Subspace Arrangements of Curve Singularities and q-Analogs of the Alexander Polynomial

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Abstract

This paper introduces q-series for a curve singularity $(C, 0)$ in the affine space $(\mathbb{C}^n, 0)$ via subspace arrangements. These q-series are certain multivariable generating functions whose coefficients are the characteristic polynomials of subspace arrangements associated with the singularity $(C, 0)$ at various orders; the q is the variable in the characteristic polynomials of the subspace arrangements, representing the equivalence class of the ground field $\mathbb{C}$ as an affine algebraic line. We show that all the q-series introduced are rational functions and satisfy some interesting properties. When $(C, 0)$ is a plane curve singularity, the intersection $S^3_\varepsilon \cap C$ defines a link L_r, where S^3_ε is a 3-sphere centered at the origin with small enough radius ε. The value of certain q-series at $q = 1$ is the Alexander polynomial of the link L_r, up to a normalization. The approach is to introduce integration on the space $\mathcal{O}_{\mathbb{C}^n,0}$ of germs via the parameterization of the singularity $(C, 0)$. The whole exposition may be extended to some higher-dimensional singularities.

1 Introduction

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Let $\varphi_i : \mathbb{C} \to \mathbb{C}^n$ be a family of holomorphic functions such that $\varphi_i(0) = 0$, $1 \le i \le r$. The image $C_i = \operatorname{Im} \varphi_i$ is a complex curve in $\mathbb{C}^n$ passing through the origin 0. Our interest is to study the local behavior of the complex curve $C = \bigcup_{i=1}^r C_i$ near the origin. To get local information we may observe how the compositions $g \circ \varphi_i$ vary for various holomorphic germs g of $\mathbb{C}^n$ at the origin. The entire observation for such holomorphic germs should, in principle, contain holomorphic information about C near the origin with respect to the family $\Phi = \{\varphi_i\}_{i=1}^r$. This paper, inspired by the work of Campillo, Delgado and Gusein-Zade [4, 5], is to apply this general philosophy in a rigorous manner. The whole exposition may be generalized to a family of parameterizations of higher dimensional singularities.

Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the ring of holomorphic germs of $\mathbb{C}^n$ at the origin 0. When a coordinate is selected, $\mathcal{O}_{\mathbb{C}^n,0}$ is the vector space of all convergent power series near the origin. Let

I_i^∞ be the ideal of germs that vanish on the curve C_i; the notation will be justified later. The collection $\mathcal{I}^\infty = \{I_i^\infty\}_{i=1}^r$ of ideals is called the *ideal arrangement of* $\mathcal{O}_{\mathbb{C}^n,0}$ *at order* ∞. Throughout the paper we denote by $\mathbb{Z}$ the set of integers and by $\mathbb{N}$ the set of nonnegative integers. For each germ $g \in \mathcal{O}_{\mathbb{C}^n,0}$, we expand the holomorphic function $g \circ \varphi_i(z)$ into power series

$$g \circ \varphi_i(z) = \varepsilon_i z^{\omega_i} + \text{higher order terms}, \;\; \varepsilon_i \neq 0. \tag{1}$$

The leading term of $g \circ \varphi_i(z)$ defines an *order function* $\omega_i : \mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{N} \cup \{\infty\}$ and a *coefficient partial function* $\varepsilon_i : \mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{C}^*$ by (1). When $g \circ \varphi_i \equiv 0$, we define $\omega_i(g) = \infty$ and keep $\varepsilon_i(g)$ undefined. We write $\omega(g) = (\omega_1(g), \dots, \omega_r(g))$ and $\varepsilon(g) = (\varepsilon_1(g), \dots, \varepsilon_r(g))$. The partial map ε is not defined for $g \in \bigcup_{i=1}^r I_i^\infty$. The *semigroup* S_Φ of the family Φ is the sub-semigroup of $\mathbb{N}^r$, consisting of the elements of the form $\omega(g) = (\omega_1(g), \dots, \omega_r(g))$ for all germs $g \in \mathcal{O}_{\mathbb{C}^n,0} - \bigcup_{i=1}^r I_i^\infty$. The *full semigroup* of Φ is the sub-semigroup $\bar{S}_\Phi = \{\omega(g) \mid g \in \mathcal{O}_{\mathbb{C}^n,0}\}$ of $(\mathbb{N} \cup \{\infty\})^r$. The *extended semigroup* $\hat{S}_\Phi$ is the sub-semigroup of $\mathbb{N}^r \times (\mathbb{C}^*)^r$, consisting of the elements of the form $(\omega(g), \varepsilon(g)) = (\omega_1(g), \dots, \omega_r(g); \varepsilon_1(g), \dots, \varepsilon_r(g))$ for all $g \in \mathcal{O}_{\mathbb{C}^n,0} - \bigcup_{i=1}^r I_i^\infty$. The semigroup operation in $(\mathbb{N} \cup \{\infty\})^r$ is the addition and in $(\mathbb{C}^*)^r$ is the multiplication; $a + \infty = \infty$ for any $a \in \mathbb{N} \cup \{\infty\}$.

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}\backslash,\prime}$. The kth jet space $J^k_{\mathbb{C}^n,0}$ is the quotient $\mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{m}^k$ whose dimension is $\binom{n+k}{k}$. For $g \in \mathfrak{m}^k$, the order $\omega(g)$ with respect to each φ_i is at least k. For $\alpha = (a_1, \dots, a_r) \in \mathbb{Z}^r$, let

$$I^\alpha = \{g \in \mathcal{O}_{\mathbb{C}^n,0} \mid \omega(g) \geq \alpha\} \quad \text{and} \quad I_0^\alpha = \{g \in \mathcal{O}_{\mathbb{C}^n,0} \mid \omega(g) = \alpha\}. \tag{2}$$

It is clear that each I^α is an ideal of $\mathcal{O}_{\mathbb{C}^n,0}$ and contains the ideals $\mathfrak{m}^k$ for large enough k; so that $\mathcal{O}_{\mathbb{C}^n,0}/I^\alpha$ is finite dimensional. The space $\mathcal{O}_{\mathbb{C}^n,0}$ of germs is decomposed into a disjoint union

$$\mathcal{O}_{\mathbb{C}^n,0} = \bigoplus_{\alpha \in \bar{S}_\Phi} I_0^\alpha$$

in which $\mathcal{O}_{\mathbb{C}^n,0}$ can be viewed as a graded semigroup under multiplication. Let the coordinate functions z_k $(1 \leq k \leq n)$ in the parameterization $\varphi_i(z) = (z_1, \dots, z_n)$ be written in the power series

