The Pick Theorem and the Proof of the Reciprocity Law for Dedekind Sums*

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Abstract

This paper is to provide some new generalizations of the Pick Theorem. We first derive a point-set version of the Pick Theorem for an arbitrary bounded lattice polyhedron, then using the idea of weight function of [2] to obtain a weighted version; other Pick type theorems known to the author for integral lattice $\mathbf{Z}^2$ are reduced to some special cases of the generalization. Finally, using the idea of Ehrhart [6] and the Pick Theorem, we give a direct proof of the reciprocity law for Dedekind sums. The ideas and methods presented here may be pushed to higher dimensions.

1 Point-Set Version of the Pick Theorem

Let P be a lattice polygon of $\mathbf{R}^2$, i.e., the vertices of P are points of the integral lattice $\mathbf{Z}^2$. Let $i(P)$ be the number of lattice points of P and $i(\partial P)$ the number of lattice points of its boundary $\partial\sigma$. The Pick Theorem says that

$$\text{area}\,(P) = i(P) - \frac{1}{2}i(\partial P) - 1. \tag{1}$$

Given a bounded lattice polyhedron X of $\mathbf{R}^2$; we denote by $\bar{X}$ the closure of X and by $\text{int}\,X$ the interior of X; the **frontier** of X is the set $\bar{X} - \text{int}X$. The **link** of X near a point $x \in \bar{X}$ is the intersection of X and a circle $S^1(x, r)$ centered at x with small enough radius r; the Euler characteristic $\chi(\text{lk}\,(x, X))$ is a local topological invariant, which plays an important role in our Pick type theorems. For an interior point $x \in \text{int}\,X$, the link $\text{lk}\,(x, X)$ is a circle and has Euler characteristic zero. If X is closed, then for any $x \in \text{fr}\,X$, the link $\text{lk}\,(x, X)$ is a collection of finite number of arcs and points, so

$$\chi(\text{lk}\,(x, X)) = \text{the number of branches near } x.$$

*Research is supported by RGC Competitive Earmarked Grant HKUST6119/00P

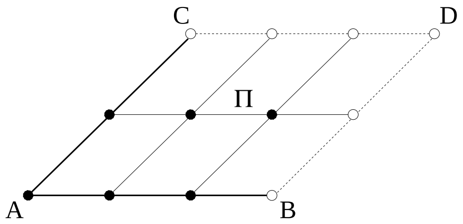

Figure 1: Π tiles the whole plane disjointly

For any positive integer m, we denote by $i(X,m)$ the number of lattice points of the dilation $mX = \{mx : x \in X\}$; and will see that $i(X,m)$ is a polynomial function of m of degree at most 2. We take $i(X,-m)$ as the value of the polynomial function $i(X,t)$ at $t = -m$; note that $i(X,-m)$ is very *different* from $i(-mX)$, which is the number of lattice points inside the set $-mX$ (the reflection of mX at the origin) and is the same as $i(mX)$.

Theorem 1.1 *Let X be a bounded lattice polyhedron of $\mathbf{R}^2$. Then*

$$\text{area}\,(X) = i(X) - \chi(X) - \frac{1}{2}\sum_{x\in\mathbf{Z}^2\cap\text{fr}\,X}\chi(\text{lk}\,(x,X)). \tag{2}$$

Proof Let ABC be a closed lattice triangle with vertices A, B, C; and denote by a, b, c, and d the numbers of lattice points inside the relative interior of the segments BC, CA, AB, and the interior of the triangle ABC, respectively. We extend the triangle ABC into a parallelogram $ABCD$; the vertex D must be a lattice point. Let Π be the half-closed and half-open parallelogram obtained from $ABCD$ by removing the segments BD and CD; the points B and C must have been removed; see Figure 1. Then, for any positive integer m,

$$i(\Pi, m) = i(m\Pi) = m^2 i(\Pi).$$

We divide Π into a disjoint union of an open triangle σ (whose closure is the triangle ABC) and the interior of the triangle BCD, the half-closed and half-open segment $[AB)$, and the two open segments (AC) and (BC). The interior of BCD is just a lattice translation of the open triangle $-\sigma$ (the reflection of σ at the origin). Assume the vertex A is at the origin. On the one hand,

$$\begin{aligned} i(\Pi, m) &= 2\,i(m\sigma) + i(m[AB)) + i(m[BC)) + i(m[CA)) - 2 \\ &= 2\,i(\sigma, m) + m\,\{i([AB)) + i([BC)) + i([CA))\} - 2; \end{aligned}$$

and on the other hand,

$$m^2 i(\Pi) = m^2\{2\,i(\sigma) + i([AB)) + i([BC)) + i([CA)) - 2\}.$$

Note that $i(\sigma) = d$, $i([AB)) = c+1$, $i([BC)) = a+1$, $i([AC)) = b+1$. It follows that

$$i(\sigma, m) = \left(d + \frac{a+b+c+1}{2}\right)m^2 - \left(\frac{a+b+c+3}{2}\right)m + 1.$$

