

Valuation

1 Euler Number

There are two principles to follow when one counts objects of finite sets. For finite sets A and B , we have

- 1) Addition Principle: $|A \cup B| = |A| + |B| - |A \cap B|$, and
- 2) Multiplication Principle: $|A \times B| = |A| |B|$.

The two principles are necessarily true when A, B are not finite sets. For instance, if A is the set of disjoint union of two disks and a zero-shape of \mathbb{R}^2 , and B is a set of disjoint union of a disk and the same zero-shape of \mathbb{R}^2 , see Figure 1, then $|A| = 3$, $|B| = 2$, $|A \cup B| = 4$, $|A \cap B| = 2$. The Addition Principle implies the contradiction

$$4 = |A \cup B| = |A| + |B| - |A \cap B| = 3 + 2 - 2 = 3.$$

However, if we think of the contribution of the zero-shape is *zero* to the counting of $A \cup B$ (the zero-shape likes the symbol 0), then the Addition Principle is still valid. Indeed, there is a generalized counting measure χ , whose values for finite sets are just counting and for infinite nice topological sets are Euler numbers. Since a finite set can be viewed as a topological space with discrete topology, the the Euler characteristic (with compact support) is a unified finitely additive measure to count both discrete and continuous sets.

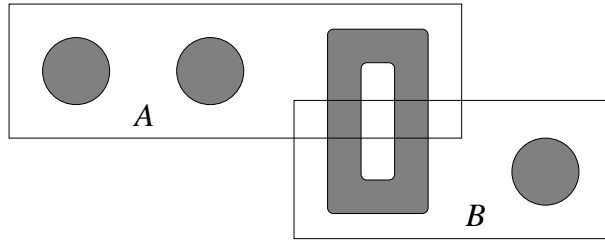


Figure 1: Three disks and a zero-shape.

Given a collection \mathcal{S} of sets containing the empty set \emptyset , closed under intersection and union, that is, if $A, B \in \mathcal{S}$ then $A \cap B, A \cup B \in \mathcal{S}$. A set function ν on \mathcal{S} with values in an abelian group is called a **valuation** (or **finitely additive measure**) on \mathcal{S} if $\nu(\emptyset) = 0$ and for $A, B \in \mathcal{S}$,

$$\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B).$$

Since there is no countable additivity, the set-function ν is not necessarily a measure. The valuation here is nothing to do with its meaning in algebra. To construct a valuation we usually begin with an **intersectional class**, a collection of sets closed under intersection. Given an intersectional class \mathcal{I} , the **relative Boolean algebra** $\mathbf{B}(\mathcal{I})$ is a minimal class of sets containing \mathcal{I} , closed under intersection, union, and relative complement.

A closed interval is a set $I = [a, b]$, where a, b are real numbers and $a \leq b$. A **parallelotope** of \mathbb{R}^n is a set $\prod_{i=1}^n I_i$, where each I_i is a closed interval. We denote by $\text{Par}(n)$ the class of parallelotopes of \mathbb{R}^n . Let $\mathbf{B}(\text{Par}(n))$ denote the relative Boolean algebra generated by $\text{Par}(n)$. A valuation ν on $\text{Par}(n)$ is **translation invariant** provided that for each $P \in \mathbf{B}(\text{Par}(n))$ and $a \in \mathbb{R}^n$,

$$\nu(P + a) = \nu(P),$$

where $P + a = \{x + a : x \in P\}$. To exclude possible pathological valuations that we have no interest, we require the translation invariant valuation ν on $\mathbf{B}(\text{Par}(n))$ to be continuous in certain sense. A valuation ν on $\mathbf{B}(\text{Par}(n))$ is said to be **continuous** if

$$\lim_{k \rightarrow \infty} \nu(I_k) = \nu(I),$$

where $I_k = [a_k, b_k]$ is a sequence of closed intervals convergent to a closed interval $I = [a, b]$, i.e., $\lim_{k \rightarrow \infty} a_k = a$ and $\lim_{k \rightarrow \infty} b_k = b$. We wish to classify all translation invariant continuous valuations on $\mathbf{B}(\text{Par}(n))$.

Note that each object S of $\mathbf{B}(\text{Par}(1))$ is a disjoint union of finitely many singletons and open intervals, that is,

$$S = \bigsqcup_{i=1}^k \{a_i\} \cup \bigsqcup_{j=1}^l (b_j, c_j) \text{ (disjoint union)}$$

Then $\chi : \mathbf{B}(\text{Par}(1)) \rightarrow \mathbb{R}$, defined by

$$\chi(S) = k - l,$$

is a translation invariant continuous valuation. Another translation invariant continuous valuation is $\ell : \mathbf{B}(\text{Par}(1)) \rightarrow \mathbb{R}$, defined by

$$\ell(S) = \text{Lesbesgue measure of } S.$$

It is easy to see that any translation invariant continuous valuation on $\mathbf{B}(\text{Par}(1))$ is a linear combination of the two valuations χ and ℓ .

Now consider the translation invariant valuation $\nu : \mathbf{B}(\text{Par}(1)) \rightarrow \mathbb{R}[t]$ such that for each closed interval $I = [a, b]$,

$$\mu(I) = 1 + (b - a)t = \chi(I) + \ell(I)t.$$

We thus have a product valuation $\mu^n : \mathbf{B}(\text{Par}(n)) \rightarrow \mathbb{R}[t]$ defined by

$$\begin{aligned} \mu^n \left(\prod_{i=1}^n I_i \right) &= \prod_{i=1}^n \mu(I_i) = \prod_{i=1}^n (1 + x_i t) \\ &= \sum_{k=0}^n \left(\sum_{J \subseteq [n], |J|=k} \prod_{j \in J} x_j \right) t^k \\ &= \sum_{k=0}^n e_k(x_1, \dots, x_n) t^k, \end{aligned}$$

where I_i are closed intervals of length x_i and e_k are the elementary symmetric polynomials of variables x_1, \dots, x_n of degree k . It follows that we have translation invariant valuations $\mu_k : \mathbf{B}(\text{Par}(n)) \rightarrow \mathbb{R}$ such that

$$\mu_k(P) = e_k(x_1, \dots, x_n)$$

for parallelotopes P of lengths x_1, \dots, x_n . The value $\mu_k(P)$ is called the k th **elementary mixed volume** of P , also called the $(n - k)$ th **quermassintegral** of P up to a universal constant. The valuations μ_i can be extended to the class $\mathbf{B}(\mathcal{K}^n)$, the relative Boolean algebra generated by \mathcal{K}^n .

Theorem 1.1.

