Week 5: Separation of Graphs

October 7, 2020

1 Cut Vertices

- A **cut vertex** of a graph G is a vertex v such that when v and the edges at v are removed, the number of components increases, i.e., $c(G) < c(G \smallsetminus v)$, where $G \smallsetminus v$ is the graph obtained from G by removing v and the edges at v.
- A **cut edge** of a graph G is an edge whose removal increases the number of components.
- A connected graph is **2-connected** if between any two vertices there are at least two internally disjoint paths. A connected graph with only one vertex is either a single loop or a single vertex.
- Let G be a connected graph without loops. Then G is 2-connected if and only if any two edges of G are contained in a cycle of G.

  Proof. The sufficiency is trivial. Given two distinct vertices u, v of G and edges e_1, e_2 at u, v respectively. Let C be a cycle containing e_1, e_2. Clearly, the cycle C contains vertices u, v and can be split into two internally disjoint paths from u to v.

  The necessity is left as a nontrivial exercise. $\square$

Theorem 1.1. *Let G be a connected graph with $n \geq 3$ vertices. Then G is 2-connected if and only if G contains no cut vertex.*

Proof. Since $|V(G)| \geq 3$, G being 2-connected is equivalent to saying that there exists two internally disjoint paths between any two vertices.

"$\Rightarrow$" Suppose G contains a cut vertex v. Then $G \smallsetminus v$ contains at least two components G_1 and G_2. Given vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Each path connecting v_1 and v_2 must pass through the vertex v. So any two paths between v_1 and v_2 must contain the common internal vertex v. This is a contradiction.

"$\Leftarrow$" Let u, v be two vertices of G and let $d(u, v)$ denote their distance. We proceed by induction on $d(x, y)$. For $d(u, v) = 1$, there exists an edge $e = uv$ in G. Note that e is not a cut edge. (Otherwise, since between u, v there exist two internally disjoint paths, there exists a third vertex w adjacent either to u or to v, say, adjacent to u. Then u is a cut vertex of G, a contradiction.) The subgraph $G \smallsetminus e$ is connected. Any uv-path in $G \smallsetminus e$ and uev are two internally disjoint paths from u to v.

Consider the case $d(u, v) = d \geq 2$. Let $P = v_0 e_1 v_1 \ldots e_d v_d$ be a path from $u (= v_0)$ to

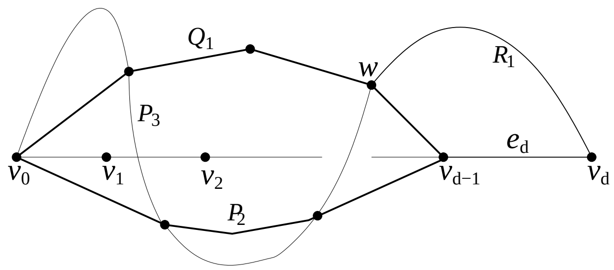

Figure 1: No cut vertex via common cycle of two vertices

$v(= v_d)$. Since $d(v_0, v_{d-1}) = d-1$, by induction there are two internally disjoint paths P_1, P_2 from v_0 to v_{d-1} in G. Since v_{d-1} is not a cut vertex of G, the subgraph $G \setminus v_{d-1}$ is connected. Let P_3 be a path from v_0 to v_d in $G \setminus v_{d-1}$. Let w be the last vertex of P_3 that meets $P_1 \cup P_2$, say, w lies in P_1. Let Q_1 denote the subpath of P_1 from v_0 to w, and R_1 the subpath of P_3 from w to v_d. Then $P_1' = Q_1 R_1$ and $P_2' = P_2 e_d v_d$ are two internally disjoint paths from u to v, see Figure 2. $\square$

2 Separation and Blocks

- A **separation** of a connected graph G is a decomposition of G into two connected subgraphs G_1 and G_2 having a unique common vertex v and disjoint nonempty edge sets; the common vertex v is called a **separating vertex** of G. A connected graph is **separable** if it contains at least one separating vertex; otherwise, it is **nonseparable**.