$$z_k = c_{k_1} z^{k_1} + c_{k_2} z^{k_2} + \cdots,$$

where $k_1, k_2, \dots$ are ascending positive integers and $c_{k_1}, c_{k_2}, \dots$ are nonzero constants, but there may be only finitely many terms. Let $d_i = \gcd(k_1, k_2, \dots)$ and $\mathbf{d} = (d_1, \dots, d_r)$. Let $\mathbf{d}_i$ be the vector of $\mathbb{Z}^r$ whose ith coordinate is d_i and zero elsewhere. Let $\mathbb{L}$ be the lattice generated by $\{\mathbf{d}_i\}_{i=1}^r$ and $\mathbb{D} = \prod_{i=1}^r [0, d_i) \cap \mathbb{Z}$. Clearly, I_0^α is nonempty if and only if $\alpha \in \mathbb{L}$; $I^{\alpha-\gamma} = I^\alpha$ for any $\alpha \in \mathbb{L}$ and $\gamma \in \mathbb{D}$. The *arrangement of* Φ *at order* α is the collection $\mathcal{I}^\alpha = \{I^{\alpha+\mathbf{d}_i}\}_{i=1}^r$ of sub-ideals in I^α, and the *projective arrangement of* Φ *at order* α is the collection $\mathbb{P}\mathcal{I}^\alpha = \{\mathbb{P}I^{\alpha+\mathbf{d}_i}\}_{i=1}^r$ of projective subspaces in $\mathbb{P}I^\alpha$. For $\alpha \in \mathbb{L}$, the sets I_0^α and $\mathbb{P}I_0^\alpha$ are the complement of the arrangement $\mathcal{I}^\alpha$ in I^α and the complement of the arrangement $\mathbb{P}\mathcal{I}^\alpha$ in $\mathbb{P}I^\alpha$, respectively; that is,

$$I_0^\alpha = I^\alpha - \bigcup_{i=1}^r I^{\alpha+\mathbf{d}_i} \quad \text{and} \quad \mathbb{P}I_0^\alpha = \mathbb{P}I^\alpha - \bigcup_{i=1}^r \mathbb{P}I^{\alpha+\mathbf{d}_i}.$$

Note that this may not be true for $\alpha \notin \mathbb{L}$. Let $L(\mathcal{I}^\alpha)$ be the semi-lattice of all possible intersections of the sub-ideals in $\mathcal{I}^\alpha$, including $I^\alpha = \bigcap_{i\in\emptyset} I^{\alpha+\mathbf{d}_i}$. The semi-lattice $L(\mathcal{I}^\alpha)$ is a poset (partially ordered set) whose partial order is the set-inclusion. Similarly, the projective semi-lattice is $L(\mathbb{P}\mathcal{I}^\alpha) = \{\mathbb{P}I \mid I \in L(\mathcal{I}^\alpha)\}$. We define *characteristic polynomials* $\chi(\mathcal{I}^\alpha, q)$ for $\mathcal{I}^\alpha$ and $\chi(\mathbb{P}\mathcal{I}^\alpha, q)$ for $\mathbb{P}\mathcal{I}^\alpha$ as follows:

$$\begin{aligned}
\chi(\mathcal{I}^\alpha, q) &= \sum_{I\in L(\mathcal{I}^\alpha)} \mu(I, I^\alpha) q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I}, \\
\chi(\mathbb{P}\mathcal{I}^\alpha, q) &= \sum_{\substack{\mathbb{P}I\in L(\mathbb{P}\mathcal{I}^\alpha),\\ I\neq \mathcal{O}_{\mathbb{C}^n,0}}} \mu(\mathbb{P}I, \mathbb{P}I^\alpha)(-q^{-1} - \cdots - q^{-\dim \mathbb{P}(\mathcal{O}_{\mathbb{C}^n,0}/I)}),
\end{aligned}$$

where μ are the Möbius functions on $L(\mathcal{I}^\alpha)$ and $L(\mathbb{P}\mathcal{I}^\alpha)$; see Section 2. The reason why we define $\chi(\mathcal{I}^\alpha, q)$ and $\chi(\mathbb{P}\mathcal{I}^\alpha, q)$ in the above form will be explained in Sections 2 and 3. The posets $L(\mathcal{I}^\alpha)$ and $L(\mathbb{P}\mathcal{I}^\alpha)$ are isomorphic; their Möbius functions satisfy the relation $\mu(I, J) = \mu(\mathbb{P}I, \mathbb{P}J)$ for $I \le J$. The polynomials $\chi(\mathcal{I}^\alpha, q)$ and $\chi(\mathbb{P}\mathcal{I}^\alpha, q)$ are actually defined for all $\alpha \in \mathbb{Z}^r$. However, they are only nonzero for $\alpha \in \mathbb{L}$. Let $\mathbf{t} = (t_1, \ldots, t_r)$ be a vector indeterminate and $\mathbf{t}^\alpha = t_1^{a_1} \cdots t_r^{a_r}$ for $\alpha = (a_1, \ldots, a_r) \in \mathbb{Z}^r$. We introduce the q-series:

$$M_\Phi(q;\mathbf{t}) = \sum_{\alpha\in\mathbb{L}} q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^\alpha} \mathbf{t}^\alpha, \tag{3}$$

$$M^0_\Phi(q;\mathbf{t}) = \sum_{\alpha\in\mathbb{L}\cap\mathbb{N}^r} \chi(\mathcal{I}^\alpha, q)\mathbf{t}^\alpha, \tag{4}$$

$$PM_\Phi(q;\mathbf{t}) = \sum_{\alpha\in\mathbb{L}, I^\alpha\neq\mathcal{O}_{\mathbb{C}^n,0}} (-q^{-1} - \cdots - q^{-\dim \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}/I^\alpha})\mathbf{t}^\alpha, \tag{5}$$

$$PM^0_\Phi(q;\mathbf{t}) = \sum_{\alpha\in\mathbb{L}\cap\mathbb{N}^r} \chi(\mathbb{P}\mathcal{I}^\alpha, q)\mathbf{t}^\alpha. \tag{6}$$

These q-series are certainly invariants for the parameterization family Φ in the sense that if $\Psi = \{\psi_i\}_{i=1}^r$ is another family such that $\psi_i(z) = \varphi_i(c_i z + \cdots)$ for some nonzero constants c_i then the q-series for Ψ are the same as the q-series for Φ. This is similar to that of [13].

There are other ways to introduce q-series for a parameterization family Φ. Like the approach in [4]; one has the linear map $\ell_\alpha : I^\alpha \to \mathbb{C}^r$, $\ell_\alpha(g) = (b_1, \ldots, b_r)$, where

$$g \circ \varphi_i(z) = b_i z^{a_i} + \text{higher order terms}, \ \ b_i \in \mathbb{C}.$$