Obviously, the coefficient of m is $-i(\partial\sigma)/2$. If the closed triangle $\bar{\sigma}$ has lattice points only at its vertices, then $a = b = c = d = 0$; and in this case the area of σ must be $1/2$ (need be checked, but easy), which is the coefficient of m^2 in $i(\sigma, m)$; such lattice triangles are called **primitive**. Since any open lattice triangle can be a disjoint union of finitely many primitive open lattice triangles, primitive open lattice segments, and some lattice points, thus for an arbitrary open lattice triangle σ,

$$i(\sigma, m) = \mathrm{area}\,(\sigma)\, m^2 - \frac{i(\partial\sigma)}{2}\, m + 1.$$

Repeat the above argument similarly for the closed lattice triangle $\bar{\sigma}$; we have

$$i(\bar{\sigma}, m) = \mathrm{area}\,(\sigma)\, m^2 + \frac{i(\partial\sigma)}{2}\, m + 1.$$

This shows that $i(\sigma, -m) = i(\bar{\sigma}, m)$. In fact, if σ is a lattice simplex of dimension at most 2, i.e., if σ is either an open lattice triangle, or an open lattice segment, or a lattice point, then

$$i(\sigma, -m) = (-1)^{\dim\sigma} i(\bar{\sigma}, m). \tag{3}$$

Now we consider a bounded lattice polyhedron X of $\mathbf{R}^2$ and a lattice triangulation Δ of X, i.e., Δ is a collection of disjoint open lattice triangles, open lattice segments, and some lattice points such that the union is the whole set X; define

$$L(X, m) = \frac{1}{2} \sum_{\sigma\in\Delta} [i(\sigma, m) + i(\sigma, -m)]. \tag{4}$$

It is clear that $L(X, m) = L(X, -m)$. In other words, the coefficient of m in $L(X, m)$ is zero. Since

$$i(\sigma, m) + i(\sigma, -m) = 2[\mathrm{area}\,(\sigma)\, m^2 + (-1)^{\dim\sigma}]$$

for any open lattice simplex σ of dimension at most 2, we have

$$L(X, m) = \mathrm{area}\,(X)\, m^2 + \chi(X). \tag{5}$$

Let us compute $L(X, m)$. It suffices to compute $\sum_{\sigma\in\Delta} i(\sigma, -m)$. In fact,

$$\begin{aligned}
\sum_{\sigma\in\Delta} i(\sigma, -m) &= \sum_{\sigma\in\Delta} (-1)^{\dim\sigma} i(\bar{\sigma}, m) \qquad \text{[by (3)]} \\
&= \sum_{\sigma\in\Delta} (-1)^{\dim\sigma} \sum_{\tau\le\sigma} i(\tau, m) \\
&= \sum_{\tau\in\bar{\Delta}} i(\tau, m) \sum_{\tau\le\sigma\in\Delta} (-1)^{\dim\sigma},
\end{aligned}$$

where $\bar{\Delta}$ is the lattice triangulation of $\bar{X}$ extended from Δ. Note that $\cup_{\tau\le\sigma\in\Delta}\sigma$ can be viewed as a star open neighborhood of any point $x \in \tau$ in X. Then for $x \in \tau$,

$$\sum_{\tau\le\sigma\in\Delta} (-1)^{\dim\sigma} = \delta(x, X) - \chi(\mathrm{lk}\,(x, X)),$$

where $\delta(x,X)=1$ for $x\in X$ and $\delta(x,X)=0$ otherwise. Thus

$$\begin{aligned}\sum_{\sigma\in\Delta} i(\sigma,-m) &= \sum_{\tau\in\bar{\Delta}} i(\tau,m)\delta(x,X)\ (x\in\tau)\\ &\quad -\sum_{\tau\in\bar{\Delta}} i(\tau,m)\chi(\mathrm{lk}\,(x,X))\ (x\in\tau)\\ &= i(X,m)-\sum_{x\in\mathbf{Z}^2\cap m\bar{X}}\chi(\mathrm{lk}\,(x,mX)).\end{aligned}$$

Substitute this into (4), one obtains

$$L(X,m)=i(X,m)-\frac{1}{2}\sum_{x\in\mathbf{Z}^2\cap m\bar{X}}\chi(\mathrm{lk}\,(x,mX)),$$

which shows that the sum (4) is independent of the lattice triangulation Δ. Set $m=1$ and make use of (5); we obtain (2) as desired. □

Corollary 1.2 *Let X be a bounded closed lattice polyhedron of $\mathbf{R}^2$. Then*

$$\mathrm{area}\,(X)\le i(X)-\chi(X)-\frac{1}{2}i(\mathrm{fr}\,X). \tag{6}$$

The equality holds if and only if X is a manifold with boundary.

Proof Since X is closed, the link $\mathrm{lk}\,(x,X)$ for any $x\in\mathrm{fr}\,X$ is a disjoint union of some closed arcs and points; thus $\chi(\mathrm{lk}\,(x,X))\ge 1$ and (6) follows immediately. Moreover, it is clear that the equality in (6) holds if and only if $\chi(\mathrm{lk}\,(x,X))=1$ for all $x\in\mathrm{fr}\,X$. This is equivalent to saying that X is a manifold with boundary. □

If X is closed and 1-dimensional, then X can be viewed as a planar graph G. Thus $\mathrm{area}\,(X)=0$, $\mathrm{fr}\,X=X$, and

$$0=i(X)-\chi(X)-\frac{1}{2}\sum_{x\in X\cap\mathbf{Z}^2}\deg\,(x).$$

It is easy to see the following inequality

$$\sum_{x\in X\cap\mathbf{Z}^2}\deg\,(x)+\#\{\text{leaves}\}\ge 2\#\{\text{vertices}\}=2i(X) \tag{7}$$

because the left side is the sum of degrees contributed at vertices, and at each vertex the contribution is at least 2, including the leaves. In other words, the left side is at least the twice of the number of vertices. We thus have $\chi(X)\le\#\{\text{leaves}\}/2$. Note that (7) is actually true for any graph, not necessary for planar graphs. Moreover, the equality in (7) holds if and only if the graph G has degree 2 at every non-leaf, which is equivalent to saying that G is a disjoint union of paths and cycles. This yields the following corollary that can be verified directly.