- A cut vertex is a separating vertex. A separating vertex is not necessarily a cut vertex. If v is a separating vertex but not a cut vertex, then there exists at least one loop at v. So if a graph has no loop at a vertex v, then v is a separating vertex if and only if v is a cut vertex.

- If a graph G is nonseparable and is not a single loop, then G contains no loops.

- A **block** of a connected graph G is a maximal nonseparable subgraph. If G contains some edges, i.e., G is not a single vertex, then each block of G is either a loop, or a cut edge, or a maximal 2-connected subgraph without loops.

Theorem 2.1. *Let G be a connected graph. Then G is nonseparable if and only if any two edges lie on a common cycle.*

Proof. "$\Leftarrow$" Suppose G is separable, i.e., G is separated at a vertex v into two connected subgraphs G_1 and G_2, such that $V(G_1) \cap V(G_2) = \{v\}$, $E(G_1) \cap E(G_2) = \varnothing$, and $E(G_i) \neq \varnothing$, $i = 1, 2$. Take edges $e_i = vv_i$ at v with $e_i \in G_i$, $i = 1, 2$. Clearly, the edges e_1, e_2 cannot be on a common cycle of G, a contradiction.

"$\Rightarrow$" If G has only one vertex, then G is a single vertex or a single loop; nothing is to be proved. Let G contain at least two vertices. Clearly, G contains no loops, i.e., all edges are links. If G is a single link edge, then again nothing is to be proved. We assume that G contain at least two edges. Given an edge $e = uv$, subdivide e into two edges by introducing a new vertex w on e to obtain a new graph G'. We claim that G' is also nonseparable. In fact, suppose G' is separable. Then G' must be separated at the vertex w into two connected subgraphs G'_u, G'_v with $u \in G'_u, v \in G'_v$. Being a middle point of e, the vertex w has to be a cut vertex of G'; consequently, the edge e is a cut edge of G. Thus $G \setminus e$ has two connected components G_u, G_v with $u \in G_u, v \in G_v$. Since G has at leat two edges, either G_u has some edges at u or G_v has some edges at v, say, it is the former case. We see that G is separated at u into G_u and $G_v \cup e \cup u$, which contradicts the non-separablility of G. So G' is nonseparable.

Now given two edges $e_i = u_i v_i$ of G, $i = 1, 2$. Subdivide e_i by introducing a new vertex w_i on e_i to obtain a new graph H. Then H is nonseparable and has at least 3 vertices. Since nonseparable graphs have no cut vertices, by Theorem 1.1 there exist two internally disjoint paths P_1, P_2 from w_1 to w_2. The closed path $P_1 P_2^{-1}$ forms a cycle of G, which contains both edges e_1, e_2. □

Corollary 2.2 (Classification of Nonseparable Graphs). *A nonseparable graph is either a single vertex, or a single loop, or a single link, or a 2-connected graph having at least two vertices and no loops.*

Theorem 2.3 (Block-Tree Decomposition). *Every connected graph G can be decomposed into blocks satisfying the following properties:*

(a) Any two blocks of G have at most one vertex in common.

(b) Every cycle is contained in a block of G.

(c) There is no sequence $B_0, B_1, \ldots, B_k$ of blocks of G such that $B_i \cap B_{i+1} \neq \varnothing$ for $i = 0, 1, \ldots, k$, where $k \geq 1$ and $B_{k+1} = B_0$.