The coefficient b_i may be zero because $b_i z^{a_i}$ may not be the leading term in the power series expansion of $g \circ \varphi_i(z)$. The image $\operatorname{Im} \ell_\alpha$ is isomorphic to the quotient $I^\alpha / I^{\alpha+\mathbf{d}}$; and the image $\ell_\alpha(I_0^\alpha)$ is a subset of $(\mathbb{C}^*)^r$. Let H_i be the coordinate hyperplane of $\mathbb{C}^r$ whose ith coordinate is zero. The arrangement $\mathcal{H}_\alpha = \{\operatorname{Im} \ell_\alpha \cap H_i\}_{i=1}^r$ in $\operatorname{Im} \ell_\alpha$ and the arrangement $\mathbb{P}\mathcal{H}_\alpha = \{\mathbb{P}\operatorname{Im} \ell_\alpha \cap H_i\}_{i=1}^r$ in $\mathbb{P}\operatorname{Im} \ell_\alpha$ are called *affine* and *projective hyperplane arrangements of* Φ *at order* α, respectively. Let $L(\mathcal{H}_\alpha)$ be the semi-lattice of all possible nonempty intersections of the hyperplanes in $\mathcal{H}_\alpha$; $L(\mathbb{P}\mathcal{H}_\alpha) = \{\mathbb{P}X \mid \{0\} \neq X \in L(\mathcal{H}_\alpha)\}$. The *characteristic polynomials* of $\mathcal{H}_\alpha$ and $\mathbb{P}\mathcal{H}_\alpha$ are defined by

$$\begin{aligned}
\chi(\mathcal{H}_\alpha, q) &= \sum_{H\in L(\mathcal{H}_\alpha)} \mu(H, \operatorname{Im} \ell_\alpha) q^{\dim H}, \\
\chi(\mathbb{P}\mathcal{H}_\alpha, q) &= \sum_{P\in L(\mathbb{P}\mathcal{H}_\alpha)} \mu(P, \mathbb{P}\operatorname{Im} \ell_\alpha)(1 + q + \cdots + q^{\dim P}),
\end{aligned}$$

where μ are the Möbius functions of $L(\mathcal{H}_\alpha)$ and of $L(\mathbb{P}\mathcal{H}_\alpha)$. We define the q-series:

$$\begin{aligned}
L_\Phi(q;\mathbf{t}) &= \sum_{\alpha\in\mathbb{L}} q^{\dim \mathrm{Im}\ell_\alpha}\mathbf{t}^\alpha, & (7)\\
L^0_\Phi(q;\mathbf{t}) &= \sum_{\alpha\in\mathbb{L}\cap\mathbb{N}^r} \chi(\mathcal{H}_\alpha, q)\mathbf{t}^\alpha, & (8)\\
PL_\Phi(q;\mathbf{t}) &= \sum_{\alpha\in\mathbb{L},\mathrm{Im}\,\ell_\alpha\neq\{0\}} (1+q+\cdots+q^{\dim \mathbb{P}\mathrm{Im}\ell_\alpha})\mathbf{t}^\alpha, & (9)\\
PL^0_\Phi(q;\mathbf{t}) &= \sum_{\alpha\in\mathbb{L}\cap\mathbb{N}^r} \chi(\mathbb{P}\mathcal{H}_\alpha, q)\mathbf{t}^\alpha. & (10)
\end{aligned}$$

The q-series introduced above encode various local information about the curve singularity $(C,0)$ near the origin along the parameterization Φ. Our first main result is the following theorem.

Theorem 1.1 *All the q-series are rational functions. More precisely, $M_\Phi(q;\mathbf{t})\prod_{i=1}^r(q^{-1}t_i^{d_i}-1)(t_i^{d_i}-1)$ and $PM_\Phi(q;\mathbf{t})\prod_{i=1}^r(q^{-1}t_i^{d_i}-1)$ are polynomials in q^{-1} and $\mathbf{t}$; $L_\Phi(q;\mathbf{t})$, $L^0_\Phi(q;\mathbf{t})$, $PL_\Phi(q;\mathbf{t})$, and $PL^0_\Phi(q;\mathbf{t})$ are rational functions in q and $\mathbf{t}$ with the same denominator $\prod_{i=1}^r(t_i^{d_i}-1)$. Moreover, $Q_\Phi(q;\mathbf{t})=(q-1)PQ_\Phi(q;\mathbf{t})$ for $Q=L,L^0,M,M^0$, and*

$$\begin{aligned}
M^0_\Phi(q;\mathbf{t}) &= \frac{M_\Phi(q;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)}{t_1^{d_1}\cdots t_r^{d_r}}, & (11)\\
L^0_\Phi(q;\mathbf{t}) &= \sum_{\alpha\in\mathbb{L}\cap\mathbb{N}^r} q^{\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}}\chi(\mathcal{I}^\alpha, q)\mathbf{t}^\alpha, & (12)\\
PL^0_\Phi(q;\mathbf{t}) &= \sum_{\alpha\in\mathbb{L}\cap\mathbb{N}^r} q^{\dim \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}}\chi(\mathbb{P}\mathcal{I}^\alpha, q)\mathbf{t}^\alpha. & (13)
\end{aligned}$$

When $q=1$, we can say more about the series. In fact, taking $q=1$ is the same as taking the Euler characteristic of the appropriate spaces constructed from the parameterization Φ; the geometric meaning is obvious. The q-series contain the original information about the curve singularity; and their values at $q=1$ are about taking their derivatives at $q=1$. Our next result is the following theorem.

Theorem 1.2 *The series $PL^0(1;\mathbf{t})$ and $PM^0(1;\mathbf{t})$ are polynomials for $r\geq 2$, but merely series for $r=1$. However, for all $r\geq 1$,*

$$\begin{aligned}
PM^0_\Phi(1;\mathbf{t}) &= \frac{PM_\Phi(1;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)}{t_1^{d_1}\cdots t_r^{d_r}} = PL^0_\Phi(1;\mathbf{t}), & (14)\\
PL_\Phi(1;\mathbf{t}) &= \frac{PM_\Phi(1;\mathbf{t})(t_1^{d_1}\cdots t_r^{d_r}-1)}{t_1^{d_1}\cdots t_r^{d_r}}, & (15)\\
PL^0_\Phi(1;\mathbf{t}) &= \frac{PL_\Phi(1;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)}{t_1^{d_1}\cdots t_r^{d_r}-1}. & (16)
\end{aligned}$$

Let $(C,0)=\bigcup_{i=1}^r(C_i,0)$ be a reducible plane curve singularity in $(\mathbb{C}^2,0)$ with the irreducible components C_i, defined by a holomorphic germ $f=\prod_{i=1}^r f_i$, where $C_i=\{f_i=0\}$. Let $\Phi=\{\varphi_i\}_{i=1}^r$ be a uniformization of $\{C_i\}_{i=1}^r$; that is, each $\phi_i:\mathbb{C}^2-\{0\}\to C_i-\{0\}$ is biholomorphic. Let S^3_ε be the 3-sphere of radius ε in $\mathbb{C}^2$ with the center at the origin. For small enough ε the intersection $S^3_\varepsilon\cap C$ defines a link L_r with r components. The

Alexander polynomial $\Delta^{L_r}(t_1, \dots, t_r)$ of L_r is defined up to the multiplication by the monomials $\pm\mathbf{t}^\alpha$. We normalize $\Delta^{L_r}(t_1, \dots, t_r)$ so that it contains no negative power terms and its constant term $\Delta^C(0, \dots, 0) = 1$. Eisenbud and Neumann [11] showed that $\Delta^{L_r}(\mathbf{t})$ can be expressed in terms of the data from the resolution of $(C, 0)$.