Corollary 1.3 *For any graph G with p vertices and q edges,*

$$p - q \leq \#\{\text{leaves}\}/2. \tag{8}$$

The equality holds if and only if G is a disjoint union of paths and cycles.

The Pick type Theorem 1.1 is in its full generality in dimension two, including the Pick type theorems of [7, 19], but not the Pick type theorem of [9], which is about abstract polygons. However, the Pick type theorem of [9] is an example of the weighted version of the Pick type theorem in the next section.

2 Weighted Version of the Pick Theorem

Let X be a compact polyhedron of $\mathbf{R}^2$. A **stratification** of X is a collection $\mathcal{D}$ of disjoint connected manifolds without boundary (called **strata**) such that the union of all strata is the whole set X. A function ω on X is called a **weight function with respect to a stratification** $\mathcal{D}$ if ω is constant on each stratum; in other word, ω is simply a function on the set $\mathcal{D}$ of strata. A better way to define weight function is not to have given the compact polyhedron X at beginning. For this purpose, we define a weight function as a function on $\mathbf{R}^2$ whose range of values is a finite set, and for each $c \in \mathbf{R}$, $\omega^{-1}(c)$ is a polyhedron. This is slightly more general than the previous one, and we use this definition throughout the whole section.

Let ω be a weight function on $\mathbf{R}^2$ with bounded support X. Let $\mathcal{D}$ be a stratification of $\bar{X}$ such that ω is constant on each stratum. We define the **weighted area**, the **weighted number of lattice points**, and the **weighted Euler characteristic** of X as

$$\begin{aligned} \operatorname{area}(X,\omega) &= \sum_{Y\in\mathcal{D}} \omega(Y) = \int_{\mathbf{R}^2} \omega(x)\mathrm{d}x, \\ i(X,\omega) &= \sum_{Y\in\mathcal{D}} \omega(Y)i(Y) = \int_{\mathbf{Z}^2} \omega(x)\mathrm{d}\#(x), \\ \chi(X,\omega) &= \sum_{Y\in\mathcal{D}} \omega(Y)\chi(Y) = \int_{\mathbf{R}^2} \omega(x)\mathrm{d}\chi(x) \end{aligned}$$

respectively, where $\#$ is the counting measure on $\mathbf{Z}^2$ and χ is the Euler measure; see [2]. For a point $x \in X$, choose a circle $S^1(x,r)$ centered at x with small enough radius r. Then $\mathcal{D}(x) = \{X_i \cap S^1(x,r) \neq \emptyset : X_i \in \mathcal{D}\}$ is a stratification of $\operatorname{lk}(x,X)$; the restriction of ω on $\operatorname{lk}(x,X)$ is a weight function with respect to $\mathcal{D}(x)$; we still use ω to denote this weight function.

Theorem 2.1 *Let ω be a weight function on $\mathbf{R}^2$ with bounded support X. Then*

$$\operatorname{area}(X,\omega) = i(X,\omega) - \chi(X,\omega) - \frac{1}{2}\sum_{x\in\mathbf{Z}^2\cap\operatorname{fr}X} \chi(\operatorname{lk}(x,X),\omega). \tag{9}$$

Proof Let $\mathcal{D}$ be a lattice stratification (each stratum is a lattice polyhedron) of $\bar{X}$ such that ω is constant on each stratum. Similar to the proof of Theorem 1.1, we define

$$L(X,\omega;m)=\frac{1}{2}\sum_{Y\in\mathcal{D}}\omega(Y)[i(Y,m)+i(Y,-m)] \tag{10}$$

for any positive integer m. Let Δ be a stratified lattice triangulation of X, i.e., each stratum of $\mathcal{D}$ is a disjoint union of some open lattice simplices of Δ. It is clear that ω is also a weight function with respect to Δ, and

$$\begin{aligned} L(X,\omega;m) &= \frac{1}{2}\sum_{\sigma\in\Delta}\omega(\sigma)[i(\sigma,m)+i(\sigma,-m)] \\ &= \operatorname{area}(X,\omega)m^2+\chi(X,\omega). \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} \sum_{\sigma\in\Delta} i(\sigma,-m)\omega(\sigma) &= \sum_{\sigma\in\Delta}(-1)^{\dim\sigma}i(\bar{\sigma},m)\omega(\sigma) \\ &= \sum_{\sigma\in\Delta}(-1)^{\dim\sigma}\omega(\sigma)\sum_{\tau\leq\sigma}i(\tau,m) \\ &= \sum_{\tau\in\bar{\Delta}}i(\tau,m)\sum_{\tau\leq\sigma\in\Delta}(-1)^{\dim\sigma}\omega(\sigma). \end{aligned}$$

