Week 9-10: Connectivity

1 Vertex Connectivity

Let $G = (V, E)$ be a graph. Given two vertices $x, y \in V$.

- Two (x, y)-path are said to be **internally disjoint** if they have no internal vertices in common.
- The **local connectivity** between two distinct vertices x and y is the maximum number of pairwise internally disjoint (x, y)-paths, denoted $p(x, y)$ or $p_G(x, y)$.
- A nontrivial graph G (i.e. having at least two vertices) is **k-connected** if $p(u, v) \geq k$ for any two distinct vertices u, v. The **connectivity** of G is the maximum value k for which G is k-connected, i.e., G is k-connected but not $(k + 1)$-connected.
- A graph is 1-connected iff it is connected. A nontrivial graph is 2-connected iff any two vertices lie on a common cycle.
- A trivial graph (i.e. one vertex graph with possible loops) is considered to be 1-connected, but not 2-connected; its connectivity is 1. A disconnected graph is 0-connected, but not 1-connected; so its connectivity is 0.
- The complete graph K_n with $n \geq 2$ has $n - 2$ internally disjoint paths of length 2 and one path of length 1. So the connectivity of K_n is $n - 1$.
- Let G be a complete graph with multiple edges. Let $\mu(x, y)$ be the number of edges between x and y. Then there are $\mu(x, y)$ paths of length 1 from x to y and $n - 2$ internally disjoint (x, y)-paths. So the local connectivity between x and y is $n - 2 + \mu(x, y)$.
- Let x, y be two nonadjacent in G. An (x, y)-vertex-cut is a subset $S \subseteq V \smallsetminus \{x, y\}$ such that x, y belong to different components of $G \smallsetminus S$. We say that such a cut **separates** x and y. We denote by $c(x, y)$ the minimum size of an (x, y)-vertex-cut.

Theorem 1.1 (Menger's Theorem). *Let $G(x, y)$ be a graph with two nonadjacent vertices x, y. Then the maximum number of pairwise internally disjoint (x, y)-paths is equal to the minimum number of vertices in an (x, y)-vertex-cut, i.e.,*

$$p(x, y) = c(x, y).$$

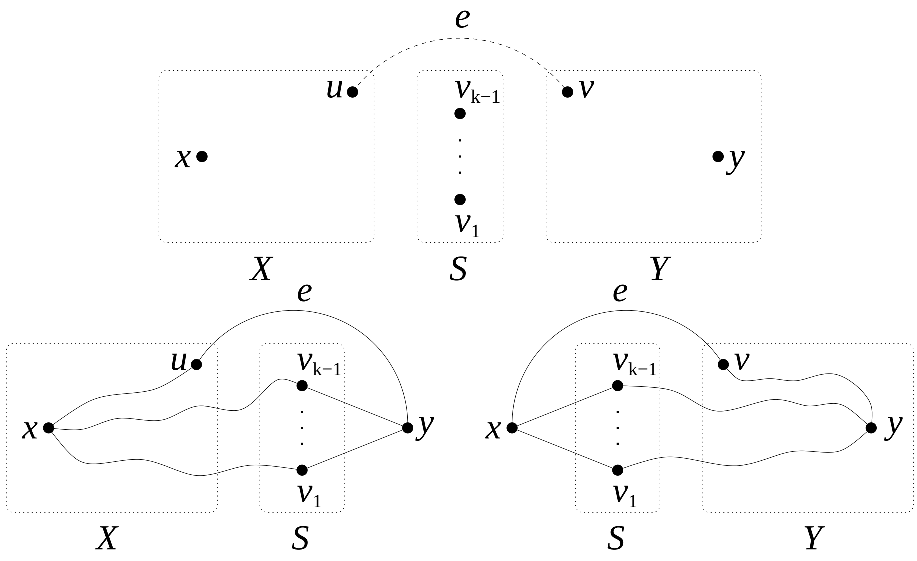

Figure 1: Proof of Menger's Theorem

Proof. Let $p = p_G(x, y)$ and $k = c_G(x, y)$. There are p internally disjoint (x, y)-paths, and a vertex k-subset $K \subseteq V \smallsetminus \{x, y\}$ that separates x and y. Since each (x, y)-path meets K at its internal vertices at least once, the p internally disjoint (x, y)-paths meet K at their internal vertices at least p times. Hence $p_G(x, y) \le c_G(x, y)$. To show that $p_G(x, y) \ge c_G(x, y)$, we proceed by induction on $|E(G)|$. We may assume that there is an edge whose end-vertices are disjoint from $\{x, y\}$. Otherwise, every edge is either incident with x or incident with y. It turns out that G is a bipartite graph with bipartition $\{x, y\} \cup (V \smallsetminus \{x, y\})$. Then all (x, y)-paths have length 2, and the conclusion is obviously true.