Let $\pi : (X, E) \to (\mathbb{C}^2, 0)$ be an embedded resolution of $(C, 0)$, where E is the exceptional divisor; the components of E are isomorphic to the complex projective line $\mathbb{CP}^1$ and their intersections are simple crossings; the components of the strict transform $\tilde{C}_i$ of the curve C_i meet E transversely and each intersection is a simple crossing. We choose a small enough neighborhood U of 0 in $\mathbb{C}^2$ so that the components of $\tilde{C} \cap \pi^{-1}(U - \{0\})$ are disjoint, each meeting E at one point. We name the components of E by $E_1, \dots, E_s$ and write $\tilde{C}_i \cap \pi^{-1}(U - \{0\}) = \bigsqcup_{k=1}^{\kappa_i} \tilde{C}_{i,k}$. The *resolution arrangement* of $(C, 0)$ is the collection $\mathcal{E} = \{\tilde{C}_{i,k}, E_j \mid 1 \le i \le r, 1 \le k \le \kappa_i, 1 \le j \le s\}$. Let $L(\mathcal{E})$ be the intersection semi-lattice of $\mathcal{E}$. The set

$$E_j^0 = E_j - \bigcup_{P < E_j, P \in L(\mathcal{E})} P$$

is called the *smooth part* of E_j and $D_j = E_j - E_j^0$ the *singular part* of E_j. The arrangement $\mathcal{E}$ can be described by a graph $\Gamma(\mathcal{E})$ whose vertex set is $\mathcal{E}$ and two vertices are adjacent if the corresponding components intersect; the graph is a tree. The order of blow-ups to obtain X induces a partial order on the vertices and the graph $\Gamma(\mathcal{E})$ turns out to be a rooted tree with the root at the exceptional divisor of the beginning blow-up. All strict transform components $\tilde{C}_{i,k}$ are leaves and the exceptional components E_j are non-leaves.

For each germ $g \in \mathcal{O}_{\mathbb{C}^2,0}$, let $m_{g,j}$ be the multiplicity of $g \circ \pi$ along the components E_j; the multiplicity of the zero germ is ∞. We write $m_{i,j}$ for the multiplicity $m_{f_i,j}$ of f_i along E_j. There is a linear map $\lambda : \mathcal{O}_{\mathbb{C}^2,0} \to \mathbb{Z}^s$ by $\lambda(g) = (m_{g,1}, \dots, m_{g,s})$. For $\beta \in \mathbb{Z}^s$, let

$$J^\beta = \{g \in \mathcal{O}_{\mathbb{C}^2,0} \mid \lambda(g) \ge \beta\} \quad \text{and} \quad J_0^\beta = \{g \in \mathcal{O}_{\mathbb{C}^2,0} \mid \lambda(g) = \beta\}.$$

Let $\mathbf{1}_j$ be the vector in $\mathbb{Z}^s$ whose jth coordinate is 1 and zero elsewhere. The collection $\{J^{\beta+\mathbf{1}_j}\}_{j=1}^s$ is a subspace arrangement of $\mathcal{O}_{\mathbb{C}^2,0}$. One can define q-series similar to L, L^0, M, M^0 and obtain results similar to Theorems 1.1 and 1.2; see Section 4. For vector indeterminate $\mathbf{u} = (u_1, \dots, u_s)$, we write $\mathbf{u}^\beta = u_1^{b_1} \cdots u_s^{b_s}$ for $\beta = (b_1, \dots, b_s)$. Set $\mathbf{m}_j = (m_{1,j}, \dots, m_{r,j})$. Our third result is the following theorem.

Theorem 1.3 *There exists a valuation (finitely additive measure) $\tilde{\nu}$ on the Boolean algebra generated by the projective subspaces of $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$, taking values in $\mathbb{Q}[q]$, such that the function $\mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)}$ from $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ to $\mathbb{Q}[q][[\mathbf{t}, \mathbf{u}]]$ is $\tilde{\nu}$-integrable, and*

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)} \mathrm{d}\tilde{\nu}(g) = \prod_{j=1}^s (1 - \mathbf{t}^{\mathbf{m}_j} u_j)^{-q-\chi(E_j^0)+1}. \tag{17}$$

In particular, taking $q = 1$, the valuation $\tilde{\nu}$ is reduced to the Euler characteristic χ on $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$, and

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)} \mathrm{d}\chi(g) = \prod_{j=1}^s (1 - \mathbf{t}^{\mathbf{m}_j} u_j)^{-\chi(E_j^0)}. \tag{18}$$

Setting $u_1 = \cdots = u_s = 1$, the formula (18) reduces to $PL_C^0(1;\mathbf{t}) = \prod_{j=1}^s (1 - \mathbf{t}^{\mathbf{m}_j})^{-\chi(E_j^0)}$, which was recently obtained by Campillo, Delgado and Gusein-Zade [4]. Using the result $\Delta^{L_r}(\mathbf{t}) = \prod_{j=1}^s (1 - \mathbf{t}^{\mathbf{m}_j})^{-\chi(E_j^0)}$ of Eisenbud and Neumann [11], it is clear that $PL^0(1;\mathbf{t}) = \Delta^{L_r}(\mathbf{t})$. This indicates that the q-series $PL_\Phi^0(q;\mathbf{t})$, as well as $PM_\Phi^0(q;\mathbf{t})$, are q-analogs of the Alexander polynomial of the algebraic link. Our approach is to study subspace arrangements and valuations (finitely additive measures) on infinite dimensional vector spaces. In Section 2 we sketch some basic ideas and necessary results on subspace arrangements that will be needed for the rest of the paper. In Section 3 we provide detailed proofs of Theorems 1.1 and 1.2. Section 4 is devoted to the proof of the q-analog product formulas (17) and (18).

2 Valuations on the Boolean Algebra of Subspaces

Let V be a finite dimensional vector space over a field $\mathbb{F}$. We denote by $L_0(V)$ ($L(V)$) the lattice of all (affine) subspaces of V. Let $B_0(V)$ ($B(V)$) be the Boolean algebra generated by $L_0(V)$ ($L(V)$). The elements of $B_0(V)$ ($B(V)$) are called *(affine) linear sets* in V. For affine linear sets X and Y in finite dimensional vector spaces V and W respectively, a map $f : X \to Y$ is called an *affine linear morphism* if its graph $\Gamma(f) = \{(x, f(x) \,|\, x \in X\}$ is an affine linear set in $V \times W$. Clearly, the Cartesian product $X \times Y$ is an affine linear set in $V \times W$. Let $g : Y \to Z$ be an affine linear morphism from Y to an affine linear set Z in a finite dimensional vector space U. The composition $g \circ f : X \to Z$ is an affine linear morphism, because its graph $\Gamma(g \circ f)$ is in one-to-one correspondence with the affine linear set $(\Gamma(f) \times \Gamma(g)) \cap H$ under the projection $\pi_W : V \times W \times W \times U \to V \times U$, where $H = \{v, w, w, u) \mid u \in U, v \in V, w \in W\}$. Thus affine linear sets and affine linear morphisms over $\mathbb{F}$ form a category $\mathcal{C}(\mathbb{F})$, called the *category of affine linear sets*. Let $K_0(\mathbb{F})$ be the ring generated by the isomorphism classes of objects in the category $\mathcal{C}(\mathbb{F})$, subject to the relations:

$$[X \cup Y] = [X] + [Y] - [X \cap Y] \tag{19}$$
$$[X][Y] = [X \times Y]. \tag{20}$$

If $\mathbb{F}$ is infinite, $K_0(\mathbb{F})$ is the ring $\mathbb{Z}[q]$ of polynomials in the variable $q = [\mathbb{F}]$. If $\mathbb{F}$ is finite, $K_0(\mathbb{F})$ is isomorphic to $\mathbb{Z}$.