The valuation ν , defined on $\mathbf{B}(\text{Par}(n))$, can be extended to the class $\mathbf{U}(\mathcal{K}^n)$ of finite union of compact convex sets of \mathbb{R}^n . In fact, if K is a compact convex set and B is the unit ball of \mathbb{R}^n then ν_i can be extended to $\mathbf{U}(\mathcal{K}^n)$ such that

$$\text{vol}_n(K + rB) = \sum_{i=0}^n \nu_i(K) \omega_{n-i} r^{n-i}, \quad (1)$$

where $K + rB = \{x + ry : x \in K, y \in B\}$, $r \geq 0$, and ω_i is the volume of the i -dimensional unit ball. The identity (1) is called the **Steiner formula**.

Exercise 1. (a) Show that χ is a translation invariant valuation on $\mathbf{B}(\text{Par}(1))$.

(b) Show that every member P of $\mathbf{B}(\text{Par}(n))$ is a disjoint union of finitely many relatively open coordinate parallelotopes

$$\prod_{i \in I} (a_i, b_i) \times \prod_{j \in J} \{c_j\},$$

where $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, n\}$.

Let V be a finite dimensional real vector space. A linear combination $t_1 v_1 + \dots + t_m v_m$ of vectors v_1, \dots, v_m in V is called an **affine linear combination** if $t_1 + \dots + t_m = 1$, and is further called a **convex linear combination** if all $t_i \geq 0$ and $t_1 + \dots + t_m = 1$. Given a nonempty subset S of V . The set S is said to be **convex** if it contains segment

$$\overline{xy} := \{tx + (1 - t)y : 0 \leq t \leq 1\}$$

for each pair of points x, y of S . The **affine span** of S , denoted $\langle S \rangle$, is the set of all possible affine linear combinations of points of S . The **convex hull** of S , denoted $\text{conv}(S)$, is the set of all possible convex linear combinations of points of S . A **convex body** is a compact convex set. A **polytope** is a convex hull of finite number of points. Given a closed convex set F . The interior of F in its affine span $\langle F \rangle$, denoted $\overset{\circ}{F}$, is called a **relatively open convex set**. The subset

$$\partial F := F \setminus \overset{\circ}{F}$$

is called the **boundary** of both F and $\overset{\circ}{F}$. The dimension of $\langle F \rangle$ is called the **dimension** of both F and $\overset{\circ}{F}$. Given a nonzero linear functional $\phi : V \rightarrow \mathbb{R}$ and a constant a . The subsets

$$H(\phi \leq a) = \{x \in \mathbb{R}^n : \phi(x) \leq a\} \quad \text{and} \quad H(\phi < a) = \{x \in \mathbb{R}^n : \phi(x) < a\}$$

are called a **closed half-space** and an **open half-space** respectively. A **convex polyhedron** (or just **polyhedron**) is an intersection of finite number of half-spaces. A **closed polyhedral convex cone** is an intersection of finite number of half-spaces whose hyperplanes pass through the origin.

Given a polyhedron P . The set P is known as a relatively open polyhedron. A **support hyperplane** of P is a hyperplane H , given by $\langle u, x \rangle = c$, such that $P \cap H \neq \emptyset$ and P is contained in the half-space $\langle u, x \rangle \leq c$. The vector u is called a **normal vector** of P and the polyhedron $P \cap H$ is called a **face** of P . We always assume that P is a face of itself. We denote by $\sigma \preceq P$ that σ is a face of P , and call relatively open polyhedron $\overset{\circ}{\sigma}$ a **face** of $\overset{\circ}{P}$; we also call $\overset{\circ}{\sigma}$ a face of P . Every polyhedron P is the intersection of the half-spaces of its support hyperplanes,

$$P = \bigcap_H H(\phi \leq c), \quad \overset{\circ}{P} = \bigcap_H H(\phi < c),$$

where the intersections are extended over all supporting hyperplanes of P . Moreover, we have the disjoint union

$$F = \bigsqcup_{\sigma \preceq P} \overset{\circ}{\sigma}.$$

We introduce the following classes of sets of \mathbb{R}^n :

- \mathcal{P}^n = the class of convex polyhedra,
- \mathcal{P}_c^n = the class of polytopes (compact convex polyhedra),
- \mathcal{K}^n = the class of compact convex sets (convex bodies),
- \mathcal{C}^n = the class of closed convex sets,
- \mathcal{O}^n = the class of relatively open convex sets.

Exercise 2. Show that a subset P of \mathbb{R}^n is a polytope if and only if P is bounded and is the intersection of finitely many closed half-spaces.

Fix two real numbers r and s such that $r + s = 1$. For each real-valued function f on \mathbb{R} , if f is continuous everywhere except discontinuous at finite number of places, and f has left and right limits at those discontinuous points (including at infinity), we introduce a so called **Euler-Schanuel integral**

$$\chi(f) = \int f(x) d\chi(x) = \sum_{x \in \mathbb{R} \cup \{\infty\}} [f(x) - rf(x^-) - sf(x^+)], \quad (2)$$

where $f(x^-)$ and $f(x^+)$ are the left and right limits of f at x , and $f(\infty) = 0$, $f(\infty^+) = \lim_{x \rightarrow -\infty} f(x)$, $f(\infty^-) = \lim_{x \rightarrow +\infty} f(x)$. The collection of Euler-Schanuel integrable functions forms a real vector space. The indicator function $1_{(a,b)}$ of an open interval (a,b) and the delta function δ_c at a point c are Euler-Schanuel integrable; moreover,

$$\chi(1_{(a,b)}) = -1, \quad \chi(\delta_c) = 1,$$

where $\delta_c(x) = 1$ if $x = c$ and $\delta_c(x) = 0$ otherwise, it is possible that $a = -\infty$ and $b = +\infty$. Since each step function is a linear combination of some delta functions and indicator functions of some open intervals, all step functions are Euler-Schanuel integrable.

More generally, for f a real-valued function on \mathbb{R}^n with bounded support, we define the the **Euler-Schanuel integral** of f as the following iterated integral

$$\chi(f) = \int f(x) d\chi(x) = \int \cdots \int f(x_1, \dots, x_n) d\chi(x_1) \cdots d\chi(x_n), \quad (3)$$

which is well-defined provided that the resulted integrand from each iterated integral is continuous, except at finite number of points and has left and right limits at those discontinuous points. Again the class of Euler-Schanuel integrable multivariable functions is a real vector space. For a set S of \mathbb{R}^n , its **indicator function** 1_S is defined as $1_S(x) = 1$ for $x \in S$ and $1_S(x) = 0$ otherwise. If 1_S is Euler-Schanuel integrable, the value $\chi(S) := \chi(1_S)$ is called the **Euler number** of S .

