Week 7-8: Flows in Network

October 28, 2020

1 Transportation Networks

- A **network** $N = (D, x, y, c)$ is a digraph $D = (V, A)$ with two distinguished vertices, a **source** x and a **sink** y, together with a nonnegative function $c : A \to \mathbb{R}_{\geq 0}$, called the **capacity function** of N. For each arc $a = (u, v)$, the value $c(a) = c(u, v)$ is called the **capacity** of a. The vertices other than x, y are called **intermediate vertices** of N.

- For each function $f : A \to \mathbb{R}$ and a vertex subset $X \subseteq V$, we define

$$f^+(X) := \sum_{a \in (X,\, X^c)} f(a), \quad f^-(X) := \sum_{a \in (X^c,\, X)} f(a).$$

  Whenever $X = \{v\}$ contains only one vertex v, we write $f^+(X)$ as $f^+(v)$ and $f^-(X)$ as $f^-(v)$.

- An (x, y)-**flow** (or just **flow**) of a network $N = (D, x, y)$ is a function $f : E(D) \to \mathbb{R}$ satisfying the **circulation condition**:

$$\sum_{a \in (v,\, v^c)} f(a) = \sum_{a \in (v^c,\, v)} f(a), \quad \text{i.e.,} \quad f^+(v) = f^-(v) \quad \forall v \in V \smallsetminus \{x, y\}.$$

  Let ω be the orientation on the underlying graph G such that $D = (G, \omega)$. Then an (x, y)-flow of N is just a real-valued function f on $E(D)$ such that for each $v \in V \smallsetminus \{x, y\}$,

$$\sum_{e \in E(D)} [v, e] f(e) = 0.$$

- The **value** of an (x, y)-flow of a network $N(x, y)$ is the flow value out of the source x, i.e.,

$$\operatorname{val}(f) := f^+(x) = f^-(y).$$

  An (x, y)-flow of a network N is said to be **feasible** if it satisfy the **capacity constraint**: $0 \leq f(a) \leq c(a)$ for all $e \in E(D)$. A flow is called a **maximum flow** if there is no flow of greater value.

Lemma 1.1. *Let f be a low of a network $N(x, y)$, and $X \subseteq V(N)$ a vertex subset such that $x \in X$ and $y \notin X$. Then*

$$\operatorname{val}(f) = f^+(X) - f^-(X).$$

Proof. By definition of flow, $\sum_{e\in E(D)}[v,e]f(e)=0$ for all $v\neq x,y$. Since $x\in X$ and $y\notin X$, we have

$$\begin{aligned}
\mathrm{val}\,(f) &= \sum_{v\in X}\sum_{e\in E(D)}[v,e]f(e)\\
&= \sum_{e\in E(D)} f(e)\sum_{v\in X}[v,e]\\
&= \left\{\sum_{e\in[X,X]}+\sum_{e\in(X,X^c)}+\sum_{e\in(X^c,X)}\right\}f(e)\sum_{v\in X}[v,e].
\end{aligned}$$

Note that $\sum_{v\in X}[v,e]=[u,e]+[w,e]=0$ for each edge $e=uw\in[X,X]$. By definition of $f^+(X)$ and $f^-(X)$, we see that $\mathrm{val}\,(f)=f^+(X)-f^-(X)$. □

- An (x,y)**-cut** (or just a **cut**) of a network $N(x,y)$ is cut $[X,X^c]$ **separating** x **from** y, i.e., $x\in X$ and $y\in X^c$. The **capacity** of such a cut $[X,X^c]$ is the value

$$c(X,X^c):=\sum_{a\in(X,X^c)}c(a).$$

- A cut $[X,X^c]$ of a network $N(x,y)$ is called a **minimum cut** if N has no cut of smaller capacity.

- Let f be a flow of a network $N(x,y)$. A cut $[X,X^c]$ is said to be f**-saturated at an edge** e if either (i) $e\in(X,X^c)$ and $f(e)=c(e)$, or (ii) $e\in(X^c,X)$ and $f(e)=0$; otherwise, it is said to be f**-unsaturated at** e, i.e., either (i) $e\in(X,X^c)$ and $f(e)<c(e)$, or (ii) $e\in(X^c,X)$ and $f(e)>0$.

  If a cut $[X,X^c]$ is f-unsaturated at its edge e, we define

$$\iota(e)=\iota(e,f):=\begin{cases} c(e)-f(e) & \text{if } e\in(X,X^c),\\ f(e) & \text{if } e\in(X^c,X).\end{cases}$$

  If $[X,X^c]$ is f-unsaturated at an edge e, then $\iota(e)>0$.

- A cut $[X,X^c]$ of a network N is said to be f**-saturated** if it is f-saturated at its every edge; otherwise, it is said to be f**-unsaturated**, i.e., if it is f-unsaturated at one of its edges.

Proposition 1.2. *For each flow f of a network $N(x,y)$ and any (x,y)-cut $[X,X^c]$,*

$$\mathrm{val}\,(f)\le c(X,X^c).$$

Moreover, the equality holds if and only if the cut $[X,X^c]$ is f-saturated.