For each $\tau\in\bar{\Delta}$, select a point $x\in\tau$; we claim that

$$\sum_{\tau\leq\sigma\in\Delta}(-1)^{\dim\sigma}\omega(\sigma)=\omega(\tau)-\chi(\operatorname{lk}(x,X),\omega).$$

In fact, it is obviously true if τ is a vertex or an open triangle of Δ. If τ is an open segment of Δ, let σ_i be the open triangles such that $\sigma_i>\tau$; we have

$$\begin{aligned} \sum_{\tau\leq\sigma\in\Delta}(-1)^{\dim\sigma}\omega(\sigma) &= -\omega(\tau)+\sum_i\omega(\sigma_i) \\ &= \omega(\tau)-\left(2\omega(\tau)-\sum_i\omega(\sigma_i)\right) \\ &= \omega(\tau)-\chi(\operatorname{lk}(x,X),\omega). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\sigma\in\Delta}i(\sigma,-m)\omega(\sigma) &= \sum_{\tau\in\bar{\Delta}}i(\tau,m)\omega(\tau)-\sum_{\tau\in\bar{\Delta}}i(\tau,m)\chi(\operatorname{lk}(x,X),\omega)\ (x\in\tau) \\ &= i(X,\omega;m)-\sum_{x\in\mathbf{Z}^2\cap m\bar{X}}\chi(\operatorname{lk}(x,mX),\omega). \end{aligned}$$

Put this in (11) and set $m=1$; we have

$$\operatorname{area}(X,\omega)+\chi(X,\omega)=i(X,\omega)-\frac{1}{2}\sum_{x\in\mathbf{Z}^2\cap\bar{X}}\chi(\operatorname{lk}(x,X),\omega).$$

Note that $\chi(\operatorname{lk}(x,X),\omega)=0$ for all $x\in\operatorname{int}X$; the formula (9) follows immediately. □

Corollary 2.2 *Let G be a graph embedded in $\mathbf{R}^2$ such that the embedding is a lattice polyhedron. Let ω be a function on $\mathbf{R}^2$ satisfying the properties:* (i) *ω is constant on all regions divided by G and vanishes on the unbounded region;* (ii) *the value of ω on an edge is the average value of the regions at both sides of the edge. Then*

$$\int_{\mathbf{R}^2} \omega(x)\mathrm{d}x = \int_{\mathbf{Z}^2} \omega(x)\mathrm{d}\#(x) - \int_{\mathbf{R}^2} \omega(x)\mathrm{d}\chi(x). \tag{12}$$

Proof Let X be the support of the weight function ω and obviously, $\mathrm{fr}\, X \subset G$. It suffices to show that the weighted Euler characteristic $\chi(\mathrm{lk}\,(x, X), \omega)$ vanishes for all $x \in G$. Let $S^1(x, r)$ be the circle centered at x of small enough radius r. Then, no matter x is a vertex or a point on an edge of G, the complement $S^1(x, r) - G$ is a collection of open arcs $S^1(x, r) \cap R_j$, $1 \le j \le n$, where R_j are some regions divided by G, R_j and R_{j+1} share a common boundary E_j, $R_{n+1} = R_1$. Then

$$\begin{aligned} \chi(\mathrm{lk}\,(x, X), \omega) &= \sum_{j=1}^{n} \omega(E_j) - \sum_{j=1}^{n} \omega(R_j) \\ &= \sum_{j=1}^{n} [\omega(R_j) + \omega(R_{j+1})]/2 - \sum_{j=1}^{n} \omega(R_j) = 0. \end{aligned}$$

□

It is interesting to notice that the weight function in Corollary 2.2 can have arbitrary values at the vertices of G; the whole plane with the given weight function is a weighted manifold (with vanishing boundary weight function) of [2]. In the following we derive the Pick type theorem of [9] as an example of Corollary 2.2 with a special weight function.

Let $\vec{P}$ be a closed oriented curve of $\mathbf{R}^2$, allowing self-intersections and even overlapping arcs; its point-set is denoted by P. If $\vec{P}$ is a smooth curve, then it is an immersion of a circle in $\mathbf{R}^2$. The complement $\mathbf{R}^2 - \vec{P}$ is a finite collection of open cells and one unbounded region. We define a function $\omega(\vec{P}, x)$ on $\mathbf{R}^2$ as follows: (i) fix the point x and take a curve $R(x)$ from x to ∞ such that $R(x)$ intersects P transversally at finitely many number of points; (ii) at each intersection point y (topologically equivalent to one of the six types in Figure 2), assign the index

$$\iota(R(x), y) = \begin{cases} 1 & \text{if } y \neq x \text{ and is of type (a)} \\ -1 & \text{if } y \neq x \text{ and is of type (b)} \\ \frac{1}{2} & \text{if } y = x \text{ and is of type (c) or (e)} \\ -\frac{1}{2} & \text{if } y = x \text{ and is of type (d) or (f)} \end{cases};$$

(iii) set

$$\omega(\vec{P}, x) = \sum_{y \in R(x) \cap \vec{P}} \iota(R(x), y), \tag{13}$$

where y is counted with multiplicity when $\vec{P}$ intersects itself. The cases (e) and (f) are special and we need to pay more attention. The case (e) means that the head of the curve $\vec{P}$ moves forward and reaches at the point x, then moves backward along