The Boolean algebra $B_0(V)$ is a subalgebra of $B(V)$. An affine linear morphism $f : X \to Y$ between linear sets is called a *linear morphism* if f satisfies $f(cx) = f(x)$ for $c \in \mathbb{F}^*$. Similarly, linear sets and linear morphisms over a fixed field $\mathbb{F}$ form a subcategory $\mathcal{C}_0(\mathbb{F})$ of the category $\mathcal{C}(\mathbb{F})$, called the *category of linear sets*. Clearly, the ring $K_0(\mathbb{F})$ is also generated by the isomorphism classes of objects in $\mathcal{C}_0(\mathbb{F})$. For a linear set $X \in B_0(V)$, the quotient $(X - \{0\})/\mathbb{F}^*$ is called the *projectivization* of X, denoted $\mathbb{P}X$; $\mathbb{P}\{0\} = \emptyset$. Let $B(\mathbb{P}V)$ be the Boolean algebra generated by projective subspaces of $\mathbb{P}V$; that is, $B(\mathbb{P}V) = \{\mathbb{P}X \mid X \in B_0(V)\}$. The elements of $B(\mathbb{P}V)$ are called *projective linear sets*. For vector spaces V and W of finite dimensions, there is a map $\pi : \mathbb{P}(V \times W) \to \mathbb{P}V \times \mathbb{P}W$ defined $[x, y] \mapsto ([x], [y])$. A subset $Z \subset \mathbb{P}V \times \mathbb{P}W$ is called a *projective linear set* if its lifting $\pi^{-1}(Z)$ is a projective linear set in $\mathbb{P}(V \times W)$. A map $f : X \to Y$ between linear projective sets is called a *projective linear morphism* if its graph $\Gamma(f) \subset \mathbb{P}V \times \mathbb{P}W$ is a projective linear set. Again, projective linear sets and projective linear maps form a category, called the *category of projective linear sets*. For a codimension one vector subspace V of a vector space W, fix a vector $w \in W - V$; one can identify $\mathbb{P}(W-V)$ with the affine space $w+V$ by $[cw+v] \mapsto w+v/c$. The identification $\mathbb{P}(W-V) \simeq (w+V)$ implies that the abelian group $K_0(\mathbb{F})$ is also generated by the isomorphism classes of projective linear sets, subject to the relation (19). However, we define a different *multiplication* by

$$[\mathbb{P}X] \circ [\mathbb{P}Y] = [\mathbb{P}(X \times Y)]. \tag{21}$$

Writing $q = [\mathbb{F}]$, we have $[V] = q^{\dim V}$ for any finite dimensional vector space V; if V is not the zero space $\{0\}$, we have $[\mathbb{P}V] = 1 + q + \ldots + q^{\dim V - 1}$. A ring homomorphism $\varrho : (K_0(\mathbb{F}), \cdot) \to K_0(\mathbb{F}), \circ)$ is induced by $\varrho([X]) = [\mathbb{P}X]$. More specifically, $\varrho(1) = 0$ and $\varrho(q^n) = 1 + q + \cdots + q^{n-1}$ for $n \geq 1$. It follows that for any polynomial $f(q)$,

$$\varrho(f(q)) = \frac{f(q) - f(1)}{q-1}. \tag{22}$$

The operator ϱ can be linearly extended to $\mathbb{Z}[q, q^{-1}]$ by setting $\varrho(q^{-n}) = -(q^{-1} + \cdots + q^{-n})$ for $n \geq 1$. The idea for defining ϱ in this way is as follows: Let Ω be an infinite dimensional vector space, and let $V_1, \ldots, V_k$ be finite codimensional vector subspaces such that $V_k \subset \cdots \subset V_1 \subset V_0 = \Omega$ and $\dim(V_{i-1}/V_i) = 1$, $1 \leq i \leq k$. Then $\mathbb{P}V_0 - \mathbb{P}V_k = \bigsqcup_{i=1}^k \mathbb{P}V_{i-1} - \mathbb{P}V_i$ and $\mathbb{P}V_{i-1} - \mathbb{P}V_i \simeq V_i$. Thus

$$[\mathbb{P}V_0] - [\mathbb{P}V_k] = [\mathbb{P}V_0 - \mathbb{P}V_k] = \sum_{i=1}^k [\mathbb{P}V_{i-1} - \mathbb{P}V_i] = \sum_{i=1}^k [V_i].$$

If we assume that Ω is normalized to a "zero-dimensional" space so that $[\Omega] = 1$, then we may assume that the subspace V_i is a "negative i-dimensional" subspace and the projectivization $\mathbb{P}\Omega$ is the empty space so that $[V_i] = q^{-i}$ and $[\mathbb{P}\Omega] = 0$. Therefore, $[\mathbb{P}V_k] = -\sum_{i=1}^k q^{-i}$. On the other hand, we still have $\varrho(q^{-n}) = (q^{-n} - 1)/(q-1)$ for $n \geq 1$. So (22) is still valid for any Laurent polynomials $f(q)$; and this kind of coincidence justifies our definition; see also Theorem 2.5 in the following for further explanation.

We are interested in the sets that can be obtained from an affine subspace by deleting finitely many of its affine subspaces. It is not hard to see that any affine linear set is a disjoint union of finitely many such sets. A *subspace arrangement* $\mathcal{A}$ in a finite dimensional vector space V over a field $\mathbb{F}$ is a collection of finitely many affine subspaces of V. The *semi-lattice* of $\mathcal{A}$ is the poset

$$L(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{E}} H \neq \emptyset \;\middle|\; \mathcal{E} \subset \mathcal{A} \right\},$$

whose partial order is the set-inclusion; here we take the convention $V = \bigcap_{H \in \emptyset} H$. A useful invariant on the poset $L(\mathcal{A})$ is its Möbius function μ. By the *Möbius function* on a locally finite poset P (each interval $[x,y] = \{z \in P \mid x \leq z \leq y\}$ is a finite set) we mean the function μ on the ordered pairs of P, defined by $\mu_P(x,x) = 1$ for $x \in P$ and defined inductively for $x < y$ by

$$\mu_P(x,y) = -\sum_{x \leq z < y} \mu_P(x,z) = -\sum_{x < z \leq y} \mu_P(z,y).$$

