space** of (G, ω), denoted $F(G, \omega)$. The edge space $\mathbb{R}^E$ has a canonical inner product

$$\langle f, g \rangle = \sum_{e \in E(G)} f(\vec{e})g(\vec{e}).$$

A **partial orientation** ω of a graph $G = (V, E)$ is a subset of $\vec{E}$ such that $\omega \cap (-\omega) = \varnothing$. A **coupling** of two partial orientations ω_1, ω_2 of G is a function $[\omega_1, \omega_2] : E(G) \to \mathbb{Z}$, defined by

$$[\omega_1, \omega_2](e) = \begin{cases} 1 & \text{if } \omega_1(e) = \omega_2(e), \\ -1 & \text{if } \omega_1(e) = -\omega_2(e), \\ 0 & \text{otherwise.} \end{cases}$$

Let C be a circuit of G. A **direction** of C is an orientation ω_C on C such that the digraph (C, ω_C) has a head and a tail at each vertex of C. A circuit with a direction is known as **directed circuit**. A **circuit flow** of a digraph (G, ω) is a coupling $[\omega, \omega_C]$, where ω_C is a direction of a circuit C of G.

Lemma 5.2. *Let (G, ω) be a digraph and (C, ω_C) a directed circuit. Then the coupling $[\omega, \omega_C]$ is a flow of (G, ω).*

Proof. The circulation law (5.1) is clearly satisfied at each vertex outside of C. If C is a loop e_0 at a vertex v, then $e_0 \in E^+(v, \omega) \cap E^-(v, \omega)$; the circulation law satisfies at v as

$$\sum_{e \in E^+(v,\omega)} [\omega, \omega_C](e) - \sum_{e \in E^-(v,\omega)} [\omega, \omega_C](e) = [\omega, \omega_C](e_0) - [\omega, \omega_C](e_0) = 0.$$

Let C be a non-loop. For each vertex $v \in C$, let e_1, e_2 be two edges of C at v, with e_1 pointing to v and e_2 pointing away from v. If both e_1, e_2 point to v in ω, then $e_1, e_2 \in E^+(v, \omega)$ and the circulation law satisfies at v as

$$\sum_{e \in E^+(v,\omega)} [\omega, \omega_C](e) - \sum_{e \in E^-(v,\omega)} [\omega, \omega_C](e) = \sum_{e \in E^+(v,\omega)} [\omega, \omega_C](e) = 1 - 1 = 0.$$

If e_1 points to v and e_2 points away from v in ω, then $e_1 \in E^+(v, \omega)$ and $e_2 \in E^-(v, \omega)$; the circulation law satisfies at v as

$$\sum_{e \in E^+(v,\omega)} [\omega, \omega_C](e) - \sum_{e \in E^-(v,\omega)} [\omega, \omega_C](e) = 1 - 1 = 0.$$

It is analogous for other two cases: the case that both e_1, e_2 point away from v in ω, and the case that e_1 points away from v and e_2 points to v in ω. □

Theorem 5.3. *Let F be a spanning forest of a graph G with an oriented graph ω. For each edge $e \in F^c$, let ω_e be a direction of the fundamental circuit $C(F, e)$ such that ω, ω_e agree on e. Then for each abelian group A,*

$$\mathrm{Flow}\,(G, \omega; A) = \bigoplus_{e \in F^c} [\omega, \omega_e] A \cong A^{E(F^c)},$$

where $[\omega, \omega_e]A = \{[\omega, \omega_e]a : a \in A\}$ with $([\omega, \omega_e]a)(e') = [\omega, \omega_e](e')a$ for $e' \in E(G)$. Moreover, for each $f \in \mathrm{Flow}\,(G, \omega; A)$,

$$f = \sum_{e \in F^c} [\omega, \omega_e] f(e).$$

Proof. Let $\sum_{e\in F^c}[\omega,\omega_e]a_e = 0$ for some $a_e \in A$. The support of $[\omega,\omega_e]$ is contained in $F \cup e$. For each $e_0 \in F^c$, we have

$$a_{e_0} = \sum_{e\in F^c}[\omega,\omega_e](e_0)a_e = \left(\sum_{e\in F^c}[\omega,\omega_e]a_e\right)(e_0) = 0.$$

Then $[\omega,\omega_e]a_{e_0} = 0$. Thus Flow $(G,\omega;A)$ is a direct sum of $[\omega,\omega_e]A$, where $e \in F^c$.