Proof. It is clear by definition that $f^+(X)\le c(X,X^c)$ and $f^-(X)\ge 0$. It follows that

$$\mathrm{val}\,(f)=f^+(X)-f^-(X)\le c(X,X^c).$$

As for the equality, the sufficiency is trivial by definition of f-saturability. For necessity, suppose $[X,X^c]$ is f-unsaturated, i.e., $[X,X^c]$ has an f-unsaturated edge e. If $e\in(X,X^c)$, then $f(e)<c(e)$; thus $\mathrm{val}\,(f)=f^+(X)-f^-(X)<c(X,X^c)-f^-(X)\le c(X,X^c)$. If $e\in(X^c,X)$, then $f(e)>0$; thus $\mathrm{val}\,(f)=f^+(X)-f^-(X)<f^+(X)\le c(X,X^c)$. In either case, we have $\mathrm{val}\,(f)<c(X,X^c)$ which is a contradiction. □

Corollary 1.3. *Let f be a flow and (X, X^c) a cut of a network $N(x, y)$. If $\mathrm{val}\,(f) = c(X, X^c)$, then f is a maximum flow and $[X, X^c]$ is a minimum cut.*

Proof. Let f^* be a maximum flow and (X^*, X^{*c}) a minimum cut of $N(x, y)$. Proposition 1.2 implies

$$\mathrm{val}\,(f) \le \mathrm{val}\,(f^*) \le c(X^*, X^{*c}) \le c(X, X^c).$$

Since $\mathrm{val}\,(f) = c(X, X^c)$, we must have $\mathrm{val}\,(f) = \mathrm{val}\,(f^*)$ and $c(C, X^c) = c(X^*, X^{*c})$. □

2 The Maxi-Flow and Min-Cut Theorem

- Let f be a flow of a network $N(x, y)$, and P an x-path (not necessarily a directed path) whose positive direction is denoted by ω_P. The **f-increment** of P is

$$\epsilon(P) = \epsilon(P, f) := \min\{\epsilon(e, f) : e \in E(P)\}, \quad \text{where}$$

$$\epsilon(e, f) := \begin{cases} c(e) - f(e) & \text{if } \vec{e} \text{ is forward arc in } P, \text{ i.e., } [\omega, \omega_P](e) = 1, \\ f(e) & \text{if } \vec{e} \text{ is a reverse arc in } P, \text{ i.e., } [\omega, \omega_P](e) = -1. \end{cases}$$

- Given a flow of network $N(x, y)$. An x-path P is said to be **f-saturated** if $\epsilon(P, f) = 0$ and **f-unsaturated** if $\epsilon(P, f) > 0$.

- An (x, y)-path is called an **f-incrementing path** if it is f-unsaturated.

Proposition 2.1. *Let f be a flow of a network $N(x, y)$ and P an (x, y)-path. Then $\epsilon(P, f) \ge 0$ and $f' := f + \epsilon(e, f)[\omega, \omega_P]$ is a flow of N with $\mathrm{val}\,(f') = \mathrm{val}\,(f) + \epsilon(P)$, where f' is explicitly given by*

$$f'(e) := \begin{cases} f(e) + \epsilon(e, f) & \text{if } \vec{e} \text{ is a forward arc in } P, \\ f(e) - \epsilon(e, f) & \text{if } \vec{e} \text{ is a reverse arc in } P, \\ f(e) & \text{otherwise.} \end{cases}$$

Consequently, if f is a maximum flow, then f has no f-incrementing path.

Proof. We only need to show that f' is a flow. Clearly, f' is feasible. Since any linear combination of flows is also a flow, it suffices to check that $[\omega, \omega_P]$ is a flow. In fact, for each internal vertex v of P,

$$\sum_{e \in E} \omega(v, e)[\omega, \omega_P](e) = \sum_{e \in E} \omega(v, e)\omega(v, e)\omega_P(v, e) = \sum_{e \in E} \omega_P(v, e)$$

which is zero at each internal vertex v of P by definition of direction of a path. This means that $[\omega, \omega_P]$ is a flow. □

Proposition 2.2. *Let f be a flow of a network $N(x, y)$, and X be the set of vertices reachable from x by f-unsaturated paths, including x itself. If there is no f-incrementing path from x to y in N, then f is a maximum flow, $[X, X^c]$ is a minimum cut, and $\mathrm{val}\,(f) = c(X, X^c)$.*

Proof. Obviously, $X \neq V$, so $[X, X^c]$ is an (x, y)-cut. We claim that $[X, X^c]$ is f-saturated. In fact, suppose $[X, X^c]$ has an f-unsaturated edge $e = uv$ with $u \in X$ and $v \in X^c$. Let P_u be an f-unsaturated path from x to u. Then $P_v := Pev$ is an f-unsaturated path from x to v, i.e., $v \in X$, which is contradictory to $v \in X^c$. Now Proposition 1.2 implies $\mathrm{val}\,(f) = c(X, X^c)$. Consequently, f is a maximum flow and (X, X^c) is a minimum cut by Corollary 1.3. □

Theorem 2.3 (Max-Flow Min-Cut Theorem)**.** *The value of a maximum flow in a network is equal to the capacity of a minimum cut.*

Proof. Let f be a maximum flow. Then there is no f-incrementing path by Proposition 2.1. Thus $[X, X^c]$ is a minimum cut and $\mathrm{val}\,(f) = c(X, X^c)$, where X is the set of vertices reachable by f-unsaturated paths, by Proposition 2.2. □

Theorem 2.4 (Ford-Fulkerson Algorithm)**.** Input: a network $N = (D, x, y)$ with a capacity function $c : A(D) \to \mathbb{R}_{\geq 0}$; a feasible flow f. Output: a maximum flow f and a minimum cut $[T, T^c]$.