Proof. (a) Let B_1, B_2 be two blocks having vertices $v_1, \ldots, v_k$ in common. It is easy to see that both B_1, B_2 have no loops. So $B := B_1 \cup B_2$ has no loops; consequently, each separating vertex of B is a cut vertex. Since B_1, B_2 are maximal nonseparable subgraphs of G, it follows that B is separable. Let B be separated at a vertex v into G_1, G_2. Note that one of $G_1 \cap B_1, G_2 \cap B_1$ contains the other; otherwise, B_1 has a separation at v into $G_1 \cap B_1$ and $G_2 \cap B_1$, contradictory to the non-separability of B_1. Thus either $G_2 \cap B_1 \subseteq G_1 \cap B_1$ or $G_1 \cap B_1 \subseteq G_2 \cap B_1$, i.e., either $B_1 \subseteq G_1$ or $B_1 \subseteq G_2$. Likewise, we have either $B_2 \subseteq G_1$ or $B_2 \subseteq G_2$. Say, $B_1 \subseteq G_1$, we must have $B_2 \subseteq G_2$. Since $B_1 \cup B_2 = G_1 \cup G_2$, it follows that $B_1 = G_1$, $B_2 = G_2$, and $k = 1$ with $v = v_1$. We have seen that B_1, B_2 have only one vertex in common.

(b) and (c) are equivalent. We prove (c). Suppose there is a sequence $B_0, B_1, \ldots, B_k$ of blocks such that $B_i \cap B_{i+1} = \{v_i\}$, $i = 0, 1, \ldots, k$, where $B_{k+1} = B_0$ and $k \geq 1$. Then $B := \bigcup_{i=0}^k B_i$ is connected and cannot be separated at its any vertex. So B is a block, contradict to the maximality of B_i. □

- Let $\mathcal{B}$ be the set of all blocks of a connected graph G, and S the set of separating vertices. Denote by $\mathrm{BT}(G)$ the bipartite graph whose vertex set has the bipartition $\{S, \mathcal{B}\}$, and whose edges are the pairs $\{v, b\}$, where $v \in S$, $v \in b \in \mathcal{B}$. Then $\mathrm{BT}(G)$ is a tree, known as the **block tree** of G.
- An **end block** of G is a block corresponding to a leaf of the block tree $\mathrm{BT}(G)$.
- Any vertex of a block of G other than the separating vertices is called an **internal vertex** of the block.

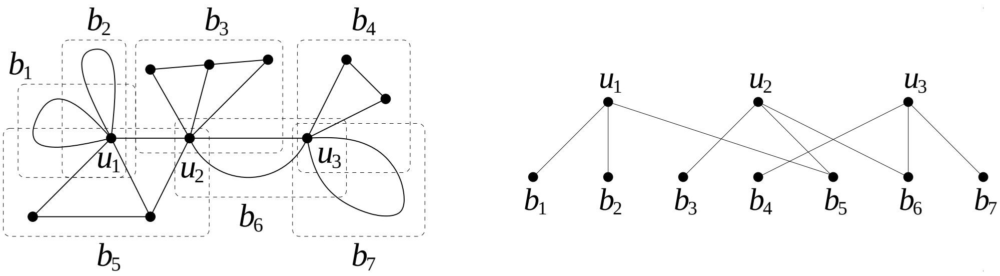

Figure 2: A block-tree decomposition and its block tree

3 Ear Decomposition

- Every nonseparable graph other than a single vertex or a link contains a cycle.
- Given a subgraph H of a graph G. An **ear** of H is a path P in G such that P is not a closed path, its initial and terminal vertices lie in H, and its edges and internal vertices lie outside H.

Proposition 3.1. *Let H be a subgraph of a nonseparable graph G. If H is not a trivial subgraph (i.e. neither a single vertex nor the whole graph G), then H has an ear in G.*

Proof. If H is a spanning subgraph, then $E(H) \subsetneq E(G)$ is a proper subset, thus each edge $e \in E(G) \smallsetminus E(H)$ is an ear of H in G. If H is not a spanning subgraph, then there exists an edge $e = uv$ with $u \in H$ and $v \in G \smallsetminus H$, since G is connected. Moreover, u is connected to some other vertices in H, as H is not a single vertex. Since G is nonseparable, the edge e cannot be a cut edge of G, so $G \smallsetminus e$ is connected. Then there exist (v, H)-paths in G. We claim that one of such (v, H)-paths is from v to a vertex w other than u. In fact, if

