Problems

Problem 1. A *polytope* of $\mathbb{R}^d$ is a convex hull of some finite number of points of $\mathbb{R}^d$. A half-space of $\mathbb{R}^d$ is a subset of the forms

$$H^-(\ell, a) = \{x \in \mathbb{R}^d : \ell(x) \leq a\}, \quad H^+(\ell, a) = \{x \in \mathbb{R}^d : \ell(x) \geq a\},$$

where $\ell(x) = a_1x_1 + \cdots + a_dx_d$ is a linear function on $\mathbb{R}^d$ and a is a constant real number. A *polyhedral set* is an intersection of finitely many half-spaces. Show that a subset $P \subset \mathbb{R}^d$ is a polytope if and only if P is bounded polyhedral set.

Problem 2. A **polyhedron** of $\mathbb{R}^d$ is a subset obtained from half-spaces by taking intersections, unions, and relative complement finitely many times, i.e., a member of the relative Boolean algebra generated by polyhedral sets.

(a) Show that every polyhedron X can be written as the form

$$X = \bigcup_{i \in I} \Big(P_i \smallsetminus \bigcup_{k \in I_i} P_{i,k} \Big),$$

where $P_i, P_{i,k}$ are polyhedral sets, the unions are finite and the union over I can be made disjoint.

(b) Let X be a bounded polyhedron. Let $X^k = X \times \cdots \times X$ (k times). $\binom{X}{k}$ denote the set of all k-subsets of X. Then $\binom{X}{k}$ can be identified as the quotient set

$$\Big(X^k \smallsetminus \bigcup_{i<j} X_{i,j} \Big) / \sim,$$

where $X_{i,j} = \{(x_1, \ldots, x_k) \in X^k : x_i = x_j\}$ with $i \neq j$, and $\sim$ is an equivalence relation generated by $(x_1, \ldots, x_k) \sim (x_{\pi(1)}, \ldots, x_{\pi(k)})$ with π a permutation of $\{1, \ldots, k\}$.

(c) Let $\mathbb{R}^d$ be linearly ordered, say, by the lexicographic order. Then $X^{(k)}$ can be further identified as the polyhedron

$$\{(x_1, \ldots, x_k) \in X^k : x_1 \prec \cdots \prec x_k\}.$$

Prove the following identity

$$\chi\left(\binom{X}{k} \right) = \binom{\chi(X)}{k},$$

where $\binom{a}{k} = a(a-1)\cdots(a-k+1)/k!$.