The *characteristic polynomial* of $\mathcal{A}$, introduced by Rota in [15] for arbitrary locally finite posets with rank functions, is the polynomial

$$\chi(\mathcal{A}, q) = \sum_{X \in L(\mathcal{A})} \mu(X, V) q^{\dim X}, \tag{23}$$

where μ is the Möbius function on $L(\mathcal{A})$. To understand the geometric meaning of the polynomial (23), we use the idea of [8] to interpret $\chi(\mathcal{A}, q)$ as the element $[V - \cup \mathcal{A}]$ in the ring $K_0(\mathbb{F})$ by setting $q = [\mathbb{F}]$. For each $X \in L(\mathcal{A})$, we define

$$X^0 = X - \bigcup_{Y < X,\, Y \in L(\mathcal{A})} Y.$$

It is clear that the set $\{X^0 \mid X \in L(\mathcal{A})\}$ is a collection of disjoint subsets of V. Then

$$[X] = \sum_{Y \leq X,\, Y \in L(\mathcal{A})} [Y^0].$$

By the Möbius inversion, we have

$$[X^0] = \sum_{Y \leq X,\, Y \in L(\mathcal{A})} \mu(Y, X)[Y].$$

In particular, $V^0 = V - \bigcup_{X \in L(\mathcal{A})} X = V - \bigcup_{X \in \mathcal{A}} X$, and

$$[V^0] = \sum_{X \in L(\mathcal{A})} \mu(X, V)[X]. \tag{24}$$

If the affine subspaces in the arrangement $\mathcal{A}$ are all vector subspaces in V, we may define a *projective arrangement* $\mathbb{P}\mathcal{A}$ in the projective space $\mathbb{P}V$, where $\mathbb{P}\mathcal{A}$ is the set of projective subspaces $\mathbb{P}X$, $X \in \mathcal{A}$, but $X \neq \{0\}$. The semi-lattice of $\mathbb{P}\mathcal{A}$ is the poset $L(\mathbb{P}\mathcal{A}) = \{\mathbb{P}X \mid X \in L(\mathcal{A}), X \neq \{0\}\}$. For projective subspaces $Y_i \in L(\mathbb{P}\mathcal{A})$ $(i = 1, 2)$, the order relation $Y_1 \leq Y_2$ is satisfied if and only if there are affine subspaces $X_i \in L(\mathcal{A})$ such that $Y_i = \mathbb{P}X_i$ and $X_1 \leq X_2$. For each $Y \in L(\mathbb{P}\mathcal{A})$, we similarly define

$$Y^0 = Y - \bigcup_{Z < Y,\, Z \in L(\mathbb{P}\mathcal{A})} Z.$$

It is clear that the set $\{Y^0 \mid Y \in L(\mathbb{P}\mathcal{A})\}$ is a collection of disjoint subsets in $\mathbb{P}V$. Thus

$$[Y] = \sum_{Z \leq Y,\, Z \in L(\mathbb{P}\mathcal{A})} [Z^0].$$

By the Möbius inversion, we have

$$[Y^0] = \sum_{Z \leq Y,\, Z \in L(\mathbb{P}\mathcal{A})} \mu(Z, Y)[Z].$$

In particular, $\mathbb{P}V^0 = \mathbb{P}V - \bigcup_{Y \in L(\mathbb{P}\mathcal{A})} Y = \mathbb{P}V - \bigcup_{Y \in \mathbb{P}\mathcal{A}} Y$, and

$$[\mathbb{P}V^0] = \sum_{Y \in L(\mathbb{P}\mathcal{A})} \mu(Y, \mathbb{P}V)[Y]. \tag{25}$$

We define the *characteristic polynomial* for the projective arrangement $\mathbb{P}\mathcal{A}$ as follows:

$$\chi(\mathbb{P}\mathcal{A}, q) = \sum_{Y \in \mathbb{P}\mathcal{A}} \mu(Y, \mathbb{P}V) \left(1 + q + \cdots + q^{\dim Y}\right).$$

Now we apply the idea of [8] by taking $\mathbb{F}$ to be either a finite field $\mathbb{F}_q$ of q elements, the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, or the division ring $\mathbb{H}$ of the quaternion. It is known from [7, 16, 19] that the combinatorial Euler characteristic is a valuation (finitely additive measure). Since $\mathbb{F}_q$ is a finite set, $\mathbb{R}$ is a 1-cell, $\mathbb{C}$ and $\mathbb{H}$ are even-dimensional cells, we have

$$\chi(\mathbb{F}) = \begin{cases} q & \text{for } \mathbb{F} = \mathbb{F}_q \\ -1 & \text{for } \mathbb{F} = \mathbb{R} \\ 1 & \text{for } \mathbb{F} = \mathbb{C}, \mathbb{H}. \end{cases}$$

We should emphasize here that the combinatorial Euler characteristic of $\mathbb{R}$ is -1, not $+1$ as is usually assumed in topology. Then for any affine linear set or projective linear set X, the isomorphism class $[X]$ is a polynomial $f(X, [\mathbb{F}])$ with integral coefficients in one variable $[\mathbb{F}]$ and $\chi(X) = f(X, \chi(\mathbb{F}))$. It can be easily proved that Grassmanians, flag varieties, partition varieties [10], and quiver varieties, etc, are all (affine) linear sets; the characteristic polynomials of any such variety over the fields $\mathbb{F}_q$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (division ring) have the same form. But this is not the issue we want to discuss in the present paper; a detailed study of the subject will be given elsewhere. We summarize the conclusions in the following theorem.

Theorem 2.1 *(a) For any affine linear set or projective linear set X, its equivalence class $[X]$ is a polynomial $f(X, q)$ with integral coefficients in one variable $q = [\mathbb{F}]$, and its (combinatorial) Euler characteristic is given by*

$$\chi(X) = f(X, \chi(\mathbb{F})).$$

(b) Let $\mathcal{A}$ be a subspace arrangement of a finite dimensional vector space V over a field $\mathbb{F}$. Writing $q = [\mathbb{F}]$, then $\chi(\mathcal{A}, q)$ is the equivalence class of the complement $V - \cup\mathcal{A}$ in the ring $K_0(\mathbb{F})$; that is,

$$\chi(\mathcal{A}, q) \;=\; \left[V - \bigcup_{X \in \mathcal{A}} X\right]. \tag{26}$$

(c) If $\mathcal{A}$ is a vector subspace arrangement, then $\chi(\mathbb{P}\mathcal{A}, q)$ is the equivalence class of the complement $\mathbb{P}V - \cup\mathbb{P}\mathcal{A}$ in $K_0(\mathbb{F})$; that is,

$$\chi(\mathbb{P}\mathcal{A}, q) \;=\; \left[\mathbb{P}V - \bigcup_{X \in \mathcal{A}} \mathbb{P}X\right], \tag{27}$$

Moreover, $\chi(\mathcal{A}, q) = (q - 1)\chi(\mathbb{P}\mathcal{A}, q)$ and $\chi(\mathbb{P}\mathcal{A}, 1) = \partial_q \chi(\mathcal{A}, 1)$.

□

The purpose of introducing the ring $K_0(\mathbb{F})$ is to construct certain valuations (finitely additive measures) that can take values in $K_0(\mathbb{F})$. Let $\mathcal{S}$ be a class of subsets of a set S, containing the empty set and closed under intersection; any such class of sets is called an *intersectional class*. The *relative Boolean algebra* generated by $\mathcal{S}$ is the class $B(\mathcal{S})$ of sets constructed from $\mathcal{S}$ by taking unions, intersections, and relative complements finitely many times. Let R be a commutative ring R with unity $1 \neq 0$. A set-function $\mu : B(\mathcal{S}) \to R$ is called a *valuation* if $\mu(\emptyset) = 0$ and

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

for $A, B \in B(\mathcal{S})$. Let M be an R-module. A function $f : S \to M$ is called *simple* with respect to $\mathcal{S}$ if $f = \sum_i v_i 1_{A_i}$ for some finitely many subsets $A_i \in \mathcal{S}$, where 1_{A_i} is the characteristic function of A_i; that is, $1_{A_i}(x) = 1$ for $x \in A_i$ and $1_{A_i}(x) = 0$ for $x \notin A_i$. Given a set-function $\mu : \mathcal{S} \to R$; one can define the integration with respect to μ for any simple function $f = \sum v_i 1_{A_i} : S \to M$ as follows:

$$\mu(f) = \int_S f(x) \mathrm{d}\mu(x) = \sum_i \mu(A_i) v_i. \tag{28}$$

The integral (28) is meaningful only if the integration for all different expressions of the same function yields the same value; and in this case μ is said to *permit an integral* for the function f. Let $F(\mathcal{S}, M)$ be the R-module of simple functions from S to M. If μ permits an integral for all functions in $F(\mathcal{S}, M)$, then μ is an R-homomorphism from $F(\mathcal{S}, M)$ to M. What conditions are needed for a set-function on an intersectional class to permit an integral for simple functions? The following proposition, essentially due to Groemer [12], gives the required condition when the integral is well-defined.