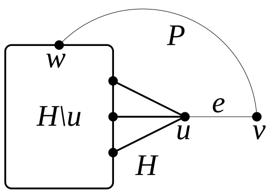

Figure 3: Existence of an ear

all (v, H)-paths are from v to u, then u is a separating vertex of G, contradictory to the non-separability of G. Each (v, H)-path P from v to a vertex w other than u is an ear of H in G. See Figure 3. $\square$

Proposition 3.2. *Let H be a subgraph of a graph G with an ear P in G. If H is nonseparable, then $H \cup P$ is also nonseparable.*

Proof. We may assume $G = H \cup P$. Suppose G is separated at a vertex v into G_1 and G_2, no one is contained in another and have exactly one vertex v in common. Since H is nonseparable, by block-tree decomposition H must be contained in either G_1 or G_2, say, $H \subseteq G_1$. Then $G_2 \subseteq P$. So G_2 is a subpath of P. This is impossible. $\square$

A sequence $G_0, G_1, \ldots, G_k$ of graphs is said to be **nested** if $G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k$. An **ear decomposition** of a nonseparable graph G is a nested sequence $G_0, G_1, \ldots, G_k$ of subgraphs of G such that

(ED1) G_0 is a cycle,

(ED2) $G_{i+1} = G_i \cup P_i$, where P_i is an ear of G_i in G for all $i = 0, 1, \ldots, k-1$, and

(ED3) $G_k = G$.

Theorem 3.3 (Ear Decomposition Theorem)**.** *Let G be a nonseparable graph. If G is neither a single vertex nor a link, then G has an ear decomposition.*

Proof. It is trivial when G is a single loop, just take G_0 to the loop. We consider that G is not a single vertex, not a single link, and not a single loop. Then G must contain at least two edges. Theorem 2.1 implies that the two edges lie on a common cycle C, which has at least two vertices. Set $G_0 := C$. If $G_0 \subsetneq G$ is a proper subgraph, then Proposition 3.1 implies that G_0 has an ear P_0 in G, since G_0 is not a single vertex. Thus $G_1 := G_0 \cup P_0$ is nonseparable. Likewise, if G_1 is a proper subgraph of G, then G_1 has an ear P_1 in G. Continue this procedure, we obtain nonseparable subgraphs G_i and their ears P_i in G such that $G_{i+1} := G_i \cup P_i$. Since G is finite, the procedure must end up with $G_k = G$ at some step k. □

Recall that a digraph D is **strongly connected** (or just **strong**) if for each proper vertex subset $X \subsetneq V(D)$, the set (X, X^c), consisting of all arcs having tails in X and heads in X^c, is nonempty. Note that (X^c, X) is also nonempty.

Proposition 3.4. *A digraph D is strongly connected if and only if for any two vertices u and v in D, there exist a directed path from u to v and a directed path from v to u, i.e., u and v are strongly connected.*

Proof. The sufficiency is trivial. For necessity, fix two vertices u and v, let V_u denote the set of all vertices w such that there exists a directed path (allowing zero length) from u to w in D. Then for each $w' \in V_u^c$ there is no directed path from u to w'. Clearly, $u \in V_u \neq \varnothing$.

If $V_u \neq V(D)$, then $(V_u, V_u^c) \neq \varnothing$, since D is strongly connected. Take an arc $e = w_1 w_2 \in (V_u, V_u^c)$ with $w_1 \in V_u$ and $w_2 \in V_u^c$, and directed path P from u to w_1. Then $Q := Pew_2$ is a directed path from u to w_2, contradictory to the fact that there is no directed path from u to w_2. Hence $V_u = V(D)$. Likewise, $V_v = V(D)$. This means that u and v are strongly connected. □

Proposition 3.5. *A connected digraph is strongly connected if and only if each of its blocks is strongly connected.*

Proof. Trivial. □

Proposition 3.6. *Let P be an ear of a sub-digraph H in a digraph D (viewed as their underlying graphs). If H is strongly connected and P is a directed path, then $H \cup P$ is also strongly connected.*