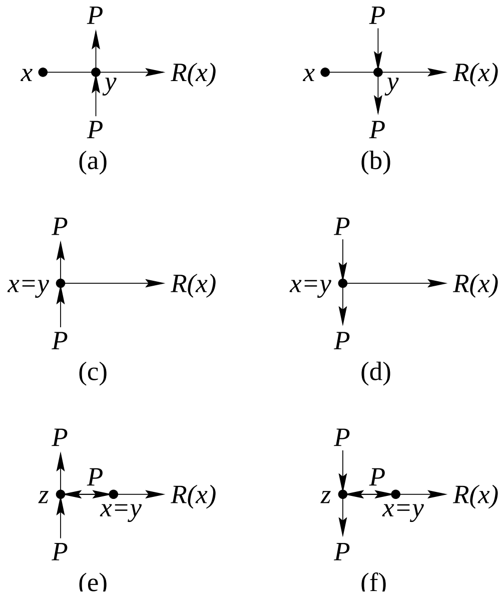

Figure 2: Six types of intersections

the original trail, and it keeps moving on the original trail until it reaches the point z, where the head starts a new trail; we call this **backtrack**, see Figure 2. The index we assigned for the cases (e) and (f) is equivalent to blowing up the overlapped arc $[zx]$ topologically, i.e., separating the overlapped arc $[zx]$ from z all the way to x and keep the local shape topologically equivalent. We call the point x in cases (e) and (f) a **whisker point** and the corresponding point z a **co-whisker point**; so a whisker point and its co-whisker point always appear in pair. We can get rid off a whisker point x and its co-whisker point z by removing the half-closed and half-open arc $[xz)$ and the curve $\vec{P}$ becomes a curve $\vec{P}'$; the function $\omega(\vec{P},\cdot)$ will remain unchanged except at x and z; and if $\omega(\vec{P},x) = \omega(\vec{P}',x) \pm 1/2$, then $\omega(\vec{P},z) = \omega(\vec{P}',z) \mp 1/2$. It should be pointed out that the idea to define the function $\omega(\vec{P},x)$ comes from the definition of the function $i(P,x)$ in [9]; and the two definitions give the same number for the cases (a), (b), (c) and (d). However, we allow backtrack case, i.e., the **whisker-free** condition of [8] is not needed in our treatment.

It is not hard to see that $\omega(\vec{P},x)$ is independent of the chosen curve $R(x)$ starting from x to ∞. In fact, if $x \notin \vec{P}$, it is just the **winding number** (need be checked, but leave it to the reader) of $\vec{P}$ at x and is constant on each cell σ_j (the unique unbounded region is also called a cell here, even it is not homeomorphic to an open disc); so $\omega(\vec{P},x)$ is well-defined. We write $\omega(\sigma_j) = \omega(\vec{P},x)$ for $x \in \sigma_j$, then $\omega(\sigma_j) = 0$ if σ_j is unbounded. Let us mark all cross points, whisker and co-whisker points on P as vertices, and take the rest of arcs as edges and loops; we obtain a planar graph G possibly with multiple edges and loops. Let x be a point on an edge (or a loop) ε between two cells σ' and σ'' (σ' and σ'' may be the same when ε bounds a whisker point), and choose the curve $R(x)$ to intersect the cell σ'; see Figure 3. Assume that there are m directed arcs of case (c)

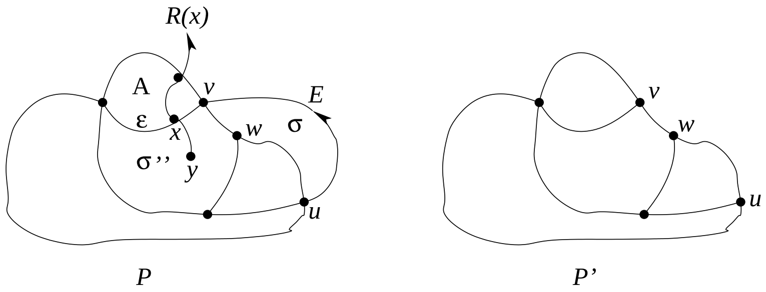

Figure 3: The arcs on E were moved to the arc uwv.

and n directed arcs of case (d) from $\vec{P}$ overlapping with ε. Then, on the one hand,

$$\omega(\vec{P},x)=\omega(\sigma')+\frac{m-n}{2},$$

and on the other hand, extending the curve $R(x)$ beyond x to a point $y\in\sigma''$, we have

$$\omega(\vec{P},y)=\omega(\sigma'')=\omega(\sigma')+m-n.$$

Thus

$$\omega(\vec{P},x)=\frac{1}{2}[\omega(\sigma')+\omega(\sigma'')],$$

which shows that $\omega(\vec{P},x)$ is well-defined. The situation for x to be a vertex of G is similar, just pay more attention to whisker points.