Let $e = uv$ be an edge such that $\{u, v\} \cap \{x, y\} = \varnothing$. Consider the subgraph $H := G \smallsetminus e$. Since $|E(H)| < |E(G)|$, by induction we have $p_H(x, y) = c_H(x, y)$. Moreover,

$$c_G(x, y) \le c_H(x, y) + 1,$$

since any (x, y)-vertex-cut of H, together with either u or v, forms an (x, y)-vertex-cut of G. Hence

$$p_G(x, y) \ge p_H(x, y) = c_H(x, y) \ge c_G(x, y) - 1 = k - 1.$$

If $p_G(x, y) = k$, then nothing is to be proved. Suppose $p_G(x, y) = k - 1$. We have $p_G(x, y) = p_H(x, y) = c_H(x, y) = k - 1$ and $c_G(x, y) = k$. Let $S = \{v_1, \ldots, v_{k-1}\}$ be an (x, y)-vertex-cut of H of minimum size. Let X be the set of vertices reachable from x in $H \smallsetminus S$, and Y the set of vertices reachable from y in $H \smallsetminus S$. Then X and Y are disjoint. Since $|S| = k - 1$ and $c_G(x, y) = k$, the set S is not an (x, y)-vertex cut of G. So there is an (x, y)-path in $G \smallsetminus S$. This path necessarily contains the edge e, otherwise, x and y are connected in $H \smallsetminus S$, which is a contradiction. The edge e must be between X and Y, say, $u \in X$ and $v \in Y$.

Now consider the graph G/Y by contracting Y to y so that y is a vertex in G/Y. Each (x, y)-vertex-cut of G/Y is an (x, y)-vertex-cut of G. We see that $c_{G/Y}(x, y) \ge$

$c_G(x,y)=k$. Since $S\cup u$ is an (x,y)-vertex-cut of G/Y, we see that $c_{G/Y}(x,y)\le k$. Thus $c_{G/Y}(x,y)=k$. Since $|E(G/Y)|<|E(G)|$, by induction there are k internally disjoint (x,y)-paths $P_1,\ldots,P_k$ in G/Y. Since $S\cup u$ is an (x,y)-vertex-cut of G/Y, each vertex of $S\cup u$ lies exactly in one of the paths $P_1,\ldots,P_k$. We may assume, without loss of generality, that $v_i\in P_i$ for $i=1,\ldots,k-1$ and $u\in P_k$. Likewise, there are k internally disjoint (x,y)-paths $Q_1,\ldots,Q_k$ in G/X such that $v_i\in Q_i$ for $i=1,\ldots,k-1$ and $v\in Q_k$. Let P_i' be the sub-path of P_i from x to v_i, Q_i' the sub-path of Q_i from v_i to y, $1\le i\le k-1$, P_k' the sub-path of P_k from x to u, and Q_k' the sub-path of Q_k from v to y. Then $P_1'Q_1',\ldots,P_{k-1}'Q_{k-1}'$, $P_k'eQ_k'$ are k internally disjoint (x,y)-paths in G, which is contradictory to $p_G(x,y)=k-1$. □

A **vertex cut** of a graph G is a vertex subset $S\subset V(G)$ which separates some nonadjacent vertices, i.e., $G\smallsetminus S$ has at least two connected components. Complete graphs have no vertex cut, and they are the only simple graphs having no vertex cut.

Recall that the connectivity of a nontrivial graph G is defined as the integer κ that there are κ internally disjoint paths between any two vertices (either adjacent or nonadjacent), and there exist two vertices between them there are no $\kappa+1$ internally disjoint paths. The following theorem says that to determine $\kappa(G)$ we only need to consider the local connectivity for the pairs of nonadjacent vertices, no need to consider the pairs of adjacent vertices.