Next it suffices to show that $f' := f - \sum_{e\in F^c}[\omega,\omega_e]f(e)$ is identically zero. It is clear that f' is an A-flow of (G,ω), since f and $[\omega,\omega_e]f(e)$ for $e \in F^c$ are A-flows. For each edge $e_0 \in F^c$, we have $f'(e_0) = f(e_0) - \sum_{e\in F^c} f(e)[\omega,\omega_e](e_0) = f(e_0) - f(e_0) = 0$. Thus $f'|_{F^c} = 0$.

Let T be a component of F. If the tree T contains edges, then T contains at least two leaves. Let $e' \in T$ be an edge whose one endpoint is a leaf v of T. The net flow of f' at v is $\pm f'(e')$, which must be zero, since f' is a flow; i.e., $f'(e') = 0$. Continue this procedure, we see that $f'|_T = 0$; consequently, $f'|_F = 0$. We have shown that $f = \sum_{e\in F^c}[\omega,\omega_e]f(e)$. □

Remark. The flow space Flow $(G,\omega;\mathbb{R})$ is the kernel of the incidence matrix $\mathbf{M}$ of the digraph (G,ω), i.e., the solution space of the matrix equation $\mathbf{Mx} = \mathbf{0}$.

6 Tension group and tension space

A **tension** valued in an abelian group A or just A**-tension** of a digraph (G,ω) is a function $g : E(G) \to A$ such that for each directed circuit (C,ω_C),

$$\langle[\omega,\omega_C], g\rangle := \sum_{e\in C}[\omega,\omega_C](e)g(e) = 0.$$

Let Tens $(G,\omega;A)$ denote the set of all A-tensions of (G,ω), which forms an abelian subgroup of A^E, called the A**-tension group** of G with respect to ω. The tension group is called **tension space** if the group A is a field.

Let $U = X, X^c$ be a cut of G. A **direction** of U is an orientation on U such that every arc of the digraph (U,ε_U) has its tail in X and head in X^c; and (U,ε_U) is called a **directed cut** of G. The **indicator function** of U is the characteristic function of the edge subset $E(U)$.

Lemma 6.1. *Let G be a graph with an orientation ω. For a cut $U = [X, X^c]$ of G with a direction ω_U, the coupling $[\omega,\omega_U]$ is an integer-valued tension of digraph (G,ω), called a* **cut tension***, or further a* **bond tension** *if U is a bond.*

Proof. Let C be a circuit of G with an orientation ω_C. Note that $[\omega,\omega_C](e)[\omega,\omega_U](e) = [\omega_C,\omega_U](e)$ fort all $e \in E(G)$. It follows that the inner product

$$\langle[\omega,\omega_C],[\omega,\omega_U]\rangle = \sum_{e\in E(G)}[\omega,\omega_C](e)[\omega,\omega_U](e) = \sum_{e\in C\cap U}[\omega_C,\omega_U](e) = 0,$$

since the number of edges of $C\cap U$ having the same orientation in ω_C, ω_U equals the number of edges having opposite orientations. So $[\omega,\omega_U]$ is a tension of (G,ω) by definition. □

Theorem 6.2. *Let F be a spanning forest of a graph G, and let ω be an orientation of G. For each edge $e \in F$, let ω_e be a direction of the fundamental bond $B(F,e)$ of G such that ω, ω_e agree on e. Then for each abelian group A,*

$$\mathrm{Tens}\,(G,\omega;A) = \bigoplus_{e\in F}[\omega,\omega_e]A \cong A^{E(F)}.$$