Lemma 1.2. *For each bounded relatively open convex set U , its Euler-Schanuel integral*

$$\chi(U) = \chi(1_U) = (-1)^{\dim U}. \quad (4)$$

Proof. It is true when $n = 1$ since either $\dim U = 0$ or $\dim U = 1$. For $n \geq 2$, consider the orthogonal projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$. It is clear that $\pi(U)$ is a bounded relatively open convex set of \mathbb{R}^{n-1} . For each point $p \in \mathbb{R}^{n-1}$, the intersection $\pi^{-1}(p) \cap U$ is either empty or a singleton or an open interval, we then have

$$\begin{aligned} \int 1_U(p, x_n) d\chi(x_n) &= \int 1_{\pi^{-1}(p) \cap U}(x_n) d\chi(x_n) \\ &= \begin{cases} 0 & \text{if } \pi^{-1}(p) \cap U \text{ is empty,} \\ 1 & \text{if } \pi^{-1}(p) \cap U \text{ is a singleton,} \\ -1 & \text{if } \pi^{-1}(p) \cap U \text{ is an open interval.} \end{cases} \end{aligned}$$

It is easy to see that $\pi^{-1}(p) \cap U = \emptyset$ if and only if $p \notin \pi(U)$. Note that if $\pi^{-1}(p) \cap U$ is a singleton for one point $p \in \pi(U)$, then $\pi^{-1}(y) \cap U$ is a singleton for all $y \in \pi(U)$; if $\pi^{-1}(p) \cap U$ is an open interval for one point $p \in \pi(U)$, then $\pi^{-1}(y) \cap U$ is an open interval for all $y \in \pi(U)$. We see that $\dim \pi^{-1}(p) \cap U = \dim U - \dim \pi(U)$ for $p \in \pi(U)$. Thus

$$\int 1_U(y, x_n) d\chi(x_n) = (-1)^{\dim U - \dim \pi(U)} 1_{\pi(U)}.$$

By induction, $\int 1_{\pi(U)}(y) d\chi(y) = (-1)^{\dim \pi(U)}$; it then follows that

$$\chi(U) = (-1)^{\dim U - \dim \pi(U)} \int 1_{\pi(U)}(y) d\chi(y) = (-1)^{\dim U}.$$

□

A subset of a real vector space of finite dimension is said to be **polyhedral** if it can be obtained by taking union, intersection, and relative complement finitely many times of half-spaces. The class of polyhedral sets of \mathbb{R}^d is the relative Boolean algebra $\mathbf{B}(\tilde{\mathcal{P}}^n)$ generated by the class \mathcal{P}^n of convex polyhedra of \mathbb{R}^n . It is known that every polyhedral set is a disjoint union of finitely many relatively open convex polyhedra.

Theorem 1.3. *Let P be a polyhedral set decomposed into disjoint relatively open polyhedra. Let α_k denote the number of k -dimensional relatively open polyhedra in the decomposition. Then*

$$\chi(P) = \sum_k (-1)^k \alpha_k. \quad (5)$$

Proof. Let P be decomposed into a disjoint relatively open convex polyhedra σ_i . Then $1_P = \sum_i 1_{\sigma_i}$ and $\chi(1_{\sigma_i}) = (-1)^{\dim \sigma_i}$. Thus

$$\chi(P) = \chi(1_P) = \sum_i \chi(1_{\sigma_i}) = \sum_i (-1)^{\dim \sigma_i} = \sum_k (-1)^k \alpha_k.$$

□

Example 1. The Euler number of an oriented surface Σ_g of genus g in \mathbb{R}^3 is

$$\chi(\Sigma_g) = 2 - 2g.$$

Assume that Σ_g is standardly sitting in \mathbb{R}^3 and its orthogonal projection to \mathbb{R}^2 is the figure in Figure 2 with $g = 3$. Taking the Euler-Schanuel integral of the indicator function 1_{Σ_3} of Σ_3 along the last coordinate x_3 , the result is a function $f(x_1, x_2)$ on \mathbb{R}^2 , whose level set is indicated in Figure 2. Taking the Euler-Schanuel integral of f along the coordinate x_2 , we obtain a function $h(x_1)$ on \mathbb{R} whose graph is given in Figure 3. Taking the Euler-Schanuel integration for h , we see that the Euler number of Σ_3 is $\chi(1_{\Sigma_3}) = 2 - 6 = -4$. The general case of $\chi(\Sigma_g) = 2 - 2g$ is analogous.

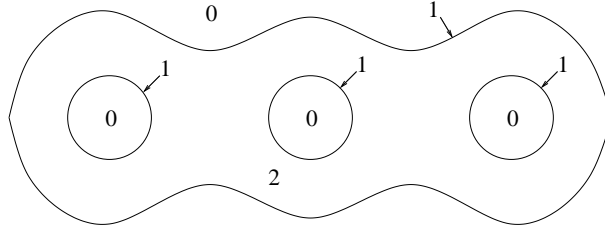


Figure 2: Projection of Σ_3 to \mathbb{R}^2 .

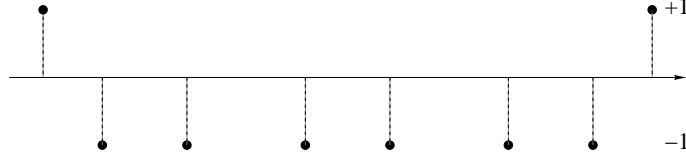


Figure 3: Projection of Σ_3 to \mathbb{R} .

2 Gram-Sommerville Theorem and Gauss-Bonnet Theorem

Recall the class \mathcal{P}^d of convex polyhedra of \mathbb{R}^d and the relative Boolean algebra $\mathbf{B}(\mathcal{P}^d)$ generated by \mathcal{P}^d . Let $\mathbf{F}(\mathcal{P}^d)$ denote the vector space of functions of the form $\sum_i a_i 1_{P_i}$, where $a_i \in \mathbb{R}$ and $P_i \in \mathcal{P}^d$. Each member f of $\mathbf{B}(\mathcal{P}^d)$ can be characterized as a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, called a **polyhedral function**, such that (i) the image of f is a finite set, (ii) for each value c of f the inverse image $f^{-1}(c)$ is a polyhedral set. Every polyhedral function is Euler-Schanuel integrable.

Let P be a polyhedral set of \mathbb{R}^d . A **face system** of P is a collection $\mathcal{F}(P)$ of relatively open convex polyhedra such that

$$\bar{P} = \bigcup_{\sigma \in \mathcal{F}(P)} \sigma \quad (\text{disjoint})$$

and the faces of each member σ of $\mathcal{F}(P)$ are also members of $\mathcal{F}(P)$. Each member of $\mathcal{F}(P)$ is called a **face** of P with respect to $\mathcal{F}(P)$.