Theorem 2.3 *Let $\vec{P}$ be a closed oriented curve of $\mathbf{R}^2$ and let $\omega(\vec{P},x)$ be the function defined by* (13)*. Let $r(\vec{P})$ be the rotation number of $\vec{P}$. Then*

$$r(\vec{P})=\int_{\mathbf{R}^2}\omega(\vec{P},x)\mathrm{d}\chi(x). \tag{14}$$

Proof We mark all cross points, whisker and co-whisker points of P to have a planar graph G. The complement of G is a finite collection of bounded open cells and one unbounded region. We proceed by induction on the number of bounded cells. If there is only one bounded cell σ, and $\vec{P}$ wraps n times around $\partial\sigma$ counterclockwise, then $r(\vec{P})=\omega(\vec{P},x)=n$ for $x\in\sigma$, and so

$$\int_{\mathbf{R}^2}\omega(\vec{P},x)\mathrm{d}\chi(x)=n\chi(\sigma)-\frac{n}{2}\chi(\partial\sigma)=r(\vec{P}).$$

In general, we choose a bounded cell σ which shares a common edge (or loop) E with the unbounded region; the common edge E bounds two end points u and v ($u=v$ if E is a loop), and is oriented with the positive orientation (counterclockwise) of $\partial\sigma$; see Figure 3.

We assume that there are m arcs on E having the same orientation as E and n arcs having the opposite orientation. We move all the $m+n$ arcs to the other side of $\partial\sigma$ so that the cell σ is connected to the unbounded region, and obtain a new closed curve $\vec{P}'$ having one less bounded cell. If E is an edge, then $r(\vec{P}) = r(\vec{P}')$,

$$\omega(\vec{P}, x) = \begin{cases} \omega(\vec{P}', x) & \text{for } x \notin \bar{\sigma} \text{ or } x \in \{u, v\} \\ \omega(\vec{P}', x) + \frac{m-n}{2} & \text{for } x \in \partial\sigma - \{u, v\} \\ \omega(\vec{P}', x) + m - n & \text{for } x \in \sigma \end{cases} ; \quad (15)$$

and $\omega(\vec{P}', x) = 0$ for $x \in \sigma \cup E$. A routine calculation shows that

$$\int_{\bar{\sigma}} [\omega(\vec{P}, x) - \omega(\vec{P}', x)] \mathrm{d}\chi(x) = 0.$$

Thus

$$\begin{aligned} \int_{\mathbf{R}^2} \omega(\vec{P}, x) \mathrm{d}\chi(x) &= \int_{\mathbf{R}^2} \omega(\vec{P}', x) \mathrm{d}\chi(x) + \int_{\bar{\sigma}} [\omega(\vec{P}, x) - \omega(\vec{P}', x)] \mathrm{d}\chi(x) \\ &= r(\vec{P}') = r(\vec{P}). \end{aligned}$$

If E is a loop, then $u = v$, (15) is still valid and $\omega(\vec{P}', x) = 0$ for $x \in \sigma \cup E$, but $r(\vec{P}) = r(\vec{P}') + (m - n)/2$. Thus

$$\begin{aligned} \int_{\mathbf{R}^2} \omega(\vec{P}, x) \mathrm{d}\chi(x) &= \int_{\mathbf{R}^2} \omega(\vec{P}', x) \mathrm{d}\chi(x) + \int_{\bar{\sigma}} [\omega(\vec{P}, x) - \omega(\vec{P}', x)] \mathrm{d}\chi(x) \\ &= r(\vec{P}') + \frac{m-n}{2} = r(\vec{P}). \end{aligned}$$

□

Remark When the directed closed path $\vec{P}$ is decomposed into some (overlapped) directed cycles, we may count the number of counterclockwise directed cycles, denoted $r^+(\vec{P})$, and the number of clockwise directed cycles, denoted $r^-(\vec{P})$. Then the rotation number $r(\vec{P})$ is given by

$$r(\vec{P}) = r^+(\vec{P}) - r^-(\vec{P}),$$

and it is independent of the decompositions of $\vec{P}$ into directed cycles.

Proposition 2.4 (Grünbaum and Shephard) *Let $\vec{P}$ be a closed oriented lattice polyhedral curve of $\mathbf{R}^2$ and let $\omega(\vec{P}, x)$ be the function defined by* (13)*. Then*

$$\int_{\mathbf{R}^2} \omega(\vec{P}, x) \mathrm{d}x = \int_{\mathbf{Z}^2} \omega(\vec{P}, x) \mathrm{d}\#(x) - r(\vec{P}).$$

Proof It follows immediately from Corollary 2.2 and Theorem 2.3. □

The Pick type theorem of Grünbaum and Shephard [9] was originally stated for an **abstract polygon**, which is just a closed oriented lattice polyhedral curve of $\mathbf{R}^2$ without whisker points. Theorem 2.1 is the two-dimensional case of the higher dimensional volume formulas of [2] in terms of weight functions; and all the volume formulas of [13, 14, 17, 18] can be induced from those volume formulas of [2] by choosing weight equal to 1. The volume formula of [12] is also a special case of the volume formula of [2] by setting weight equal to 1.

There should be a higher dimensional analog of Theorem 2.3. Let M be a closed oriented smooth n-manifold and is immersed in $\mathbf{R}^{n+1}$ by a smooth map ϕ. The image $\phi(M)$ divides $\mathbf{R}^{n+1}$ into finite number of regions. We similarly define the function $\omega(\phi, x)$ on $\mathbf{R}^{n+1}$ in the following steps:

(i) Choose an orientation ε of $\mathbf{R}^{n+1}$.

(ii) Fix a point $x \in \mathbf{R}^{n+1}$ and take a smooth curve $R(x)$ from x to ∞ such that $R(x)$ intersects $\phi(M)$ transversally at finitely many number of points.