Moreover, for each $g \in \mathrm{Tens}\,(G,\omega;A)$,

$$g = \sum_{e\in F}[\omega,\omega_e]g(e).$$

Proof. Let $\sum_{e\in F}[\omega,\omega_e]a_e = 0$ for some $a_e \in A$. The support of $[\omega,\omega_e]$ is contained in $F^c \cup e$. For each $e_0 \in F$, we have

$$a_{e_0} = \sum_{e\in F}[\omega,\omega_e](e_0)a_e = \left(\sum_{e\in F}[\omega,\omega_e]a_e\right)(e_0) = 0.$$

Then $[\omega,\omega_e]a_{e_0} = 0$. Thus $\mathrm{Tens}\,(G,\omega;A)$ is a direct sum of $[\omega,\omega_e]A$, where $e \in F$.

Next it suffices to show that $g' := g - \sum_{e\in F}[\omega,\omega_e]g(e)$ is identically zero. It is clear that f' is an A-tension of (G,ω), since g and $[\omega,\omega_e]g(e)$ for $e \in F$ are A-tensions. For each edge $e_0 \in F$, we have $g'(e_0) = g(e_0) - \sum_{e\in F^c} g(e)[\omega,\omega_e](e_0) = g(e_0) - g(e_0) = 0$. Thus $g'|_F = 0$.

For each $e' \in F^c$, let $\omega_{e'}$ be a direction of the fundamental circuit $C(F,e')$. By definition of tension, we have

$$g'(e') = \langle[\omega,\omega_{e'}], g'\rangle = 0.$$

This means that $g'|_{T^c} = 0$. We have shown that $g = \sum_{e\in F}[\omega,\omega_e]g(e)$. □

Corollary 6.3. *The flow space* $\mathrm{Flow}\,(G,\omega;\mathbb{R})$ *and the tension space* $\mathrm{Tens}\,(G,\omega;\mathbb{R})$ *are orthogonal complement of each other in $\mathbb{R}^{E(G)}$. In particular,*

$$\mathbb{R}^{E(G)} = \mathrm{Flow}\,(G,\omega;\mathbb{R}) \oplus \mathrm{Tens}\,(G,\omega;\mathbb{R}).$$

However, $\mathrm{Flow}\,(G,\omega;\mathbb{Z}) \oplus \mathrm{Tens}\,(G,\omega;\mathbb{Z})$ *is a full rank lattice of $\mathbb{Z}^{E(G)}$.*

Proof. Trivial. □

7 Cycle Group and Cocycle Group

A **chain** of a graph $G = (V,E)$ is a function $c : \vec{E}(G) \to \mathbb{Z}$ such that $c(-e) = -c(e)$ for $e \in \vec{E}(G)$. Given an orientation ω, a chain c is usually written as

$$c = \sum_{e\in\omega} c(e)e.$$

There is a bilinear **pairing** $\langle,\rangle : C_1(G) \times C_1(G) \to \mathbb{Z}$, defined by

$$\langle c_1, c_2\rangle = \sum_{e\in\omega} c_1(e)c_2(e). \tag{7.1}$$

Set of all chains forms an abelian group $C_1(G)$. Let $C_0(G)$ be the group of all integer-valued functions defined on $V(G)$. There is a **boundary operator** $\partial : C_1(G) \to C_0(G)$, defined by

$$\partial c = \sum_{e \in \omega} c(e) \partial e, \tag{7.2}$$

where $\partial e = u - v$ if $e = \vec{uv}$, i.e., the oriented edge e has its tail at u and heat at v. It is clear that ∂ is a group homomorphism so that $\ker \partial$ is a subgroup of $C_1(G)$, called the **cycle group** of G, denoted $Z(G)$.

There is a group homomorphism $\delta : C_0(G) \to C_1(G)$, called the **co-boundary** (or **difference**) **operator** of G, defined for each oriented edge $e = \vec{uv}$ by

$$(\delta p)(e) = p(u) - p(v). \tag{7.3}$$

A chain $c \in C_1(G)$ is called a **tension** of G if for every cycle C and its direction ω_C,

$$\langle c, \omega_C \rangle = 0. \tag{7.4}$$

The set of all tensions of G forms a subgroup of $C_1(G)$, called the **tension group** of G, denoted $T(G)$.