Step 1 *Initialize a tree $T := x$, set $\iota(x) := \infty$, then go to* Step 2.

Step 2 *If $y \in T$, set $f := f + \iota(y)[\omega, \omega_P]$ with P the unique path from x to y in T, then return to* Step 1.

If $y \notin T$, got to Step 3.

Step 3 *If $[T, V(T)^c]$ is f-saturated,* Stop. *(f is a miximum flow and $[T, V(T)^c]$ is a minimum cut.)*

If $[T, V(T)^c]$ is f-unsaturated, select an f-unsaturated arc $e = uv \in [T, V(T)^c]$ with $u \in T$ and $v \in V(T)^c$, add ev to T, set $\iota(ev) := \min\{\iota(u), \iota(e)\}$, then go to Step 2.

Proof. Trivial with previous preparations. □

Theorem 2.5 (Labeling Procedure)**.** Input: a network $N = (D, x, y, c)$ with $c(a) \geq 0$ for each $a \in A(D)$. Output: a maximum flow f and a minimum cut $[T, V(T)^c]$

Step 1 *Start with a feasible flow f. Initially, $f = 0$. Label the source x with $(\emptyset, \Delta_x)$, where $\Delta_x = \infty$.*

Step 2 *For each arc $\overrightarrow{uv}$ that u is labeled and v is not labeled, label v as follows:*

If $c(\overrightarrow{uv}) - f(\overrightarrow{uv}) > 0$, set $\Delta_v := \min\{\Delta_u, c(\overrightarrow{uv}) - f(\overrightarrow{uv})\}$ and label v with (u, Δ_v).

If $c(\overrightarrow{uv}) - f(\overrightarrow{uv}) = 0$, leave v unlabeled.

Step 3 *For each arc $\overrightarrow{vu}$ that u is labeled and v is not labeled, label v as follows:*

If $f(\overrightarrow{vu}) > 0$, set $\Delta_v := \min\{\Delta_u, f(\overrightarrow{vu})\}$ and label v with (u, Δ_v).

If $f(\overrightarrow{vu}) = 0$, leave v unlabeled.

STEP 4 *If the sink y is labeled with Δ_y, then an f-incrementing (x, y)-path P is found by chasing back to the source x. Set $f(a) := f(a) + \Delta_y$ if the arc a is forward in P, $f(a) := f(a) - \Delta_y$ if the arc a is backward in P, and keep $f(a)$ unchanged if $a \notin P$. Return to* STEP 1.

If the sink y is unlabeled, then f is a maximum flow, and the arcs between labeled vertices and unlabeled vertices form a minimum cut.

Example 2.1. Consider the following network with capacity function specified on the edges inside the parenthesis. The following Figures demonstrate how a maximum flow and minimum cut are found by the Labeling Algorithm.

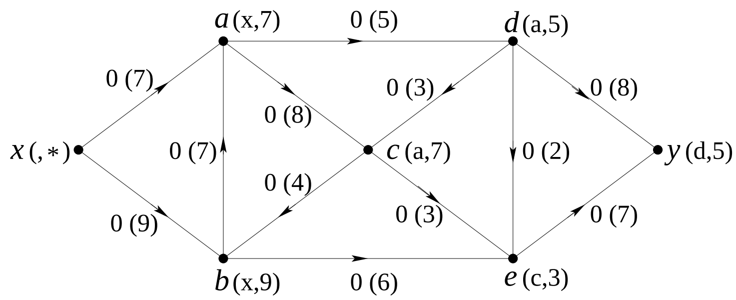


Figure 1: A directed path $xady$ of value 5 is found.

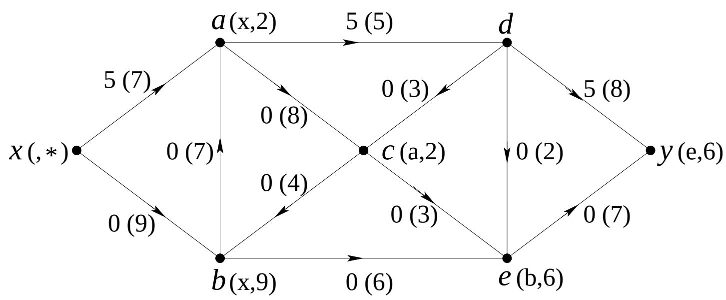


Figure 2: A directed path $xbey$ of value 6 is found.