(iii) For each intersection point $y \in R(x) \cap \phi(M)$ and its any inverse image $p \in \phi^{-1}(y)$, the orientation of the tangent space T_pM of M at p induces an orientation on the tangent space $T_y\phi(M)$ of $\phi(M)$ at y; this orientation of $T_y\phi(M)$ together with the direction vector of $R(x)$ at y form an orientation ε_p of $\mathbf{R}^{n+1}$.

(iv) Define the the function

$$\iota(R(x), p) = \begin{cases} 1 & \text{if } x \neq \phi(p) \in R(x), \varepsilon_p = \varepsilon \\ -1 & \text{if } x \neq \phi(p) \in R(x), \varepsilon_p = -\varepsilon \\ \frac{1}{2} & \text{if } \phi(p) = x, \varepsilon_p = \varepsilon \\ -\frac{1}{2} & \text{if } \phi(p) = x, \varepsilon_p = -\varepsilon \end{cases}.$$

(v) Define the function

$$\omega(\phi, x) = \sum_{\phi(p) \in R(x)} \iota(R(x), p).$$

On the other hand, for any $p \in M$ and its induced orientation ε_p of the tangent space $T_{\phi(p)}\phi(M)$, there is a unique unit vector v_p normal to $T_{\phi(p)}\phi(M)$ such that the orientation ε_p together with v_p gives the chosen orientation ε of $\mathbf{R}^{n+1}$. This defines a smooth map ψ from M to the unit n-sphere S^n. We state the following conjecture.

Conjecture 2.5 *Let M be a closed oriented smooth n-manifold and let $\phi : M \longrightarrow \mathbf{R}^{n+1}$ be a smooth immersion, $n \geq 1$. Let $\omega(\phi, x)$ and ψ be defined as above.*

1. *If n is odd, then*
$$\deg \psi = \int_{\mathbf{R}^{n+1}} \omega(\phi, x) d\chi(x).$$
In particular, if ϕ is a smooth embedding, then M is a closed hyper-surface of $\mathbf{R}^{n+1}$, ψ is the Gauss map, and
$$\deg \psi = \chi(M_+),$$
where M_+ is the bounded component of the complement $\mathbf{R}^{n+1} - M$.

2. *If n is even, then*
$$\int_{\mathbf{R}^{n+1}} \omega(\phi, x) d\chi(x) = 0.$$

The conjecture may be stated in a more general setting, but the present version is good enough for testing. For $n = 1$, it is Theorem 2.3. For even n, it is easily verified when ϕ is an embedding.

3 The Reciprocity Law of Dedekind Sums

Counting the number of lattice points inside the dilation mX of a lattice polyhedron X by a positive integer m was systematically studied by Ehrhart [6] in higher dimensions. Let σ be an open lattice triangle of $\mathbf{R}^2$ with the vertices α_1, α_2 and the origin. Let m be a positive integer and α a point of $m\bar{\sigma}$. Write $\alpha = x_1\alpha_1 + x_2\alpha_2$ with real numbers $x_1 \geq 0, x_2 \geq 0$ such that $x_1 + x_2 \leq m$; then by the Division Algorithm,

$$\alpha = (k_1\alpha_1 + k_2\alpha_2) + (u_1\alpha_1 + u_2\alpha_2),$$

where k_1 and k_2 are non-negative integers, $0 \leq u_1 < 1$ and $0 \leq u_2 < 1$, $u_1+u_2+k_1+k_2 \leq m$. We define the *determined set* for σ:

$$D(\sigma) = \{u_1\alpha_1 + u_2\alpha_2 \in \mathbf{Z}^2 : 0 \leq u_1 < 1, 0 \leq u_2 < 1\}.$$

Then $\alpha \in \mathbf{Z}^2$ if and only if $\gamma = u_1\alpha_1 + u_2\alpha_2 \in \mathbf{Z}^2$. Write $|\gamma| = u_1 + u_2$; then we have $k_1 + k_2 \leq m - |\gamma|$, which is equivalent to $k_1 + k_2 \leq m - \lceil|\gamma|\rceil$, where $\lceil|\gamma|\rceil$ is the smallest integer greater than or equal to $|\gamma|$. The number of tuples (k_1, k_2) of non-negative integers such that $k_1 + k_2 \leq m - \lceil|\gamma|\rceil$ is the same as the number of non-negative integer solutions of the equation $k_1 + k_2 + k_3 = m - \lceil|\gamma|\rceil$, which turns out to be the binomial coefficient

$$\binom{m - \lceil|\gamma|\rceil + 2}{2} = \frac{1}{2}m^2 + \left(\frac{3}{2} - \lceil|\gamma|\rceil\right)m + \frac{1}{2}(\lceil|\gamma|\rceil^2 - 3\lceil|\gamma| + 2\rceil).$$

Thus the constant coefficient c_0 of m in $i(\bar{\sigma}, m)$ is given by

$$c_0 = \frac{1}{2}\sum_{\gamma\in D(\sigma)} (\lceil|\gamma|\rceil^2 - 3\lceil|\gamma|\rceil + 2).$$

Now we consider the special lattice triangle $\sigma(a, b)$ with the vertices $(0, 0)$, $(a, 0)$, $(0, b)$, where a and b are coprime positive integers. Then

$$c_0 = \frac{1}{2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\left(\left\lceil\frac{i}{a} + \frac{j}{b}\right\rceil^2 - 3\left\lceil\frac{i}{a} + \frac{j}{b}\right\rceil + 2\right). \tag{16}$$