Lemma 7.1. *For each potential function $p : V(G) \to \mathbb{Z}$, the image δp is a tension of G.*

Proof. For each direction ω_C of a cycle C, the direction ω_C can be represented by a path closed path $v_0 e_1 v_1 e_1 \cdots e_n v_n$ with $v_0 = v_n$. Let ω be an orientation of G such that ω agrees ω_C on C. Then

$$\langle \delta p, \omega_C \rangle = \sum_{e \in \omega} (\delta p)(e) \omega_C(e) = \sum_{e \in \omega_C} (\delta p)(e) = \sum_{i=1}^{n} \delta p(e_i) = \sum_{i=1}^{n} [v_{i-1} - v_i] = 0.$$

This means that δp is a tension. □

Theorem 7.2. $T(G) = \operatorname{im} \delta$.

Proof. Tension Lemma 7.1 implies $\operatorname{im} \delta \subseteq T(G)$. For each tension $g \in T(G)$, we construct a potential p such that $\delta p = g$. We may assume that G is connected. Fix a vertex v_0 and set $p(v_0)$ arbitrarily.

Now for each vertex v, let $W = v_0 e_1 v_1 e_2 \cdots e_m v_m$ be a shortest path from v_0 to $v_m = v$. Assume that p have been defined on $v_0, v_1, \ldots, v_{m-1}$. We then define p at v as

$$p(v_m) = p(v_0) + \sum_{i=1}^{m} g(e_i).$$

We see that $g(e_i) = p(v_i) - p(v_{i-1})$, $i = 1, \ldots, m$.

To show that p is well-defined at v. Let $W' = v'_0 e'_1 v'_1 e'_2 \cdots e'_n v'_n$ be another shortest path from $v_0 = v'_0$ to $v = v'_n$. Then

$$p(v'_n) = p(v'_0) + \sum_{j=1}^{n} g(e'_j).$$

Note that $W'W^{-1}$ is a closed path, its edges forms a cycle C with direction $W'W^{-1}$. Thus

$$p(v_n') - p(v_n) = \sum_{j=1}^{n} g(e_j') - \sum_{i=1}^{m} g(e_i) = \sum_{j=1}^{n} g(e_j') + \sum_{i=1}^{m} g(-e_i) = \sum_{e \in W'W^{-1}} g(e) = 0.$$

Hence p is well-defined at v. It is clear that $\delta p = g$ by definition of p. □

Question. The cycle group $Z(G)$ and the cocycle group $T(G)$ are subgroups of $C_1(G)$, and $Z(G) + T(G)$ is a full rank subgroup. What is the cardinality $|C_1(G)/(Z(G) + T(G))|$?

8 Cayley's Labeled Tree formula

Theorem 8.1 (Cayley's Labeled Tree Formula). *Let $t(K_n)$ denote the number of labeled spanning trees of the complete graph K_n on n vertices. Then $t(K_n) = n^{n-2}$.*

Proof. Recall that a **labeled branching** is an oriented labeled rooted tree such that there is no edge head at the root and exactly one edge head at each of other vertices. Since each labeled tree of K_n gives rise to n labeled branchings, the number of labeled branchings of K_n is $n \cdot t(K_n)$. It suffices to show that the number of labeled branchings of K_n is n^{n-1}.