Proof. Let $P = v_0 e_1 v_1 \cdots e_n v_n$ be directed path from v_0 to v_n. For two vertices u and v of $H \cup P$, if $u, v \in H$, nothing is to be proved, since H is strongly connected. If $u, v \in P$, say, $u = v_i$ and $v = v_j$ with $i < j$, take a directed path P_0 in H from v_n to v_0. Write $P = P_1 P_2 P_3$, where P_2 is the directed subpath from v_i to v_j. Then $Q := P_3 P_0 P_1$ is a directed path from v_j to v_i. So u and v are strongly connected.

If $u \in P$ and $v \in H$ with $u = v_i$, take a directed path P_0 in H from v_n to v and a directed path Q_0 in H from v to v_0. Then $v_i e_{i+1} v_{i+1} \cdots e_n P_0$ is a directed path from u to v, and $Q_0 e_1 v_1 \cdots e_i v_i$ is a directed path from v to u. Thus u and v are strongly connected.

Proposition 3.4 implies that $H \cup P$ is strongly connected. □

Theorem 3.7. *Every connected graph G without cut edge has a strong orientation.*

Proof. It suffices to show that each block of G has a strong orientation. We may assume that G is nonseparable. If G is a single vertex or a loop, it is trivial that G has a strong orientation. Note that G cannot be a single link. So we assume that G is not a vertex, not a link, and not a loop. The Ear Decomposition Theorem 3.3 implies that G has an ear decomposition $(G_0, G_1, \ldots, G_k)$, where $G_{i+1} = G_i \cup P_i$ and P_i is an ear of G_i in G, $i = 0, 1, \ldots, k-1$. We now orient the edges of G_0 and P_i so that G_0 becomes a directed cycle and P_i becomes directed paths. Initially, G_0 is strongly connected. Proposition 3.6 implies that all $G_{i+1} = G_i \cup P_i$ are strongly connected. Hence $G = G_k$ is strongly connected. □

4 Ear Decomposition of Digraphs

A **directed ear** of a sub-digraph H in a digraph D is a directed path P whose distinct initial and terminal vertices lie in H and internal vertices lie outside H.

Proposition 4.1. *Let H be a sub-digraph of a nonseparable and strongly connected digraph D. If H is strongly connected, but not a vertex, not a loop, and not the whole digraph D, then H has a directed ear in D.*

Proof. Since D is nonseparable and H is nontrivial subgraph of D, then H has some ears in D by Proposition 3.1. Among these ears we choose one ear P having minimum number of reversing arcs relative to the order of P. We claim that such an ear P is actually a directed ear of H in D. Write P as the sequence $v_0e_1v_1\cdots e_kv_k$ of vertices and arcs in D. If $k=1$, then either P or P^{-1} is a directed ear of H in D.

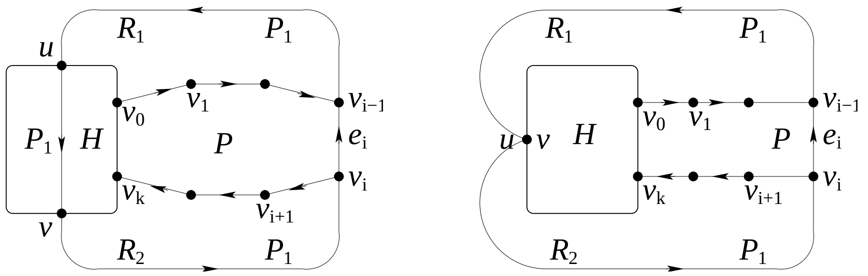

Figure 4: Existence of directed ear

Suppose P is not a directed path. Clearly, $k \geq 2$. Let e_i be an arc having its tail at v_i and head at v_{i-1}. Then one of v_i and v_{i-1} is outside H. Since D is strongly connected, there is a directed path P_1 from v_{i-1} to v_i in D. Then

$$Q_1 := v_0e_1v_1\cdots e_{i-1}P_1e_{i+1}v_{i+1}\cdots e_kv_k$$

is a walk from v_0 to v_k, having less number of reversing arcs comparing with the walk P.