A **cone** with apex 0 is a subset C of \mathbb{R}^d such that if $v \in C$ then $tv \in C$ for all $t > 0$. The **tangent cone of P at a point p** of \mathbb{R}^d is the set

$$\text{cone}(P, p) = \{v \in \mathbb{R}^d : \exists \varepsilon > 0 \text{ s.t. } p + tv \in P \forall 0 < t < \varepsilon\}. \quad (6)$$

The tangent cone is empty if $p \notin \bar{P}$. The **tangent cone** of P at a face ρ with respect to a face system $\mathcal{F}(P)$ is

$$\text{cone}(P, \rho) := \text{cone}(P, p), \quad p \in \rho. \quad (7)$$

The **tangent indicator** of P at ρ is the function

$$T_\rho(P) := 1_{\text{cone}(P, \rho)}. \quad (8)$$

Given a relatively open convex polyhedron σ . The **tangent cone of σ at its face ρ** is

$$\text{cone}(\sigma, \rho) := \text{cone}(\sigma, p), \quad p \in \rho.$$

Let f be a polyhedron function on \mathbb{R}^d . A **face system** of f is a face system of $\overline{\text{supp}}(f)$, the closure of the support $\{x \in \mathbb{R}^d : f(x) \neq 0\}$ of f , such that $f(x)$ is constant on each σ of $\mathcal{F}(f)$. The **tangent indicator** of f at its face ρ with respect to a face system $\mathcal{F}(f)$ is

$$T_\rho(f) = \sum_{\sigma \in \mathcal{F}(f)} f(\sigma) T_\rho(\sigma),$$

where $f(\sigma) = f(x)$ for any $x \in \sigma$, and the **tangent indicator** of f at ∞ is

$$T_\infty(f) = \sum_{\sigma \in \mathcal{F}(f)} f(\sigma) T_\infty(\sigma).$$

One can define tangent indicator intrinsically for polyhedral functions without face system. The **tangent indicator of f at a point x** is a function $T_x(f) : \mathbb{R}^d \rightarrow \mathbb{R}$, defined for $v \in \mathbb{R}^d$ by

$$T_x(f)(v) = \lim_{t \rightarrow 0^+} f(x + tv). \quad (9)$$

Each vector v can be viewed as a linear operator on $\mathbf{F}(\mathcal{P}^d)$ such that $v(f)(x) = T_x(f)(v)$. The **tangent cone of f at x** is the set

$$\text{cone}(f, x) = \{v \in \mathbb{R}^d : T_x(f)(v) \neq 0\}. \quad (10)$$

The map $T(f) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $(x, v) \mapsto f_v(x)$, is called the **tangent indicator** of f . The **tangent curvature** of f at a face ρ with respect to a face system $\mathcal{F}(f)$ and at a point $x \in \rho$ is

$$\tau_\rho(f) = \tau_x(f) = \frac{1}{\text{vol}(B^d)} \int_{B^d} T_x(f)(v) dv, \quad (11)$$

where B^d is the unit ball centered at the origin of \mathbb{R}^d .

Lemma 2.1. *Let $f \in \mathbf{F}(\mathcal{P}^d)$ and $v \in \mathbb{R}^d$. The map $T(f) : \mathbb{R}^d \rightarrow \mathbf{F}(\mathcal{P}_0^d)$, $x \mapsto T_x(f)$, is a polyhedral map, and $f_v : \mathbb{R}^d \rightarrow \mathbb{R}$, $x \mapsto T_x(f)(v)$, is a polyhedral function. Moreover,*

$$T(f) = \sum_{\rho \in \mathcal{F}(f)} 1_\rho \cdot T_\rho(f) = \sum_{\rho \in \mathcal{F}(f)} 1_\rho \cdot \sum_{\sigma \succeq \rho} f(\sigma) 1_{\text{cone}(\sigma, \rho)},$$

where $T_\rho(f) = T_x(f)$ with $x \in \rho$; and $T(f)$ and $T(f)(v)$ are integrable with respect to the Euler valuation χ .

Proof. Given a face system $\mathcal{F}(f)$ of f . Notice that $f = \sum_{\sigma \in \mathcal{F}(f)} f(\sigma) 1_\sigma$, where $f(\sigma) = f(x)$ with $x \in \sigma$, and $T_\sigma(f) = T_x(f)$ for each $\sigma \in \mathcal{F}(f)$ and all $x \in \sigma$. Then for $v \in \mathbb{R}^d$,

$$T_x(f)(v) = \sum_{\sigma \in \mathcal{F}(f)} f(\sigma) T_x(1_\sigma)(v).$$

Let σ' denote the side of σ that can be seen from $-\infty v$. Then σ' is a disjoint union of some faces of σ , and $T(1_\sigma)(v) = 1_{\sigma \cup \sigma'}$ is a polyhedral function of variable x . By linearity $f_v := T(f)(v)$ is a polyhedral function of variable x . Of course, f_v is integrable with respect to χ . Moreover,

$$T(f) = \sum_{\rho \in \mathcal{F}(f)} 1_\rho \cdot T_\rho(f) = \sum_{\rho \in \mathcal{F}(f)} 1_\rho \cdot \sum_{\sigma \succeq \rho} f(\sigma) 1_{\text{cone}(\sigma, \rho)},$$

which can be viewed as a function with values in $\mathbf{F}(\mathcal{P}_0^d)$. □

Theorem 2.2 (Gram-Sommerville Formula). *Let f be a polyhedral function on \mathbb{R}^d with bounded support and a face system $\mathcal{F}(f)$. Then*

$$\int_{\mathbb{R}^d} T_x(f) d\chi(x) = \sum_{\rho \in \mathcal{F}(f)} (-1)^{\dim \rho} T_\rho(f) = \chi(f) 1_{\{0\}}, \quad (12)$$

where $1_{\{0\}}$ is the indicator function of the set whose only element is the origin 0 of \mathbb{R}^d .