Note that each labeled branching on n vertices can be build up, one at a time to add an edge, starting with the empty graph on n vertices. In order to end up with a branching, the subgraph constructed at each stage must be a branching forest (each of its components is a branching). Initially, this branching forest has n components, each consists of an isolated vertex. At each stage, we add a new directed edge $\vec{e}$ joining a vertex u of one branching to the root v of another branching with $\vec{e} = \overrightarrow{uv}$, resulting that the number of components decreases exactly by one. If there are k components at a stage, where $2 \le k \le n$, the number of ways to add a new edge $\vec{e} = \overrightarrow{uv}$ is $n(k-1)$: u can be any one of the n vertices, whereas v must be the root of a branching that does not contain the vertex u; there are exactly $k-1$ choices for v. Thus the total number of ways of constructing a labeled branching on n vertices in this ways is

$$\prod_{k=2}^{n} n(k-1) = n^{n-1}(n-1)!.$$

On the other hand, each individual labeled branching on n vertices contains exactly $n-1$ edges, and adding these $n-1$ edges in any order results the same labeled branching. So each labeled branching can be constructed in exactly $(n-1)!$ ways. It follows that the number of labeled branchings is n^{n-1}. □

Proposition 8.2. *Let G be a connected graph and e a link edge. Let $t(G)$ denote the number of spanning trees of labeled graph G. Then*

$$t(G) = t(G \backslash e) + t(G/e).$$

Proof. The spanning trees of G that do not contain the edges e are exactly the spanning trees of $G \backslash e$. The spanning trees T of G that do contain the edges e are exactly the spanning trees T/e of G/e. The formula follows immediately. □

Exercises. Ch4: 4.3.1; 4.3.2; 4.3.3; 4.3.7; 4.3.8; 4.3.9; 4.3.10.

9 Laplacian matrix

Let $G = (V, E)$ be a graph with adjacency matrix $\mathbf{A}$ and diagonal degree matrix $\mathbf{D}$. Recall that each entry a_{uv} of $\mathbf{A}$ is the number of edges between u and v, and each diagonal entry $\deg_G(v)$ of $\mathbf{D}$ is the number edges at v, where each loop is counted twice. Let $\mathbf{M}$ be the incidence matrix of a digraph (G, ω). The **Laplacian** of G is

$$\mathbf{L} := \mathbf{M}\mathbf{M}^T = \mathbf{D} - \mathbf{A}.$$

Choose a vertex v, let $\mathbf{M}_v$ denote the submatrix obtained from $\mathbf{M}$ by deleting the v-row. The **truncated Laplacian** of G relative to v is the matrix

$$\mathbf{L}_v := \mathbf{M}_v {\mathbf{M}_v}^T,$$

which is the submatrix obtained from $\mathbf{L}$ by deleting the v-row and v-column.

Theorem 9.1. *For each vertex v of a connected graph G,*

$$\det \mathbf{L}_v = t(G).$$

Example 9.1. The complete graph K_n on the vertex set $\{1, 2, \ldots, n\}$ has the Laplacian

$$\mathbf{L} = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}_{n\times n}.$$

Deleting the last row and column, we obtain an $(n-1) \times (n-1)$ matrix $\mathbf{L}_v$. Adding up all rows of $\mathbf{L}_v$ other than the top row to the top, then adding the top row to each row row below, we have $(n-1) \times (n-1)$ matrices

$$\mathbf{L}_v \rightsquigarrow \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix}_{(n-1)\times(n-1)}.$$

We see that $\det \mathbf{L}_v = n^{n-2}$.

A **unimodular matrix** is a square integer matrix having determinant either 1 or -1. A **totally unimodular matrix** is a rectangular integer matrix whose all square submatrices have determinant 1 or 0 or -1.

Lemma 9.2. *The incidence matrix of any digraph is totally unimodular.*

Proof. We prove it by induction on the size of square submatrices. For 1×1 submatrix, there is one vertex v and one edge e; if v is not an endpoint of e or e is a loop, then the 1×1 submatrix is 0; if v is an endpoint of e and e is not a loop, then the submatrix is of the form $[\pm 1]$. It is true clearly in all cases.