Recall the *Dedekind sum* $s(q, p)$ of coprime positive integers p and q, which is defined by

$$s(q, p) = \sum_{k=1}^{p-1}\left(\left(\frac{qk}{p}\right)\right)\left(\left(\frac{k}{p}\right)\right),$$

where $((t))$ is the function

$$((t)) = \begin{cases} t - \lceil t\rceil + \frac{1}{2} & \text{for } t \notin \mathbf{Z} \\ 0 & \text{for } t \in \mathbf{Z}. \end{cases}$$

If $i/a + j/b = k$ is an integer, i.e., $bi + aj = kab$, then $a|i$ and $b|j$, and it forces that $(i, j) = (0, 0)$. Thus, if $(i, j) \neq (0, 0)$,

$$\left\lceil\frac{i}{a} + \frac{j}{b}\right\rceil = \left(\frac{i}{a} + \frac{j}{b}\right) - \left(\left(\frac{i}{a} + \frac{j}{b}\right)\right) + \frac{1}{2}.$$

Set $u = i/a + j/b$ and pay attention to the sum (16) at $(i, j) = (0, 0)$; we further have

$$c_0 = \frac{5}{8} + \frac{1}{2} \sum \left[u^2 - 2u((u)) + ((u))^2 - 2u + 2((u)) + \frac{3}{4} \right], \tag{17}$$

where the sum is extended over $0 \le i \le a-1$, $0 \le j \le b-1$. A trivial calculation shows that

$$\begin{aligned} \sum u &= ab - \frac{a+b}{2}, \\ \sum u^2 &= \frac{7ab}{6} - (a+b) + \frac{1}{6}\left(\frac{a}{b} + \frac{b}{a}\right) + \frac{1}{2}. \end{aligned}$$

To figure out the other terms in the sum (17), we need the formulas

$$\begin{aligned} \sum_{k=0}^{p-1} \left(\left(\frac{k}{p}\right)\right) &= 0, \\ \sum_{k=0}^{p-1} \left(\left(\frac{k+t}{p}\right)\right) &= ((t)), \\ \sum_{k=0}^{p-1} \left(\left(\frac{k}{p}\right)\right)^2 &= \frac{p}{12} + \frac{1}{6p} - \frac{1}{4}. \end{aligned}$$

which can be checked directly by the properties of the function $((t))$; see [16]. Sine a and b are coprime integers, the integers $bi + aj$ for $0 \le i \le a-1$ and $0 \le j \le b-1$ (mod ab) range from 0 to $ab-1$ and they must be distinct residues modulo ab. Thus

$$\begin{aligned} \sum ((u)) &= 0, \\ \sum ((u))^2 &= \frac{ab}{12} + \frac{1}{6ab} - \frac{1}{4}. \end{aligned}$$

For the term $\sum u((u))$, we need to use Dedekind sums as follows:

$$\begin{aligned} \sum u((u)) &= \sum_{i=0}^{a-1} \frac{i}{a} \sum_{j=0}^{b-1} \left(\left(\frac{bi/a + j}{b}\right)\right) + \sum_{j=0}^{b-1} \frac{j}{b} \sum_{i=0}^{a-1} \left(\left(\frac{i + aj/b}{a}\right)\right) \\ &= \sum_{i=0}^{a-1} \frac{i}{a} \left(\left(\frac{bi}{a}\right)\right) + \sum_{j=0}^{b-1} \frac{j}{b} \left(\left(\frac{aj}{b}\right)\right) \\ &= \sum_{i=0}^{a-1} \left[\left(\left(\frac{i}{a}\right)\right) + \frac{1}{2}\right] \left(\left(\frac{bi}{a}\right)\right) + \sum_{j=0}^{b-1} \left[\left(\left(\frac{j}{b}\right)\right) + \frac{1}{2}\right] \left(\left(\frac{aj}{b}\right)\right) \\ &= s(b, a) + s(a, b). \end{aligned}$$

Substitute these into the constant coefficient formula (17) and simplify it carefully, we obtain c_0 explicitly as

$$c_0 = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) + \frac{3}{4} - s(a, b) - s(b, a).$$

Use the fact $c_0 = 1$ by the Pick Theorem; the sum $s(a, b) + s(b, a)$ has a rational expression of a and b as given in the following.

Theorem 3.1 (Reciprocity Law of Dedekind Sums) *For coprime positive integers a and b,*

$$s(a,b)+s(b,a)=\frac{1}{12}\left(\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right)-\frac{1}{4}.$$

The above direct proof for the reciprocity law of Dedekind sums is a special case of [3] for the computation of the co-dimension two coefficient of a special lattice simplex. In higher dimensions, the coefficient formulas similar to (17) have been given in [3, 4]; and one can apply those coefficient formulas to an n-dimensional lattice simplex to obtain Zagier's reciprocity law of higher dimensional Dedekind sums; see [10, 20]. However, I would like to mention another proof given by Beck [1] for the reciprocity law of Dedekind sums, using the generating functions of Ehrhart polynomials of [5]. Finally, it should be pointed out that the idea to realize the reciprocity law of Dedekind sums by a lattice simplex is from the work of Diaz and Robins [5], Kantor and Khovanskii [11], Pommersheim [15], though they employed more advanced tools.

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