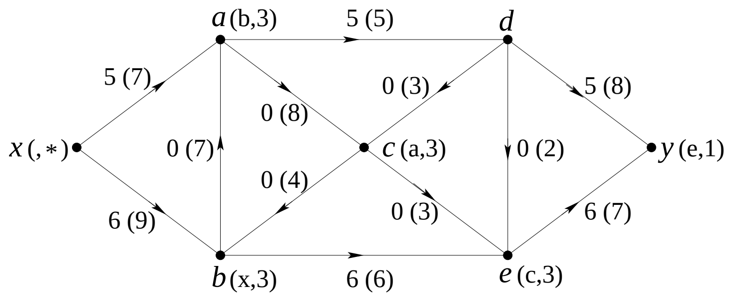


Figure 3: A directed path $xbacey$ of value 1 is found.

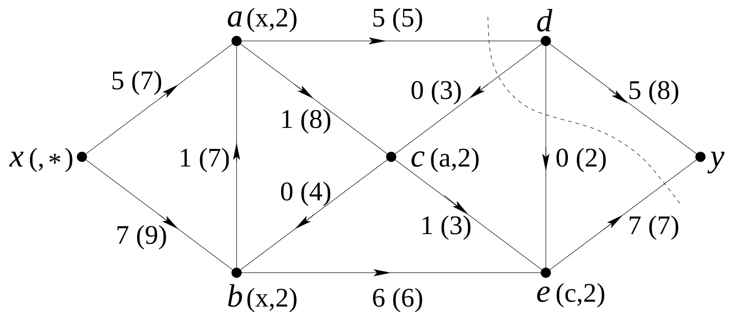


Figure 4: A minimum cut $\{ad, cd, ed, ey\}$ is found.

The maximum flow is f with $f(x,a) = 5$, $f(x,b) = 6$, $f(b,a) = 0$, $f(a,d) = 5$, $f(a,c) = 0$, $f(c,b) = 0$, $f(d,c) = 0$, $f(c,e) = 0$, $f(d,y) = 5$, $f(e,y) = 7$, and maximum flow value is $\text{val}(f) = 12$.

Remark. The Step 3 has to be performed sometimes. For instance, when one label the vertices in order x, a, d, c, b, e, y, an incrementing path $xadcey$ is found starting with the zero flow. See Figure 5, where the first entry of the vector on an arc denotes the capacity of the arc, the subsequent entries are value of flows at the arc.

Another way to find a maximum flow and a minimum cut:

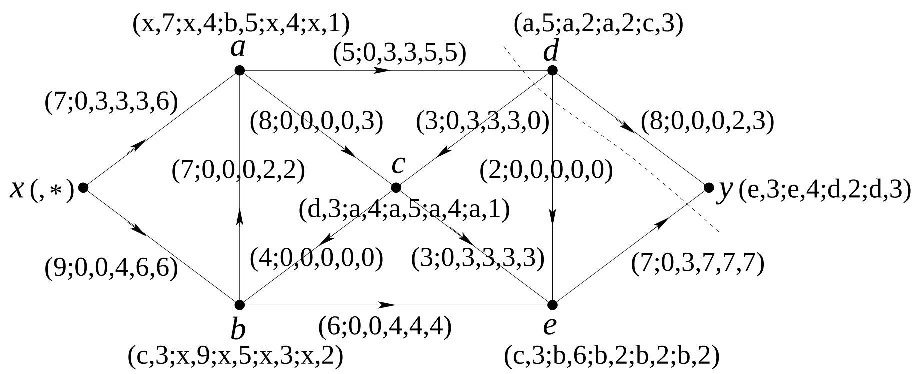


Figure 5: A dif and only iferent maximum flow and the same minimum cut are found.

3 Arc Disjoint Paths

Proposition 3.1. *Given a digraph D and function $f : A(D) \to \mathbb{R}$. The* **support** *of f is the set* $\operatorname{supp}(f)$ *of arcs a such that $f(a) \neq 0$.*

(a) If f is a nonzero flow, then $\operatorname{supp}(f)$ *contains a cycle.*

(b) If f is a nonzero and nonnegative flow, then $\operatorname{supp}(f)$ *contains a directed cycle.*

Proof. Let D_1 be a connected nontrivial component of the sub-digraph of D induced by the arc set $\operatorname{supp}(f)$. Recall that the values of f on the arcs at each fixed vertex add up to zero. Choose an arc $a_1 = (v_0, v_1) \in D_1$ (with $f(a_1) > 0$ when $f \geq 0$). If $v_1 = v_0$, then $v_0 a_1 v_1$ is a (directed) loop. If $v_1 \neq v_0$, since the contribution of f on the arc a_1 at v_1 is nonzero (negative when $f \geq 0$), the contribution of f must be nonzero (positive when $f \geq 0$) on another arc a_2 at v_1. Then there exists an arc $a_2 = (v_1, v_2)$ such that $f(a_2) \neq 0$ ($f(a_2) > 0$ when $f \geq 0$), of course, $a_2 \in D_1$.