Let $\mathbf{M}_1$ be a square submatrix indexed by a vertex subset V_1 and edge subset E_1 with $|V_1| = |E_1| \geq 2$. If there exists a vertex $v \in V_1$ not incident with any edge of E_1, then the v-row of $\mathbf{M}_1$ is zero; consequently, $\det \mathbf{M}_1 = 0$. We may assume that all vertices of V_1 are incident with some edges of E_1. If there exists a loop $e \in E_1$ or an edge $e \in E_1$ not incident with any vertex in V_1, then the e-column is zero; consequently, $\det M = 0$. If there exists an edge $e \in E_1$ incident with only one vertex v in V_1 and e is not a loop, then the e-column has only one nonzero entry 1 or -1. Using cofactor expansion formula for determinant, we see that $\det \mathbf{M}_1$ is 1 or 0 or -1 by induction.

Now we may assume that endpoints of edges of E_1 are contained in V_1. Then $G_1 = (V_1, E_1)$ is a subgraph. If G_1 contains a cycle, then the columns of $\mathbf{M}_1$ are linearly dependent, so $\det \mathbf{M}_1 = 0$. If G_1 contains no cycle, i.e., G_1 is a forest. Then $|E_1| = |V_1| - c(G_1)$, which is contradictory to $|V_1| = |E_1|$. □

Proposition 9.3 (Cauchy-Binet Formula). *Let A be an $m \times n$ matrix and B an $n \times m$ matrix. If $m \leq n$, then*

$$\det(AB) = \sum_{S \subseteq [n],\, |S|=m} \det(A|_S) \det(B|_S), \tag{9.1}$$

where $[n] = \{1, 2, \ldots, n\}$, $A|_S$ is the $m \times m$ submatrix of A whose column index set is S, and $B|_S$ is the $m \times m$ submatrix of B whose row index set is S.

Proof. Let $A = [a_{ik}]_{m\times n}$, $B = [b_{kj}]_{n\times m}$, and $C = AB = [c_{ij}]_{m\times m}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Then

$$\begin{aligned}
\det(C) &= \det \begin{pmatrix} \sum_{k_1=1}^{n} a_{1k_1} b_{k_1 1} & \cdots & \sum_{k_m=1}^{n} a_{1k_m} b_{k_m m} \\ \vdots & \ddots & \vdots \\ \sum_{k_1=1}^{n} a_{mk_1} b_{k_1 1} & \cdots & \sum_{k_m=1}^{n} a_{mk_m} b_{k_m m} \end{pmatrix} \\
&= \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} \det \begin{pmatrix} a_{1k_1} b_{k_1 1} & \cdots & a_{1k_m} b_{k_m m} \\ \vdots & \ddots & \vdots \\ a_{mk_1} b_{k_1 1} & \cdots & a_{mk_m} b_{k_m m} \end{pmatrix} \\
&= \sum_{k_1,\ldots,k_m=1}^{n} \det \begin{pmatrix} a_{1k_1} & \cdots & a_{1k_m} \\ \vdots & \ddots & \vdots \\ a_{mk_1} & \cdots & a_{mk_m} \end{pmatrix} b_{k_1 1} \cdots b_{k_m m}.
\end{aligned}$$

Rewrite the nonzero terms in the above expansion of $\det(C)$, we obtain

$$\begin{aligned}
\det(C) &= \sum_{1 \leq k_1, \ldots, k_m \leq n,\, k_i \neq k_j} \det(A|_{\{k_1,\ldots,k_m\}})\, b_{k_1 1} \cdots b_{k_m m} \\
&= \sum_{1 \leq t_1 < \cdots < t_m \leq n} \sum_{\sigma \in S_m} \det(A|_{\{t_{\sigma(1)},\ldots,t_{\sigma(m)}\}})\, b_{t_{\sigma(1)} 1} \cdots b_{t_{\sigma(m)} m} \\
&= \sum_{1 \leq t_1 < \cdots < t_m \leq n} \det(A|_{\{t_1,\ldots,t_m\}}) \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma)\, b_{t_{\sigma(1)} 1} \cdots b_{t_{\sigma(m)} m},
\end{aligned}$$

where $\mathfrak{S}_m$ is the set of all permutations of $\{1, \ldots, m\}$. Set $S = \{t_1, \ldots, t_m\}$, we have $\det(C) = \sum_{S \subseteq [n],\, |S|=m} \det(A|_S) \det(B|_S)$. □