Proposition 2.2 *Let $\mathcal{S}$ be an intersectional class of subsets of a set S and let M be an R-module. Let $\mu : \mathcal{S} \to R$ be a set-function such that $\mu(\emptyset) = 0$. If μ satisfies the Inclusion-Exclusion formula*

$$\mu(A_1 \cup \cdots \cup A_n) = \sum_i \mu(A_i) - \sum_{i<j} \mu(A_i \cap A_j) + \sum_{i<j<k} \mu(A_i \cap A_j \cap A_k) - \cdots, \tag{29}$$

where $A_1, \ldots, A_n, A_1 \cup \cdots \cup A_n$ are sets in $\mathcal{S}$, then μ permits an integral by (28) for simple functions from S to M with respect to the class $\mathcal{S}$.

Proof. We first note that permitting an integral by (28) is equivalent to

$$\sum w_j 1_{B_j} = 0, \ B_j \in \mathcal{S} \ \Rightarrow \ \sum \mu(B_j) w_j = 0. \tag{30}$$

Let $[n] = \{1, \ldots, n\}$ and $A_I = \bigcap_{i \in I} A_i$ for $I \subset [n]$. Then $\sum_{I \subset [n]} (-1)^{\#I} 1_{A_I} = 0$. Thus if the module M contains R, the condition (29) is also necessary. Now suppose that μ does not permit an integral by (28); that is, there exist sets $Y_1, \ldots, Y_k$ such that (30) is not satisfied. We linearly order the nonempty subsets of $[n]$ as follows: $I_1, \ldots, I_p$, where $p = 2^k - 1$; i-subsets are ahead of $(i+1)$-subsets; i-subsets are ordered arbitrarily. Consider the intersections $Y_{I_1}, \ldots, Y_{I_p}$, some of them may be the same sets. Since (30) is not satisfied, it follows that there are elements $v_i \in M$ such that

$$\sum_{i \geq r} v_i 1_{Y_{I_i}} = 0, \ v_r \neq 0 \quad \text{but} \quad \sum_{i \geq r} v_i \mu(Y_{I_i}) \neq 0. \tag{31}$$

Since $1 \leq r \leq p$ we may assume that the elements v_i are selected so that the integer r is maximal. However, r can not be p; if so, then $v_p 1_{Y_1 \cap \cdots \cap Y_k} = 0$ and $v_p \neq 0$ imply that

$Y_1 \cap \cdots \cap Y_p = \emptyset$; hence $v_p\mu(Y_1 \cap \cdots \cap Y_p) = 0$. Because of (31) it is impossible to find points $x \in S$ such that $x \in Y_{I_r}$ and $x \notin Y_{I_i}$ for all $i > r$. Thus every point of Y_{I_r} is contained in some Y_{I_i} with $i > r$. We have

$$Y_{I_r} = Y_{I_r} \cap (Y_{I_{r+1}} \cup \cdots \cup Y_{I_p}) = (Y_{I_r} \cap Y_{I_{r+1}}) \cup \cdots \cup (Y_{I_r} \cap Y_{I_p}).$$

Since both the characteristic functions and the set-function μ satisfy the Inclusion-Exclusion formula it follows that

$$\begin{aligned} 1_{Y_{I_r}} &= \sum_{r<i\le p} 1_{Y_{I_r}\cap Y_{I_i}} - \sum_{r<i<j\le p} 1_{Y_{I_r}\cap Y_{I_i}\cap Y_{I_j}} + \cdots, \\ \mu(Y_{I_r}) &= \sum_{r<i\le p} \mu(Y_{I_r}\cap Y_{I_i}) - \sum_{r<i<j\le p} \mu(Y_{I_r}\cap Y_{I_i}\cap Y_{I_j}) + \cdots. \end{aligned}$$

Each intersection in these two equalities is some Y_{I_s} with $s > r$. Collecting the like terms with the same index I_s, there are elements $d_s \in R$ such that

$$1_{Y_{I_r}} = \sum_{s>r} d_s 1_{Y_{I_s}} \quad \text{and} \quad \mu(Y_{I_r}) = \sum_{s>r} d_s \mu(Y_{I_s}). \tag{32}$$

Substitute (32) into (31) and collect the like terms with the same index I_i; we obtain

$$\sum_{i\ge r+1} w_i 1_{Y_{I_i}} = 0 \quad \text{but} \quad \sum_{i\ge r+1} w_i \mu(Y_{I_i}) \neq 0, \tag{33}$$

where $w_i \in M$. If $w_i = 0$ for all $i \ge r+1$, then $\sum_{i\ge r+1} w_i\mu(Y_{I_i}) = 0$, a contradiction. This means that not all w_i are zero; let r' be the smallest integer such that $w_{r'} \neq 0$. We thus have an expression of the same type as (31) but with a larger integer r', a contradiction. □

Groemer's original statement assumed that the commutative ring R is the real field $\mathbb{R}$, the R-module M is real vector space, the set-function μ is from $\mathcal{S}$ to M, and simple functions are from S to $\mathbb{R}$. But his proof does not depend on these assumptions. Our modification makes it convenient for application because we often have the situation where the set-function and the function to be integrated are taking values in different algebraic structures. We are only interested in the case where the set S is a vector space Ω over a field $\mathbb{F}$ and the intersectional class $\mathcal{S}$ is the lattice $L(\Omega)$ of all affine subspaces of Ω or the sub-lattice $L_0(\Omega)$ of all vector subspaces of Ω. The Boolean algebras generated by $L(\Omega)$ and $L_0(\Omega)$ are denoted by $B(\Omega)$ and $B_0(\Omega)$, respectively. We want to find the special form of the Inclusion-Exclusion condition (29) for a set-function on $L_0(\Omega)$ or $L(\Omega)$. It turns out that in this special case the Inclusion-Exclusion condition (29) is automatically satisfied when the field $\mathbb{F}$ is infinite.