Proof. First Proof. For each face ρ of f with respect to $\mathcal{F}(f)$, note that $T_\rho(f) = T_x(f)$ for $x \in \rho$. For each fixed vector $v \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} T_x(f)(v) d\chi(x) = \int \sum_{\rho \in \mathcal{F}(f)} T_\rho(f)(v) 1_\rho d\chi(x) = \sum_{\rho \in \mathcal{F}(f)} (-1)^{\dim \rho} T_\rho(f)(v),$$

which is the first identity in (12). To see the second identity, note that $T_x(f)(v) = f_v(x)$ and

$$\int_{\mathbb{R}^d} T_x(f)(v) d\chi(x) = \int_{\mathbb{R}^d} f_v(x) d\chi(x).$$

If $v \neq 0$, then for each point $p \in \mathbb{R}^d$, the support of f_v on the line $p + \mathbb{R}v$ is a disjoint union of finitely many half-closed and half-open intervals, and f_v is constant on each of such intervals. In order to compute the integral $\int f_v(x) d\chi(x)$, we choose a frame (v_1, \dots, v_d) of \mathbb{R}^d with $v_d = v$. We see that

$$\int f_v(x_1 v_1 + \dots + x_d v_d) d\chi(x_d) = 0,$$

consequently, $\int_{\mathbb{R}^d} f_v(x) d\chi(x) = 0$. If $v = 0$, then $f_v(x) = f(x)$, consequently, $\int f_v(x) d\chi(x) = \chi(f)$.

Second Proof. Since $f = \sum f(\sigma) 1_\sigma$, it suffices to show that it is true for each indicator 1_σ . Clearly, $(1_\sigma)_v(x)$ is either 0 or 1 for all x . If $v \neq 0$, then $(1_\sigma)_v(x) = 1$ iff $v \in \text{cone}(\sigma, x)$, i.e., $\sigma \cap (x + \mathbb{R}v)$ is an open

segment (p, q) with direction v and $x \in [p, q)$. It follows that if $v \neq 0$, the support of $(1_\sigma)_v$ is the union of σ and the boundary side of σ that can be seen from the direction v , and this boundary side is homeomorphic to a cell of dimension $\dim \sigma - 1$. Thus $\chi((1_\sigma)_v) = 0$ if $v \neq 0$. If $v = 0$, then $(1_\sigma)_v = 1_\sigma$, consequently, $\chi((1_\sigma)_0) = \chi(1_\sigma)$. Now by linearity of χ , we have

$$\begin{aligned} \int_{\mathbb{R}^d} f_v(x) d\chi(x) &= \sum_{\sigma} f(\sigma) \int_{\mathbb{R}^d} (1_\sigma)_v(x) d\chi(x) \\ &= \sum_{\sigma} f(\sigma) \chi(1_\sigma) 1_{\{0\}} = \chi(f) 1_{\{0\}}. \end{aligned}$$

□

Corollary 2.3. *Let f be a polyhedral function on \mathbb{R}^d with bounded support. Then*

$$\int \tau_x(f) d\chi(x) = \sum_{\rho \in \mathcal{F}(f)} (-1)^{\dim \rho} \tau_\rho(f) = 0.$$

Let \mathcal{P}_0^d denote the class of closed polyhedral convex cones of \mathbb{R}^d with apex 0. Let $\mathbf{F}(\mathcal{P}_0^d)$ denote the vector space generated by indicator functions of polyhedral cones with apex 0. There is a linear operator $N : \mathbf{F}(\mathcal{P}_0^d) \rightarrow \mathbf{F}(\mathcal{P}_0^d)$, defined for $g \in \mathbf{F}(\mathcal{P}_0^d)$ and $u \in \mathbb{R}^d$ by

$$N(g)(u) = \chi(g \cdot 1_{\{v \in \mathbb{R}^d : \langle u, v \rangle \leq 0\}}) = \int_{\{v \in \mathbb{R}^d : \langle u, v \rangle \leq 0\}} g(x) d\chi(x). \quad (13)$$

Let C be a relatively open convex cone of \mathbb{R}^d with apex 0. The **normal cone** of C is the closed convex cone

$$C^* = \{u \in \mathbb{R}^d : \langle u, v \rangle \leq 0 \text{ for all } v \in C\}.$$

Lemma 2.4. *For each relatively open convex cone $C \in \mathcal{P}_0^d$, we have*

$$N(1_C) = (-1)^{\dim C} 1_{C^*}.$$

Proof. The map N is well-defined and is linear since χ is a linear functional. Given C a relatively open convex cone. For each $u \in C^*$, it is clear that $C \subseteq H_u^-$, we then have

$$N(1_C)(u) = \chi(C \cap H_u^-) = \chi(C) = (-1)^{\dim C}.$$

For $u \notin C^*$, we have either $C \cap H_u^- = \emptyset$ or $C \cap H_u^- \neq \emptyset$. In latter case the set $C \cap H_u^-$ is a half-closed and half-open convex cone. Thus $N(1_C)(u) = \chi(C \cap H_u^-) = 0$. □

Let $f \in \mathbf{F}(\mathcal{P}^d)$ be a polyhedron function with a face system $\mathcal{F}(f)$. The **normal indicator** of f at a face ρ with respect to a face system $\mathcal{F}(f)$ and at a point $x \in \rho$ is

$$K_\rho(f) = K_x(f) := N(T_x(f)). \quad (14)$$

The **Gaussian curvature** of f at a face ρ with respect to $\mathcal{F}(f)$ and at point $x \in \rho$ is

$$\kappa_\rho(f) = \kappa_x(f) := \frac{1}{\text{vol}(B^d)} \int K_x(f)(v) dv. \quad (15)$$

Theorem 2.5 (Generalized Gauss-Bonnet Formula). *Let f be a polyhedron function on \mathbb{R}^d with bounded support and a face system $\mathcal{F}(f)$. Then*

$$\int_{\mathbb{R}^d} K_x(f) d\chi(x) = \sum_{\rho \in \mathcal{F}(f)} K_\rho(f) = \chi(f) 1_{\mathbb{R}^d}, \quad (16)$$

where $1_{\mathbb{R}^d}$ is the indicator function of the whole space \mathbb{R}^d .

Proof. For each face $\rho \in \mathcal{F}(f)$, note that $T_x(f) = T_\rho(f)$, we have $K_x(f) = K_\rho(f)$. Then

$$\begin{aligned} \int_{\mathbb{R}^d} K_x(f) d\chi(x) &= \sum_{\rho \in \mathcal{F}(f)} (-1)^{\dim \rho} K_\rho(f) \\ &= \sum_{\rho \in \mathcal{F}(f)} (-1)^{\dim \rho} N(T_\rho(f)) \\ &= N\left(\sum_{\rho \in \mathcal{F}(f)} (-1)^{\dim \rho} T_\rho(f)\right) \\ &= N(\chi(f) 1_{\{0\}}) = \chi(f) 1_{\mathbb{R}^d}. \end{aligned}$$