Theorem 1.2. *Let G be a graph having at least one pair of nonadjacent vertices. Then the connectivity of G is*

$$\kappa(G)=\min\{p_G(u,v):u,v\in V(G),u\neq v,uv\notin E(G)\}.$$

Proof. We may assume that G is simple. Otherwise, take an edge e which is either a loop or one of multiple links. Clearly, the graph $H:=G\smallsetminus e$ has at least one pair of nonadjacent vertices. By induction on the number of edges,

$$\kappa(H)=\min\{p_H(u,v):u,v\in V(H),u\neq v,uv\notin E(H)\}.$$

Let x,y be nonadjacent vertices in H such that $\kappa(H)=p_H(x,y)$. Since any k internally disjoint (x,y)-paths in G can be adapted to k internally disjoint (x,y)-paths in H, we see that $p_G(x,y)=p_H(x,y)$. Note that $\kappa(H)\le\kappa(G)\le p_G(x,y)$, and x,y are nonadjacent in G by the choice of e. It turns out that

$$\kappa(G)=p_G(x,y)=p_H(x,y)=\kappa(H),\quad xy\notin E(G).$$

We have seen that the theorem is true for G. This is why we only need to consider simple graphs.

Recall that $\kappa(G)=\min\{p_G(u,v):u,v\in V(G),u\neq v\}$. Let x,y be vertices such that $p_G(x,y)=\kappa(G)$. If x,y are nonadjacent in G, nothing is to be proved. So we assume that xy is an edge of G. Consider the graph $H:=G\smallsetminus xy$. Clearly, $p_G(x,y)=p_H(x,y)+1$.

Since x, y are not adjacent in H, by Menger's Theorem, $p_H(x, y) = c_H(x, y)$. Let S be a minimum (x, y)-vertex-cut in H. Then

$$p_H(x, y) = c_H(x, y) = |S|, \quad p_G(x, y) = |S| + 1.$$

If $V(G \smallsetminus S) = \{x, y\}$, then

$$\kappa(G) = p_G(x, y) = |S| + 1 = |V| - 2 + 1 = |V| - 1.$$

It turns out that G must be complete, which is contrary to the hypothesis. So we may assume that $G \smallsetminus S$ contains at least three vertices x, y, z. The component of z in $H \smallsetminus S$ should be different from one of the two components of x and y, say, different from the component of x in $H \smallsetminus S$. Since $\{x, y\} \cap S = \varnothing$, it forces that x, z are nonadjacent in G. Since $G \smallsetminus (S \cup y) = H \smallsetminus (S \cup y)$, the set $S \cup y$ is a vertex cut in both G and H separating x and z. Thus

$$c_G(x, z) \leq |S \cup y| = |S| + 1 = p_G(x, y).$$

Now by Menger's Theorem, $p_G(x, z) = c_G(x, z)$. We then have $p_G(x, z) \leq p_G(x, y)$. By the minimality $p_G(x, y)$ among all pairs $\{u, v\}$, we have $p_G(x, z) = p_G(x, y)$. Thus we obtain

$$\kappa(G) = p_G(x, y) = p_G(x, z), \quad xz \notin E(G).$$

We have proved that the theorem is true for G. □

Corollary 1.3. *If a graph G has at least one pair of nonadjacent vertices, then*

$$\kappa(G) = \min\{c(u, v) : u, v \in V(G), u \neq v, uv \notin E(G)\},$$

i.e., $\kappa(G)$ is equal to the size of a minimum vertex cut of G.

Corollary 1.4. *Let G be a graph having at leat two vertices not adjacent. If G is k-connected with $k \geq 1$, then $G \smallsetminus S$ is connected for any $(k-1)$-subset $S \subset V(G)$.*

2 The Fan Lemma

Many properties about connectivity can be derived from Menger's Theorem.

Lemma 2.1. *Let G be a k-connected graph. Let H be a graph obtained from G by adding a new vertex v and joining it to at least k vertices of G. Then H is also k-connected.*

Proof. If H is the complete graph K_n, then $\kappa(H) = n - 1 = |V(G)| \geq k$. If H is not a complete graph then H has at leat one pair of nonadjacent vertices. It suffices to show that $H \smallsetminus S$ is connected for any $(k-1)$-subset $S \subset V(H)$. Fix a $(k-1)$-subset $S \subset V(H)$. If $v \in S$, then $H \smallsetminus S = G - (S \smallsetminus v)$ and is connected, since $|S \smallsetminus v| = k - 2$. If $v \notin S$, i.e., $S \subset V(G)$, then $G \smallsetminus S$ is connected and is contained in $H \smallsetminus S$, for G is k-connected. Since v is adjacent to at least k vertices of G, there is at least one vertex of $G \smallsetminus S$ adjacent to v in $H \smallsetminus S$. So $H \smallsetminus S$ is connected. □