If $v_2 \in \{v_0, v_1\}$, we obtain a (directed) cycle either $v_0 a_1 a_2 v_2$ or $v_1 a_2 v_2$ in D. If $v_2 \notin \{v_0, v_1\}$, likewise, there exists another arc $a_3 = (v_2, v_3)$ such that $f(a_3) \neq 0$ ($f(a_3) > 0$ when $f \geq 0$), of course, $a_3 \in D_1$. If $v_3 \in \{v_0, v_1, v_2\}$, then we obtain a (directed) cycle either $v_0 a_1 v_1 a_2 v_2 a_3 v_3$ or $v_1 a_2 v_2 a_3 v_3$ or $v_2 a_3 v_3$ in D. Continue this procedure, since D is a finite digraph, eventually, we reach a situation $v_k \in \{v_0, \ldots, v_{k-1}\}$, say, $v_k = v_i$ with $i < k$. Then $v_i a_{i+1} v_i \cdots a_k v_k$ is a (directed) cycle in D. $\square$

Proposition 3.2. *Let $D = (V, A)$ be a digraph whose underlying graph is $G = (V, E)$. Let W be a directed walk in the digraph $(V, \vec{E})$. Then the function $f_W : A \to \mathbb{Z}$, defined by*

$$f_W(a) = \sum_{\vec{e} \in W} [a, \vec{e}],$$

where W is considered as multiset on $\vec{E}$ and

$$[a, \vec{e}] = \begin{cases} 1 & \text{if } a = \vec{e} \\ -1 & \text{if } a = -\vec{e} \\ 0 & \text{otherwise} \end{cases}$$

satisfies the circulation condition at all vertices, except the initial and terminal vertices of W. Moreover,

(a) If W is a closed walk, then f_W is a flow of D.

(b) If W has no opposite arcs, then either $f_W \geq 0$ or $f_W \leq 0$ on D with

$$\operatorname{supp}(f_W) = \{a : a \in W\}.$$

Proof. Let W be written as the vertex-arc sequence $v_0 a_1 v_1 a_2 v_2 \cdots a_l v_l$. Give a vertex v other than v_0, v_l. If $v \notin W$, it is clear that f satisfies the circulation condition at v, since f is zero on all arcs at v. If $v \in W$ but $v \neq v_0, v_l$, say, $v_{i_1} = \cdots = v_{i_k} = v$, where $1 \leq i_1 < \cdots < i_k \leq l-1$, then

$$\begin{aligned} f_W^+(v) - f_W^-(v) &= \sum_{a \in (v, \{v\}^c)} f_W(a) - \sum_{a \in (\{v\}^c, v)} f_W(a) \\ &= \sum_{a \in (v, \{v\}^c)} \sum_{a' \in W} [a, a'] - \sum_{a \in (\{v\}^c, v)} \sum_{a' \in W} [a, a'] \\ &= \sum_{a' \in W} \Bigg(\sum_{a \in (v, \{v\}^c)} [a, a'] - \sum_{a \in (\{v\}^c, v)} [a, a'] \Bigg). \end{aligned}$$

Since $[a, a']$ is zero unless the underline edges of a, a' are the same and a is at the vertex v, it follows that we only need to consider those $a' \in W$ incident with v. Thus

$$\begin{aligned} f_W^+(v) - f_W^-(v) &= \sum_{j=1}^{k} \Bigg(\sum_{a \in (v, \{v\}^c)} [a, a_{i_j}] - \sum_{a \in (\{v\}^c, v)} [a, a_{i_j}] \Bigg) \\ &\quad + \sum_{j=1}^{k} \Bigg(\sum_{a \in (v, \{v\}^c)} [a, a_{i_j+1}] - \sum_{a \in (\{v\}^c, v)} [a, a_{i_j+1}] \Bigg) \\ &= -k + k = 0. \end{aligned}$$

If W is closed, then f_W is clearly a flow of D. If W is not closed, i.e., $v_0 \neq v_l$, then

$$f_W^+(v_0) - f_W^-(v_0) = 1, \quad f_W^+(v_l) - f_W^-(v_l) = -1.$$

□

Theorem 3.3. *Let f be a nonzero integral flow of a digraph $D = (V, A)$. Let $D' = (V, A')$ be the digraph obtained from D by reversing the orientations of the arcs a that $f(a) < 0$.*

(a) *Then $|f|$ is a flow of D'.*

(b) *If* $\mathrm{supp}\,(f)$ *is connected, then there exists a directed closed walk W in D' such that*

$$f_W = f.$$

(c) *There exists a directed closed walk W in $(V, \vec{E})$ such that $f_W = f$.*

Proof. (a) Trivial.