It is trivial when $v = 0$, the values at v of both sides are the same. Assume $v \neq 0$. Let $\pi : \mathbb{R}^d \rightarrow L^\perp$ and $\pi_v : \mathbb{R}^d \rightarrow L$ be the orthogonal projections, where $L = \mathbb{R}v$. Let X be decomposed into frame cells σ_i with respect to a frame, where v is a member of the frame. For the cell σ_i , we may further assume that $\bar{\sigma}_i \cap \pi^{-1}(a_{i+1})$ is a closed frame cell; otherwise we may divide $\bar{\sigma}_i \cap \pi_v^{-1}(a_{i+1})$ so that \cdot . Thus $K_x(\sigma_i)(v) = \chi(\sigma_i)$ if $x \in \pi_v^{-1}(a_{i+1})$ and $K_x(\sigma_i)(v) = 0$ otherwise. Then

$$\int K_x(\sigma_i)(v) d\chi(x) = \chi(\sigma_i).$$

Hence

$$\int K_x(X)(v) d\chi(x) = \sum \int K_x(\sigma_i)(v) d\chi(x) = \sum \chi(\sigma_i) = \chi(X).$$

$\pi : \mathbb{R}^d \rightarrow L$ be the orthogonal projection, where $L = \mathbb{R}v$. Then \mathbb{R} can be divided into finite singletons $\{a_i\}$ and open intervals (a_i, a_{i+1}) so that $\pi^{-1}(a_i, a_{i+1}) \simeq (a_i, a_{i+1}) \times \pi^{-1}(a)$, where $a \in (a_i, a_{i+1})$ is a fixed number. Let $\pi^{-1}(a)$ be decomposed into disjoint cells σ_{ij} . Then $K_x(\sigma_{ij})(v) = 0$ for $x \in \pi^{-1}(a_i, a_{i+1})$ and $K_x(\sigma_{ij})(v) = (-1)^{\dim \sigma_{ij}}$ for $x \in \bar{\sigma}_{ij} \cap \pi^{-1}(a_{i+1})$. \square

Corollary 2.6. *Let f be a polyhedral function on \mathbb{R}^d with bounded support. Then*

$$\int \kappa_x(f) d\chi(x) = \sum_{v \in V(f)} \kappa_v(f) = \chi(f).$$

$$K_\infty(f)(v) =$$

Proposition 2.7 (Frame Dependence). *Given a function on a real vector n -space V . If $\chi(f, v) = \chi(f, v_\pi)$ for each frame $v = (v_1, \dots, v_n)$ and all permutations π of $\{1, \dots, n\}$, where $v_\pi = (v_{\pi(1)}, \dots, v_{\pi(n)})$, then $\chi(f, v)$ is independent of the chosen frame v .*

Proof. Given two frames u and v of V , there exists nonsingular $n \times n$ matrix A such that $u = vA$. Note that A can be decomposed into a product of elementary matrices. For $A = \text{Diag}[1, \dots, a, \dots, 1]$ with $a \neq 0$, we have $u = (v_1, \dots, av_i, \dots, v_n)$. For each fixed coordinate $x = (x_{i+1}, \dots, x_n)$, the function

$$g(x_i, x) := \int f(x_1 u_1 + \dots + x_i u_i + \dots + x_n u_n) d\chi(x_1) \cdots d\chi(x_{i-1})$$

of one variable x_i is discontinuous at (finitely many) points $p_k(x)$. Then the function

$$\begin{aligned} g_1(x_i, x) &= \int f(x_1 u_1 + \dots + x_i (a u_i) + \dots + x_n u_n) d\chi(x_1) \cdots d\chi(x_{i-1}) \\ &= g(ax_i, x) \end{aligned}$$

of one variable x_i is discontinuous at points of $p_k(x)/a$. Thus

$$\begin{aligned} \int g(x_i, x) d\chi(x_i) &= \sum_k \left(g(p_k(x), x) - r g(p_k(x)^-, x) - s g(p_k(x)^+, x) \right) \\ &= \sum_k \left(g_1(p_k(x)/a, x) - r g_1((p_k(x)/a)^-, x) - s g_1((p_k(x)/a)^+, x) \right) \\ &= \int g_1(x_i, x) d\chi(x_i). \end{aligned}$$

It follows that

$$\chi(f, u) = \int g(x_i, x) d\chi(x_i) d\chi(x) = \int g(x_i, x) d\chi(x_i) d\chi(x) = \chi(f, v).$$

Let A be an elementary matrix such that $u = (v_1, \dots, v_i, \dots, v_j + av_i, \dots, v_n)$ and $i < j$. Then $\|v_j\| = b\|v_j + av_i\|$ for some $b \neq 0$. Note that for $x = (x_{j+1}, \dots, x_n)$,

$$g_2(x_j, x) := \int f(x_1 v_1 + \dots + x_j (v_j + av_i) + \dots + x_n v_n) d\chi(x_1) \cdots d\chi(x_{j-1}) = g(bx_j, x).$$

Likewise,

$$\chi(f, u) = \int g(x_j, x) d\chi(x_j) d\chi(x) = \int g_2(x_j, x) d\chi(x_j) d\chi(x) = \chi(f, v).$$

Now if the matrix A is such that $u = (v_1, \dots, v_j, \dots, v_i, \dots, v_n)$ with $i < j$, then we have $\chi(f, u) = \chi(f, v)$ by assumption. It follows from the previous argument that $\chi(f, u) = \chi(f, v')$, where $v' = (v_1, \dots, v_j, \dots, v_i + av_j, \dots, v_n)$. Switching the entries at i and j in v' , we see that $\chi(f, v') = \chi(f, v'')$, where $v'' = (v_1, \dots, v_i + av_j, \dots, v_j, \dots, v_n)$. We thus have $\chi(f, v) = \chi(f, v'')$. We have shown that the Euler-Schanuel integral of f is independent of the selection of frames. \square

Example 2. The iterated Euler-Schanuel integral depends on the order of a basis. Consider the function

$$f(x, y) = \begin{cases} y^2 & \text{if } x^2 + y^2 < 4, \\ 0 & \text{otherwise.} \end{cases}$$

The Euler-Schanuel integral of f with respect to the frame (e_1, e_2) is

$$\begin{aligned} \iint f d\chi(x) d\chi(y) &= \int_{(-2,2)} y^2 d\chi(y) \int_{(-\sqrt{4-y^2}, \sqrt{4-y^2})} d\chi(x) \\ &= \int_{(-2,2)} (-y^2) d\chi(y) = 4. \end{aligned}$$

However, the Euler-Schanuel integral of f with respect to the frame (e_2, e_1) is

$$\begin{aligned} \iint f d\chi(y) d\chi(x) &= \int_{(-2,2)} d\chi(x) \int_{(-\sqrt{4-x^2}, \sqrt{4-x^2})} y^2 d\chi(y) \\ &= \int_{(-2,2)} (-(4-x^2)) d\chi(x) = 0. \end{aligned}$$