Proposition 2.2. *Let G be a k-connected graph with $k \geq 1$. Let X, Y be two vertex subsets of cardinality at least k. Then there exists in G a family of k pairwise disjoint (X, Y)-paths.*

Proof. Let H be a graph obtained from G by adding two new vertices x, y and joining x to each vertex of X and y to each vertex of Y. Then H is still k-connected by Lemma 2.1. Thus there exist in H internally disjoint (x, y)-paths $P_1, \ldots, P_k$. Delete the first and the last edges of these paths, we obtain a family of k disjoint (X, Y)-paths in G. □

A **k-fan** from a vertex x to a vertex subset Y in a graph G, where $x \notin Y$, is a family of k pairwise disjoint (x, Y)-paths except their initial vertex x.

Corollary 2.3. *Let G be a k-connected graph. Given a vertex x and vertex subset Y of G such that $x \notin Y$. Then for each $d \leq \min\{k, |Y|\}$ there exists a d-fan from x to Y.*

Proof. Let X be the set of neighbors of x in G. Clearly, $|X| \geq k$. Extend Y to a k-set if $|Y| < k$. Then there are k pairwise disjoint (X, Y)-paths in G of shortest total length. Connecting x to each of these paths, we obtain k pairwise disjoint paths from x to Y except their common initial vertex x. □

Proposition 2.4. *Let G be a k-connected graph with $k \geq 2$. The any k vertices of G lie on a common cycle of G.*

Proof. We apply induction on k. For $k = 2$, it is trivially true. For $k \geq 3$, assume that it is true for $(k-1)$-connected graphs. Given k distinct vertices $v_1, \ldots, v_k$ of G. Since G is automatically $(k-1)$-connected, there exists a cycle C in G such that C contains the vertices $v_1, \ldots, v_{k-1}$. We may assume that $v_1, \ldots, v_{k-1}$ are arranged in order along a direction of C. Then there exists a $(k-1)$-fan from v_k to $\{v_1, \ldots, v_{k-1}\}$, i.e., internally disjoint paths P_i from v_k to v_i, $i = 1, \ldots, k-1$. Now replacing the path segment of C from v_{k-1} to v_1 by the path $P_{k-1}^{-1} P_1$, we obtain a cycle C' which contains all the vertices $v_1, \ldots, v_k$. □

Corollary 2.5. *Let G be a k connected graph with $k \geq 2$. Then for any distinct vertices $v_0, v_1, \ldots, v_k$, there exists a cycle C containing $v_1, \ldots, v_k$ and k internally disjoint paths from v_0 to $v_1, \ldots, v_k$.*

3 Edge Connectivity

- The **local edge connectivity** between two distinct vertices x, y in a graph G is the maximum number of pairwise edge-disjoint (x, y)-path, denoted $p'_G(x, y)$.
- A nontrivial graph G is **k-edge-connected** if $p'(u, v) \geq k$ for all two distinct vertices u, v of G. By convention, a trivial graph (i.e. having one vertex) is considered to be **1-edge-connected**.

- The **edge connectivity** $\kappa'(G)$ of a graph G is the maximum value of k for which G is k-edge-connected.

- For two distinct vertices x, y of a graph G, let $c'(x, y)$ or $c'_G(x, y)$ denote the minimum size of an edge cut $[X, X^c]$ that separates x and y, i.e., $x \in X$ and $y \in X^c$.

Theorem 3.1 (Menger's Theorem, Edge Version)**.** *For any graph G with two distinct prescribed vertices x, y,*

$$p'(x, y) = c'(x, y).$$

The vertex connectivity κ, the edge connectivity κ', and the minimum degree δ of a graph G are related by

$$\kappa \leq \kappa' \leq \delta.$$

The second inequality is trivial; the first one is also trivial when G has a pair of nonadjacent vertices.

A **trivial edge cut** is one associated with a single vertex. A k-edge-connected graph is said to be **essentially** $(k + 1)$**-edge-connected** if all its k-edge cuts are trivial.

4 Connectivity in Digraphs

Theorem 4.1 (Menger's Theorem, Directed Version)**.** *Let $D(x, y)$ be a digraph with two prescribed vertices x, y, where $(x, y) \notin S(D)$. Then the maximum number of pairwise internally disjoint directed (x, y)-paths is equal to the minimum number of vertices in an (x, y)-vertex cut.*

Exercises

Ch9: 9.1.1; 9.1.8; 9.1.9; 9.2.1; 9.3.2; 9.3.7.