(b) We may assume $f \geq 0$, and consequently, $D' = D$. We apply induction on $\|f\| := \sum_{a \in A} f(a)$. Choose an arc $a_1 = (v_0, v_1) \in \mathrm{supp}\,(f)$. If $v_1 = v_0$, then $v_0 a_1 v_1$ is a (directed) loop. If $v_1 \neq v_0$, since the contribution of f on the arc a_1 at v_1 is negative, the contribution of f must be positive on another arc at v_1, then there exists an arc $a_2 = (v_1, v_2) \in \mathrm{supp}\,(f)$ at v_1 other than a_1. If $v_2 \in \{v_0, v_1\}$, we obtain a directed cycle either $v_0 a_1 a_2 v_2$ or $v_1 a_2 v_2$ in D. If $v_2 \notin \{v_0, v_1\}$, likewise, there exists an arc $a_3 = (v_2, v_3) \in \mathrm{supp}\,(f)$ at v_2 other than a_2. If $v_3 \in \{v_0, v_1, v_2\}$, then we obtain a directed cycle either $v_0 a_1 v_1 a_2 v_2 a_3 v_3$ or $v_1 a_2 v_2 a_3 v_3$ or $v_2 a_3 v_3$ in D. Continue this procedure, since D is a finite digraph, eventually, we reach a situation $v_l \in \{v_0, \ldots, v_{l-1}\}$, say, $v_l = v_i$ with $i < l$. Then $W_0 := v_i a_{i+1} v_i \cdots a_l v_l$ is a directed closed walk in D. Thus f_{W_0} is a nonnegative integral flow of D and $f_{W_0} \leq f$.

Now set $f' := f - f_{W_0}$, which is a nonnegative integral flow of D. Decompose f' into $f' = \sum_{i=1}^{k} f_i$ of nonzero integral flows $f_i \geq 0$ such that f_i cannot be further decomposed. Then $D_i := \operatorname{supp}(f_i)$ are connected components and $\operatorname{supp}(f') = D_1 \cup \cdots \cup D_k$. By induction there exist directed closed walks W_i in D such that $f_i = f_{W_i}$. Clearly, $f = \sum_{i=0}^{k} f_{W_i}$. Since $\operatorname{supp}(f)$ is connected, W_0 intersections each of $W_1, \ldots, W_k$, say, at the vertices $u_1, \ldots, u_k$ respectively. The vertices $u_1, \ldots, u_k$ appear in W_0 in some linear order, which may be assumed to be in the same order $u_1, \ldots, u_k$ without loss of generality.

We are ready to construct a directed closed walk W in D as follows: Start at u_1 to finish the walk W_1 first, next finish the segment from u_1 to u_2 on W_0, again finish the walk W_2, then finish the segment from u_2 to u_3 on W_0, and continue this procedure; when W_k is finished, we then finish the segment from u_k to u_1, returned back to the starting vertex u_1. Clearly, $\sum_{i=0}^{k} f_{W_i} = f_W$. We have $f = f_W$.

(c) For directed closed walks constructed inside the connected components of $\operatorname{supp}(f)$, we use directed paths P and their reverses P^{-1} to connect them to construct the required directed closed walk W in $(V, \vec{E})$. □

Proposition 3.4. *Let f be a flow of D. Let C be a directed cycle in $(V, \vec{E})$, written as a directed closed walk W. The flow f_W of D is called the* **directed cycle flow** *associated with C. Moreover,*

(a) Every nonnegative flow of D is a nonnegative linear combinations of directed cycle flows.

(b) If f is a (nonnegative) integer-valued flow of D, then f is a (nonnegative) integer linear combination of directed cycle flows.

Proof. (a) Let $P = v_0 a_1 v_1 a_2 \cdots a_n v_n$, where $a_i = (v_{i-1}, v_i)$ and $i = 1, \ldots, n$. Then $(\partial 1_P)(v_i)$ is the addition of contributions of a_i and a_{i+1} at v_i, which are -1 and 1 respectively, added up to zero, for all $i = 1, \ldots, n-1$.

(b) □

Corollary 3.5. *Let $N = (D, x, y, c)$ be a network with source x, sink y, and constant capacity function $c \equiv 1$. Then N has an (x, y)-flow of value k if and only if N has k arc-disjoint directed (x, y)-path.*

Proof. The sufficiency is trivial, since each directed path P produces an (x, y)-flow f_P of value 1. The k arc-disjoint directed (x, y)-paths produce an (x, y)-flow of value k.

For necessity, let f be an (x, y)-flow of N having flow value k. When $k = 0$, nothing is to be proved. For $k \geq 1$, we claim that $\operatorname{supp} f$ contains a directed (x, y)-path. Let $X \subsetneq V(D)$ be such that $x \in X$ and $y \in X^c$. Recall that

$$f^+(X) \geq f^+(X) - f^-(X) = \operatorname{val}(f) > 0.$$

Clearly, $[X, X^c]$ contains an arc $a = uv \in$ with tail $u \in X$ and head $v \in X^c$. Initially, we may take $X = \{x\}$; since $x \in X$ and $y \in X^c$, there exists an arc $a = xx' \in (X, X^c)$. Set $X := \{x, x'\}$; there exists a directed path from x to each vertex of X on $\operatorname{supp} f$. If $y \in X$, we already see that there exists a directed (x, y)-path on $\operatorname{supp} f$. If $y \notin X$, there exists an arc

$a = uv \in (X, X^c)$ so that we enlarge X by setting $X := X \cup v$, and there exists a directed path from x to eah vertex of the new X on $\operatorname{supp} f$. Continue this procedure, we eventually reach the situation that $y \in X$, as N is finite. We then obtain a directed (x, y)-path P such that $f(a) > 0$ for all $a \in P$. Since $f(a) \le c(a) = 1$, we see that $f = 1_P$ on P.