Suppose $Q_1 \cap H = \varnothing$. Then Q_1 is disjoint from H, except $v_0, v_k \in H$. Let Q be a path followed the walk Q_1 from v_0 to v_k, having no repeating vertices. Then Q is a directed ear of H in D, having less number of reversing arcs comparing with P, which is contradictory to the choice of P. So we must have $Q_1 \cap H \neq \varnothing$.

Let u be the first vertex and v the last vertex of Q_1 such that $u, v \in H$. Let R_1 be the subpath of Q_1 from v_{i-1} to u, and R_2 the subpath of Q_1 from v to v_i. If $u \neq v$, then $Q := R_2e_iR_1$ is a directed ear of H in D; see the left of Figure 4 below. If $u = v$, we have two cases: $u \neq v_0$ and $u \neq v_k$. In the former case, the directed walk $v_0e_1v_1\cdots v_{i-2}e_{i-1}R_1$ contains a directed ear Q of H in D. In the latter case, the directed walk $R_2e_{i+1}v_{i+1}\cdots e_kv_k$ contains a directed ear Q of H in D. See the right of Figure 4. In both cases the directed path Q has less number of reversing arcs comparing with P, which is contradictory to the choice of P. □

A **directed ear decomposition** of a nonseparable strong digraph D is a nested sequence $D_0, D_1, \ldots, D_k$ of nonseparable strong sub-digraphs of D such that

(DED1) D_0 is a directed cycle,

(DED2) $D_{i+1} = D_i \cup P_i$, where P_i is a directed ear of D_i in D, $i = 0, 1, \ldots, k-1$, and

(DED3) $D_k = D$.

Theorem 4.2. *Let D be a nonseparable and strongly directed digraph. If D is nontrivial, i.e., not a single vertex, then D has a directed ear decomposition.*

Proof. It is trivial when D is a directed loop. When D is not a loop, D must have at least two vertices, say, u and v. Let P be a directed path from u to v and Q a directed path from v to u in D. Then $W := PQ^{-1}$ is a closed directed walk containing both u and v. Of course, W contains a directed cycle D_0, whose all edges are links. Clearly, D_0 is nonseparable and strongly connected. If $D_0 \subsetneq D$, then D_0 has a directed ear P_0 in D by Proposition 4.1, and $D_1 := D_0 \cup P_0$ is nonseparable and strongly connected by Propositions 3.2 and 3.6. Continue this procedure, we obtain a nested sequence $D_0, D_1, \ldots, D_k$ of nonseparable and strongly connected sub-digraphs of D such that $D_{i+1} = D_i \cup P_i$, where P_i is a directed ear of D_i in D for $i = 0, 1, \ldots, k-1$, and $D_k = D$. This is a directed ear decomposition of D. $\square$

- A **feedback set** of a digraph D is an arc subset S of D such that $D \smallsetminus S$ contains no directed cycle.
- A feedback set S of a digraph D is **minimal** if for each arc $e \in S$ the sub-digraph $(D \smallsetminus S) \cup e$ contains some directed cycles. Each such directed cycle intersects S at the unique arc e, and is called a **fundamental directed cycle** of D with respect to S.
- A minimal feedback set S of a digraph D is **coherent** if each arc of D is contained in some fundamental directed cycles of D with respect to S.
- If a digraph D admits a coherent feedback set, then every component of D must be strongly connected, for every edge is contained in a fundamental directed cycle and the union of directed cycles is obviously strongly connected.