3 Another Eulerian valuation

Let P be a relatively open convex polyhedron, possibly unbounded. We introduce **tangent cone of P at ∞** as the set

$$\text{cone}(P, \infty) = \{-v \in V : v \neq 0, \exists p \in P \text{ s.t. } p + tv \in P \forall t > 0\}. \quad (17)$$

The **tangent indicator of σ at ∞** is the function

$$T_\infty(P) = (-1)^{\dim P - 1} 1_{\text{cone}(P, \infty)}.$$

We shall see that

$$T_\infty(P) + \sum_{\sigma \preceq P} (-1)^{\dim \sigma} T_\sigma(P) = \chi(P) 1_{\{o\}}.$$

Given a function $f \in \mathbf{F}(\mathcal{P}^d)$. We introduce a **tangent indicator of f at ∞** is a function $T_\infty(f) : \mathbb{R}^d \rightarrow \mathbb{R}$, defined for nonzero $v \in \mathbb{R}^d$ by

$$T_\infty(f)(-v) = - \lim_{r \rightarrow \infty} \int_{\text{cone}(o, B(rv)) \setminus o * B(rv)} f(x) d\chi(x) \quad (18)$$

and $T_\infty(f)(0) = 0$, where $B(rv)$ is the unit open ball of \mathbb{R}^d centered at rv , $o * B(rv)$ is the join of o and $B(rv)$, and $\text{cone}(o, B(rv))$ is the open convex cone generated by $B(rv)$.

Lemma 3.1. *Let P be a relatively open convex polyhedron of \mathbb{R}^d , given as the intersection of half-spaces $H(\phi_i < a_i)$ and hyperplanes $H(\psi_j = b_j)$. Then $\text{cone}(P, \infty)$ is a convex polyhedral cone and*

$$\text{cone}(P, \infty) = \bigcap_{i,j} H(\phi_i \geq 0) \cap H(\psi_j = 0) \setminus \{0\}.$$

Moreover, the tangent indicator of 1_P at infinity is given by

$$T_\infty(1_P) = (-1)^{\dim P - 1} 1_{\text{cone}(P, \infty)}.$$

Proof. Let $-v \in \text{cone}(P, \infty)$, i.e., $v \neq 0$ and $p + \mathbb{R}_+ v \subseteq P$ for some $p \in P$. Clearly, it is equivalent to $p + \mathbb{R}_+ v \subseteq H(\phi_i \leq a_i)$ for all i . By translation it is further equivalent to $\mathbb{R}_+ v \subseteq H(\phi_i \leq 0)$, i.e., $v \in \bigcap_i H(\phi_i \leq 0)$. This the same same $-v \in \bigcap_i H(\phi_i \geq 0)$.

As for the tangent indicator at infinity, since $p + \mathbb{R}_+ v$ and $\mathbb{R}_+ v$ are parallel half-lines of same direction, we see that $p + \mathbb{R}_+ v$ intersects the truncated infinite cone $\text{cone}(o, B(tv)) \setminus o * B(tv)$ for all $t > 0$, of course P intersects the truncated cone. So $T_\infty(1_P)(-v) = (-1)^{\dim P - 1}$.

Let $-v \notin \text{cone}(P, \infty)$ and $v \neq 0$. Then there exists an i such that $v \notin H(\phi_i \geq 0)$, i.e., $\phi_i(v) < 0$. Let t_0 a positive real number be such that the distance of $t_0 v$ to $H(\phi_i = a_i)$ is larger than t_0 . Then $H(\phi_i < a_i)$ is disjoint from the truncated cones $\text{cone}(o, B(tv)) \setminus o * B(tv)$ for all $t > t_0$. Of course, P is disjoint from the truncated cones. It follows that $T_\infty(1_P)(-v) = 0$. \square

We refer each ordered basis of a finite dimensional vector space a **frame**. A real-valued function f on a real vector n -space V is **Euler-Schanuel integrable with respect to the frame** $\mathbf{v} = (v_1, \dots, v_n)$ the function $F(x) = f(x_1v_1 + \dots + x_nv_n)$ with $x = (x_1, \dots, x_n)$ is Euler-Schanuel integrable. If so,

$$\chi(f, \mathbf{v}) = \int_{\mathbb{R}^n} F(x) d\chi(x), \quad (19)$$

is called the **Euler-Schanuel integral** of f with respect to the frame \mathbf{v} . If f is Euler-Schanuel integrable with respect to all frames, we simply say that f is **Euler-Schanuel integrable**. Let $\mathbf{E}_{r,s}(V)$ denote the set of Euler-Schanuel integrable functions on a real vector n -space V . It is clear that $\mathbf{E}_{r,s}(V)$ forms a vector space under ordinary additions and scalar multiplication of functions.

Theorem 3.2. *Let E be a subset of real vector n -space V . If 1_E is Euler-Schanuel integrable, then its Euler valuation $\chi(E, \mathbf{v})$ with respect to each frame \mathbf{v} is an integer, and is independent of the numbers r, s such that $r + s = 1$.*

Proof. We proceed by induction on the dimension n . For $n = 1$, the set E is a disjoint union of finitely many open intervals (may be unbounded) and singleton sets. Each open interval interval has Euler measure -1 and each singleton has Euler measure 1 . The Euler-Schanuel integral is an integer.

Assume that it is true for Euler sets of real vector spaces of dimension $n-1$. Given a frame $\mathbf{v} = (v_1, \dots, v_n)$ of V . For each fixed $x \in \mathbb{R}$, the function $F_x : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, defined by $F_x(x_1, \dots, x_{n-1}) = 1_E(x_1v_1 + \dots + x_{n-1}v_{n-1} + xv_n)$, is Euler-Schanuel integrable with respect to the frame (v_1, \dots, v_{n-1}) . Then the function

$$f(x) = \int F_x(x_1, \dots, x_{n-1}) d\chi(x_1) \cdots d\chi(x_{n-1})$$

is integer-valued by induction and is Euler-Schanuel integrable. It follows that $f(x)$ has the form

$$f = \sum_i \alpha_i 1_{(a_i, b_i)} + \sum_j \beta_j 1_{\{c_j\}},$$

where α_i and β_j are integers independent of r and s such that $r + s = 1$. Hence

$$\int 1_E(x_1v_1 + \dots + x_nv_n) d\chi(x_1) \cdots d\chi(x_n) = \int f(x) d\chi(x)$$

is an integer and is independent of r and s such that $r + s = 1$. □

Definition 3.3. **Frame cellular sets** are defined inductively as follows:

- (FC1) Every open segment or a singleton of any one-dimensional real vector space L is a **frame cell** with respect to each frame of L .
- (FC2) Let $\mathbf{v} = (v_1, \dots, v_n)$ be a frame of real vector d -space V , $H_i = \text{span}\{v_1, \dots, \hat{v}_i, \dots, v_n\}$, and let $\pi_i : V \rightarrow H_i$ be the projection with $\ker(\pi_i) = \text{span}\{v_i\}$. A subset $\sigma \subseteq V$ is a **frame cell** with respect to \mathbf{v} provided that $\pi_i(\sigma)$ is a frame cell with respect to the frame $(v_1, \dots, \hat{v}_i, \dots, v_n)$, and either $\pi_i^{-1}(x) \cap \sigma$ are open intervals for all $x \in \pi_i(\sigma)$ or are singletons for all $x \in \pi_i(\sigma)$.
- (FC3) A subset X of a real vector n -space V is **frame cellular** if X can be decomposed into a disjoint union of finitely many frame cells with respect to each frame of V .

Proposition 3.4. *The indicator function of any frame cellular set E is Euler-Schanuel integrable. Moreover, if $\{\sigma_i\}$ is a frame cellular decomposition of E with respect to a frame \mathbf{v} , then*

$$\chi(E) = \sum_i (-1)^{\dim \sigma_i}. \quad (20)$$

Proof. Let E be a frame cellular set of a real vector n -space V , and let $\mathbf{v} = (v_1, \dots, v_n)$ be a frame of V . It is clear that $\{\sigma_i\}$ is a frame cellular decomposition of E with respect to the frame $(v_{\pi(1)}, \dots, v_{\pi(n)})$ for all permutations π of $\{1, \dots, n\}$. Since $\chi(1_{\sigma_i}, \mathbf{v}) =$

Thus

$$\chi(1_E, \mathbf{v}) = \sum_i (-1)^{\dim \sigma_i} = \chi(1_E, v_\pi).$$

It follows from Proposition 2.7 that 1_E is Euler-Schanuel integrable, its integral is independent of the selection of frames, and $\chi(E, \mathbf{v}) = \sum_i (-1)^{\dim \sigma_i}$. □

Let σ be a frame cell of \mathbb{R}^d . The tangent cone

Let X be a nonempty closed subset of \mathbb{R}^d . A **smooth stratification** of X is a finite collection $\{X_a | a \in P\}$ of subsets of \mathbb{R}^d , such that the following three conditions are satisfied.

- Each X_a is a smooth submanifold of \mathbb{R}^d , called a **pure stratum** of X .
- $X = \bigcup_{a \in P} X_a$ (disjoint).
- If $X_a \cap \bar{X}_b$ then $\bar{X}_a \subseteq \bar{X}_b$; we say that X_a is a **face** of X_b if $\bar{X}_a \subseteq \bar{X}_b$, written $X_a \leq X_b$. So P becomes a poset, called the **poset of strata**.

The space X together with a stratification $\{X_a | a \in P\}$ is called a **stratified space**.

Given a nonzero vector $v \in \mathbb{R}^d$; let $L = \mathbb{R}v$ be the subspace spanned by v and $V = L^\perp$ the orthogonal complement of L . Let $\phi : \mathbb{R}^d \rightarrow L$ be the orthogonal projection. A nonempty closed subset X of \mathbb{R}^d is said to be **cylindric** along v if L can be divided into

set function $\phi : L \rightarrow V$

Given a nonzero vector $v \in \mathbb{R}^d$; we say that X is

- **locally finite at a point x along v** if the intersection $X \cap \{x + tv : 0 \leq t < \varepsilon\}$ is either an empty set, or a singleton, or an half-closed and half-open segment, when ε is small enough.
- **finite with respect to v** if for each point $x \in \mathbb{R}^d$, the intersection $X \cap \{x + tv : t \in \mathbb{R}\}$ is a finite union of singletons, or closed segments, or half-closed and half-open segments.
- **finite along v** if $X \cap \{x + tv : t \in \mathbb{R}\}$ is a finite union of singletons, or closed segments, or half-closed and half-open segments.

4 Group Arrangements

5 Grassmannian

Let $\text{Gr}_n(\mathbb{R}^{n+k})$ be the set of all n -dimensional vector subspaces of \mathbb{R}^{n+k} , and $\text{Gr}_n(\mathbb{R}^\infty)$ the set of all n -dimensional vector subspaces of \mathbb{R}^∞ . We shall make both $\text{Gr}_n(\mathbb{R}^{n+k})$ and $\text{Gr}_n(\mathbb{R}^\infty)$ into CW-complexes so that $\text{Gr}_n(\mathbb{R}^{n+k})$ is a subcomplex of $\text{Gr}_n(\mathbb{R}^\infty)$.

Let B^d denote the *unit ball* of \mathbb{R}^d , consisting of all vectors v with $\|v\| = 1$. The *interior* of B^d is defined to be the subset B_0^d consisting of all vectors v with $\|v\| < 1$. For the special case $p = 0$, both B^d and B_0^d consist of the single zero vector.

Any space homeomorphic to B^p is called a *closed p -cell*, and any space homeomorphic to B_0^p is called an *open p -cell*. For instance \mathbb{R}^p is an open p -cell.

A **CW complex** is a Hausdorff space X , called the **underlying space**, together with a partition of X into a collection $\{e_\alpha\}$ of disjoint subsets, such that the following four conditions are satisfied.

CW1 Each e_α is a topologically an open cell of dimension $d(\alpha) \geq 0$, that is, there exists a continuous map

$$\phi_\alpha : B^{d(\alpha)} \rightarrow X,$$

called the **characteristic map** for the cell e_α , such that $\phi_\alpha(\hat{B}^{d(\alpha)}) = e_\alpha$ and the restriction $\phi_\alpha : \hat{B}^{d(\alpha)} \rightarrow e_\alpha$ is a homeomorphism.

CW2 If $e_\alpha \cap \bar{e}_\beta \neq \emptyset$ then $\bar{e}_\alpha \subseteq \bar{e}_\beta$; we say that e_α is a **face** of e_β if $\bar{e}_\alpha \subseteq \bar{e}_\beta$. If $\{e_\alpha\}$ contain only finite number of cells, we say that X satisfying CW1 and CW2 is a **finite CW complex**.

CW3 **Closure Finiteness**. Each point of X is contained in a finite subcomplex.

CW4 **Whitehead Topology**. The space X is topologized as the direct limit of its finite subcomplexes, that is, a subset of X is closed if and only if its intersection with each finite subcomplex is closed in the finite subcomplex.