Now $f' := f - 1_P$ is an (x, y)-flow of value $k - 1$ and $f' = 0$ on P. We think of f' as an (x, y)-flow on the digraph D', obtained from D by deleting the arcs of P. By induction, there are $k - 1$ arc-disjoint (x, y)-paths in D'. Consequently, the $k - 1$ arc-disjoint (x, y)-paths in D' plus P constitute k arc-disjoint (x, y)-paths in D. □

Theorem 3.6 (Menger's Theorem). *(a) In any digraph $D(x, y)$, the maximum number of arc-disjoint directed (x, y)-paths is equal to the minimum of forward arcs in an (x, y)-cut.*

(b) In any graph $G(x, y)$, the number of edge-disjoint (x, y)-paths is equal to the minimum number of edges in an (x, y)-cut.

Proof. (a) Trivial by the Max-Flow Min-Cut Theorem.

(b) First of all, its is clear that the number of edge-disjoint (x, y)-paths is always less than or equal to the number of edges in any (x, y)-cut, since each such path crosses such an cut at least once. We claim that the equality holds when one takes max in one side and min in the other side of the inequality.

Let D be the digraph obtained from G by replacing each edge $e = uv$ of G with two directed arcs (u, v) and (v, u). Clearly, edge-disjoint path can be transformed into arc-disjoint directed paths. Likewise, maximum number of arc-disjoint directed (x, y)-paths of shortest total length can be transformed into edge-disjoint paths just by ignoring their orientations on the edges. In fact, let k be the maximum flow value. Suppose two such arc-disjoint paths P, Q have a common edge $e = uv$ with opposite orientations, say, $(u, v) \in P$ and $(v, u) \in Q$. Let P_1 be the directed sub-path of P from x to u, and P_2 the directed sub-path from v to y. Let Q_1 be the directed sub-path of Q from x to v, and Q_2 the directed sub-path from u to y. Then $P' := P_1 Q_2$ and $Q' := P_2 Q_1$ are arc-disjoint directed (x, y)-paths. Replace P, Q by P', Q' respectively, we obtain the same maximum number of arc-disjoint directed (x, y)-paths of shorter total length, which is a contradiction. □

4 Matchings in Bipartite Graphs

- A **matching** in a graph G is a subset $M \subset E$ of link edges such that no two edges of M share a common vertex. The two vertices of an edge of M are said to be **matched under** M, and the vertices incident with edges of M are said to be **covered by** M.

- A matching in G is said to be **maximum** if it covers as many vertices as possible; the number of edges of such a matching is called the **matching number** of the graph, denoted $\alpha'(G)$.

- A matching is said to be **perfect** if every vertex is incident with an edge of the matching.

- Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \cup Y$ and $|X| \leq |Y|$. The edge set E can be viewed as a binary relation R of X to Y, where

$$R = \{(u, v) : u \in X, v \in Y, uv \in E\}.$$

A matching of G is said to be **complete** if every vertex of X is incident with an edge of the matching. A complete matching is perfect if and only if $|X| = |Y|$. Finding a complete matching of G is equivalent to finding an injective mapping $f : X \to Y$ such that $f(x) \in R(x)$, where $R(x) = \{y \in Y : (x, y) \in R\}$.

Theorem 4.1 (Hall's Theorem). *Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \cup Y$ and $|X| \leq |Y|$, which can be considered as a binary relation R from X to Y. There exists a complete matching of X to Y if and only if for each subset $A \subseteq X$,*

$$|A| \leq |R(A)|,$$

where $R(A)$ is the set of vertices v such that there exists an edge $e = uv$ with $u \in A$.

Proof. The necessity is trivial. For sufficiency, let $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$, and $m \leq n$. We construct a network N with a source x, a sink y, and intermediate vertex set V, where (x, x_i) and (y_j, y) have capacity 1, (x_i, y_j) has capacity at least m. It is easy to see that having a complete matching in G is equivalent to having a flow of N that uses all arcs (x, x_i), $1 \leq i \leq m$, i.e., to having a flow of value m. We claim that the maximum flow value of N is m. To this end, it suffices to show that $c(S, S^c) \geq m$ for any (x, y)-cut $[S, S^c]$ of N, and the equality holds for at least one such cut.

We only need to show the inequality since the equality holds when $S = \{x\}$. Given an (x, y)-cut $[S, S^c]$, set $A := S \cap X$ and $B := S \cap Y$. Then $S = \{x\} \cup A \cup B$, $S^c = (X \smallsetminus A) \cup (Y \smallsetminus B) \cup \{y\}$, and

$$(S, S^c) = (x, X \smallsetminus A) \cup (A, Y \smallsetminus B) \cup (B, y).$$

If $(A, Y \smallsetminus B) \neq \emptyset$, it is clear that $c(S, S^c) \geq m$, as each arc in $(A, Y \smallsetminus B)$ has capacity m. If $(A, Y \smallsetminus B) = \emptyset$, i.e., $B \supseteq R(A)$, then

$$c(S, S^c) = |X \smallsetminus A| + |B| \geq |X \smallsetminus A| + |R(A)| \geq |X \smallsetminus A| + |A| = m.$$