Theorem 4.3. *Let D be a nontrivial (i.e. not a single vertex) connected digraph. Then D is strongly connected if and only if D admits a coherent feedback set.*

Proof. The sufficiency is trivial. For necessity, let D be a strongly connected digraph with at least two vertices. If D is separable, then each its block is strongly connected, and we may consider each of its blocks. So we may assume that D is nonseparable. By Theorem 4.2, D has a directed ear decomposition $D_0, D_1, \ldots, D_k$, where D_0 is a directed cycle, $D_{i+1} = D_i \cup P_i$, P_i is a directed ear of D_i in D, $i = 0, 1, \ldots, k-1$, and $D_k = D$. Now choose an arc e_0 from D_0 and set $S_0 := \{e_0\}$. If $D_1 \smallsetminus S_0$ contains no directed cycles, set $S_1 := S_0$. If $D_1 \smallsetminus S_0$ contains a directed cycle, then the directed cycle must contain the directed path P_0; choose an edge $e_1 \in P_0$ and set $S_1 := S_0 \cup e_1$. Thus $D_1 \smallsetminus S_1$ contains no directed cycles.

In general for $i \geq 1$, the digraph $D_{i-1} \smallsetminus S_{i-1}$ contains no directed cycle, and $D_i \smallsetminus S_{i-1}$ may and may not contain directed cycles. If $D_i \smallsetminus S_{i-1}$ contains a directed cycle C, then C must contain the directed path P_{i-1}. Let

$$S_i = \begin{cases} S_{i-1} & \text{if } D_i \smallsetminus S_{i-1} \text{ contains no directed cycle,} \\ S_{i-1} \cup e_i & \text{otherwise, choose an edge } e_i \in P_{i-1}. \end{cases}$$

We see that $D_i \smallsetminus S_i$ contains no directed cycle. Finally, we have constructed a coherent feedback set $S = S_k$ for D. $\square$

Proposition 4.4. *Every strongly connected digraph D has a strongly connected spanning sub-digraph of at most* $2\big(|V(D)|-1\big)$ *arcs.*

Proof. Delete all loops of D if necessary. We may assume that D contains no loops. If D is a single vertex, it is clearly true. We assume that D has at least two vertices and no loops. Then each block of D is strongly connected, containing at least two vertices. For each block B of D, consider a directed ear decomposition of B. Delete from B the edges in the directed ears of length one. We obtain a nonseparable and strongly connected spanning sub-digraph H of B, and a directed ear decomposition $D_0, D_1, \ldots, D_k$ of H, where D_0 is a directed cycle, $D_{i+1} = D_i \cup P_i$, P_i is a directed ear of D_i in D of length at least two, $i = 0, 1, \ldots, k-1$, and $D_k = H$. Since each ear contains at least one internal vertex and $|V(D)| \geq 2$, we have

$$k \leq |V(H)| - |V(D_0)| \leq |V(H)| - 2.$$

Since D_0 is a cycle and P_i is a path, we have

$$|E(D_0)| = |V(D_0)|, \quad |E(P_i)| = |V(P_i)| - 1, \quad i = 0, 1, \ldots, k-1.$$

It follows that

$$\begin{aligned} |E(H)| &= |E(D_0)| + \sum_{i=0}^{k-1} |E(P_i)| = |V(D_0)| + \sum_{i=0}^{k-1} \big(|V(P_i)| - 1\big) \\ &= |V(D_0)| + \sum_{i=0}^{k-1} \Big[\big(|V(P_i)| - 2\big) + 1\Big] = |V(H)| + k \\ &\leq 2|V(H)| - 2 \quad (\because k \leq |V(H)| - 2). \end{aligned}$$

Now the union of the strongly connected sub-digraphs H, one for each block of D, is a strongly connected spanning sub-digraph of D. It then follows that

$$\Big|E\big(\bigcup H\big)\Big| = \sum_H |E(H)| \leq 2 \sum_H \big(|V(H)| - 1\big) = 2\big(|V(D)| - 1\big),$$

by induction on the number of H's. $\square$

Exercises

Ch5: 5.1.2; 5.2.1; 5.2.2; 5.2.5; 5.3.3; 5.3.6; 5.3.8; 5.4.2; 5.4.4; 5.4.5.