Note that by the Labeling Algorithm the integer-valued maximum flow can be attained whenever the capacity function is integer-valued. Let f be a maximum integer-valued flow of the network N. Since $\mathrm{val}(f) = m$, we must have $f(x, x_i) = 1$ for all $x_i \in X$. At each vertex x_i, there exists an edge (x_i, y_j) such that $f(x_i, y_j) = 1$, since f satisfies the circulation condition at x_i. Suppose there there are two vertices x_i, x_j such that $f(x_i, y_k) = f(x_j, y_k) = 1$, then f cannot satisfy the circulation condition at y_k, since $1 \geq f^+(y_k) = f^-(y_k) \geq 2$, which is a contradiction. So the support of f on E is a complete matching of G. □

- Let M be a matching in a graph G. An M-**alternating path** (**cycle**) in G is a path (cycle) P whose edges along its direction are alternating between M and M^c. The starting (ending) edge of an M-alternating path may or may not be an edge of M.

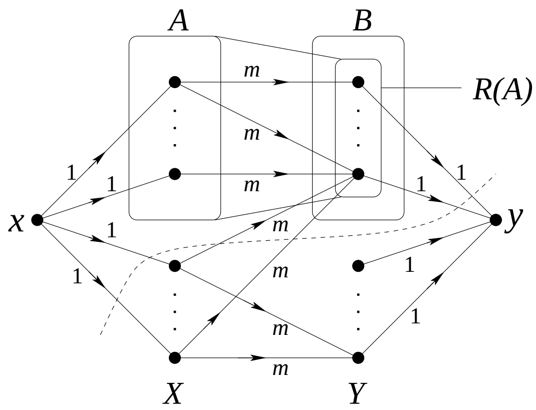


Figure 6: An arbitary (x, y)-cut of the bipartite graph

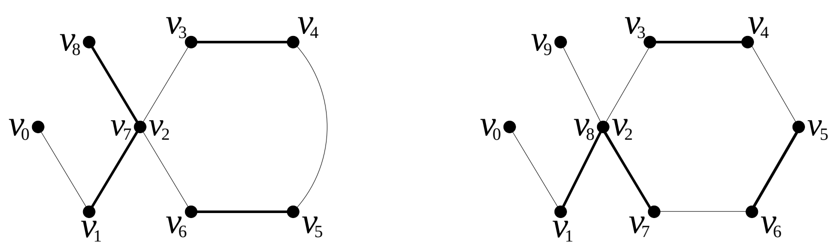
- An M**-augmenting path** is an M-alternating path in which none of its initial and terminal vertices is covered by M.

Lemma 4.2. *Let M be a matching in a graph G, and P am M-alternating trail. We have*

(a) P has no self-section vertices.

(b) If P is an M-augmenting path, then $M' := M\Delta P$ is a matching and $|M'| = |M| + 1$.

Proof. (a) Suppose that the path $P = v_0v_1 \ldots v_{2m+1}$ has self-intersection, i.e., two of the vertices $v_0, v_1, \ldots, v_{2m+1}$ are the same, say, $v_i = v_j$ with $i < j$. There are two possibilities: $j - i$ is odd and $j - i$ is even. In the former case, we see that either the edges $v_{i-1}v_i, v_jv_{j+1}$ belong to M or the edges $v_iv_{i+1}, v_{j-1}v_j$ belong to M. This is a contradiction since two edges of M share the common vertex $v_i(v_j)$. In the latter case, we see that either the edges $v_{i-1}v_i, v_{j-1}v_j$ belong to M or the edges v_iv_{i+1}, v_jv_{j+1} belong to M, so two edges of M share the common vertex $v_i(v_j)$.



(b) Since M is a matching, no two edges of $M \smallsetminus P$ share a common vertex. Since P has no self-intersection, no two edges of $P \smallsetminus M$ share a common vertex.

Note that the vertices of $M \cap P$ are internal vertices of P, and neither the initial vertex nor the terminal vertex of P is an endpoint of M. We see that the endpoints of $M \smallsetminus P$ are disjoint from P, of course, disjoint from $P \smallsetminus M$. Thus the symmetric difference $M' = M\Delta P := (M \smallsetminus P) \cup (P \smallsetminus M)$ is a matching. Clearly, $|M'| = |M| + 1$. $\square$

Theorem 4.3 (Berge's Theorem)**.** *A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.*

Proof. "⇒" Suppose that G contains an M-augmenting path P. Then P has more edges in M^c than in M, and the initial and terminal vertices are not covered by M. Thus $M' := M\Delta E(P)$ is a matching in G with $|M'| > |M|$, contradictory to the maximality of $|M|$.

"⇒" Suppose that M is not a maximum matching. Given a maximum matching M^*. The subgraph $H := G(M\Delta M^*)$ has degree at most 2 at every vertex. Thus H is a vertex-disjoint union of path and cycles, whose edges are alternating between M and M^*. Since $|M^*| > |M|$, H contains more edges of M^* than of M. It follows that H has at least one path component P, whose initial and terminal vertices are covered by M^*, i.e., not covered by M). So P is an M-augmenting path, which is a contradiction. □

Exercises

Ch7: 7.1.4; 7.2.2; 7.3.1; 7.3.3.