Definition 2.3 *(a) A partial function $f : S \to M$ is called* measurable *with respect to an intersectional class $\mathcal{S}$ if (i) the image $Im\, f$ is a countable set; (ii) the complement of $S - Dom\, f$ belongs to $B(\mathcal{S})$; (iii) for each $v \in Im\, f$ the inverse image $f^{-1}(v)$ belongs $B(\mathcal{S})$.*

(b) Let M be a complete topological R-module and $\mu : B(\mathcal{S}) \to R$ a valuation. A measurable function $f : S \to M$ is called integrable *with respect to μ if $Im\, f$ is finite, or $Im\, f$ is countably infinite but for any sequence $\{v_k\}_{k=1}^{\infty} = Im\, f$, the following sequence*

$$S(f, v_1, \ldots, v_n) = \sum_{k=1}^{n} v_i \mu(f^{-1}(v_i))$$

is convergent in M

Proposition 2.4 *Let Ω be a vector space over an infinite field $\mathbb{F}$, either finite or infinite dimensional. Then any set-function μ on $L_0(\Omega)$ ($L(\Omega)$) satisfies the Inclusion-Exclusion formula (29); hence μ can be extended uniquely to a valuation on the Boolean algebra $B_0(\Omega)$ ($B(\Omega)$). In particular, any set function μ on $L_0(\Omega)$ can be uniquely extended to a translation invariant valuation on $B(\Omega)$.*

Proof. Let $V, V_1, \ldots, V_n$ be vector subspaces of Ω such that $V = V_1 \cup \cdots \cup V_n$. We first claim that $V = V_i$ for some i. This is obviously true when V is finite dimensional because any finitely many proper subspaces of V can not cover the whole space V. Let V be infinite dimensional. Suppose that it is not true; that is, each V_i is a proper subspace of V. Then there are vectors $v_i \in V$ such that $v_i \notin V_i$. Let W be the vector subspace spanned by $v_1, \ldots, v_n$. Obviously, $W = (W \cap V_1) \cup \cdots \cup (W \cap V_n)$, and each $W \cap V_i$ is a proper subspace of W because $v_i \notin W \cap V_i$. This is contrary to that of the finite dimensional case.

Now we claim that μ satisfies (29). Let $[n] = \{1, \ldots, n\}$. The formula (29) can be written as $\sum_{I\subset[n]}(-1)^{\#I}\mu\left(\bigcap_{i\in I} V_i\right) = 0$. We preceed by induction on n. It is obviously true for $n = 1$. For arbitrary n, among the subspaces $V_1, \ldots, V_n$ there is at least one subspace which is actually the whole space V, say, $V_n = V$. Then

$$\sum_{I\subset[n]}(-1)^{\#I}\mu\left(\bigcap_{i\in I} V_i\right) = \sum_{I\subset[n-1]}(-1)^{\#I}\mu\left(\bigcap_{i\in I} V_i\right) - \sum_{I\subset[n-1]}(-1)^{\#I}\mu\left(\bigcap_{i\in I} V_i \cap V_n\right) = 0.$$

The proof for the case of the lattice $L(\Omega)$ is exactly the same. □

Let Γ be the weighted directed graph defined as follows: The vertex set of Γ is $L_0(\Omega)$; for vertices corresponding to subspaces V and W such that $V \subset W$ and $\dim W/V < \infty$, there is a directed edge from V to W with the weight $\dim W/V$ and a directed edge from W to V with the negative weight $-\dim W/V$. Then Γ is decomposed into disjoint connected components Γ_σ. Recall that the *length* of a path is the number of edges in the path; the distance between two vertices is the minimal length of the paths between the two vertices.

Theorem 2.5 *For any vector space Ω over an infinite field $\mathbb{F}$, either finite or infinite dimensional. Then*

(a) every connected component Γ_σ is a lattice; that is, $V, W \in \Gamma_\sigma$ imply $V \cap W \in \Gamma_\sigma$ and $V + W \in \Gamma_\sigma$;

(b) there exists a translation invariant valuation $\mu : B(\Omega) \to \mathbb{Z}$ such that $\mu(W - V) = \dim W/V$ for any vector subspaces V and W satisfying $V \subset W$ and $\dim W/V < \infty$;

(c) the valuation in (b) is unique in the sense that if ν is another such valuation, there are numbers $c(\Gamma_\sigma)$ corresponding to connected components Γ_σ, such that $\nu(V) = \mu(V) + c(\Gamma_\sigma)$ for all $V \in \Gamma_\sigma$.

Proof. (a) We proceed by induction on the length $\ell(V, W)$ for $V, W \in \Gamma_\sigma$. It is obviously true when $\ell(V, W) = 1$ because we either have $V \subset W$ or $W \subset V$; so $V \cap W$ and $V + W$ belong to Γ_σ. For the case $\ell(V, W) = 2$, if we have a path like $V \hookrightarrow U \hookrightarrow W$ or $V \hookleftarrow U \hookleftarrow W$ then we have either $V \subset W$ or $V \supset W$; so $\ell(V, W) = 1$, a contradiction.

Thus the path of length 2 must be of the form $V \hookrightarrow U \hookleftarrow W$ or $V \hookleftarrow U \hookrightarrow W$. This means either $V + W \subset U$ or $U \subset V \cap W$; so either $\dim(V + W)/V \le \dim U/V < \infty$ or $\dim V/(V \cap W) \le \dim V/U < \infty$; that is, either $V + W$ or $V \cap W$ belongs to Γ_σ. In the first case, we have vectors $v_1, \ldots, v_k$ in V and vectors $w_1, \ldots, w_l$ in W such that $V \oplus \mathrm{span}\{w_1, \ldots, w_l\} = V + W = W \oplus \mathrm{span}\{v_1, \ldots, v_k\}$; hence $V + W = (V \cap W) \oplus \mathrm{span}\{v_1, \ldots, v_k\} \oplus \mathrm{span}\{w_1, \ldots, w_l\}$; it is clear that $\dim V/(V \cap W) < \infty$; so $V \cap W \in \Gamma_\sigma$. In the latter case, there are vectors $v_1, \ldots, v_k$ in V and vectors $w_1, \ldots, w_l$ in W such that $V = (V \cap W) \oplus \mathrm{span}\{v_1, \ldots, v_k\}$ and $W = \mathrm{span}\{w_1, \ldots, w_l\} \oplus (V \cap W)$; then $V + W = \mathrm{span}\{v_1, \ldots, v_k, w_1, \ldots, w_l\} \oplus (V \cap W)$; thus $\dim(V + W)/V < \infty$; therefore $V + W \in \Gamma_\sigma$.

Now if $\ell(V, W) = n \ge 3$, there are vectors $V_1, \ldots, V_{n-1}$ such that $\ell(V_i, V_{i+1}) = 1$ for $0 \le i \le n - 1$, here $V_0 = V$ and $V_n = W$. Then $V \cap V_2$ and $V + V_2$ are in Γ_σ and $\ell(V, V \cap V_2) = \ell(V, V + V_2) = 1$. Since either $V_2 \subset V_3$ or $V_2 \supset V_3$, we have either $\ell(V \cap V_2, V_3) = 1$ or $\ell(V + V_2, V_3) = 1$. Therefore $\ell(V, V_3) \le 2$; subsequently, $\ell(V, W) \le n - 1$, a contradiction. We have shown that each components Γ_σ is a lattice.

(b) Choose a subspace V_σ from each connected component Γ_σ and assign an integer $\mu(V_\sigma)$ arbitrarily. For any vertex W in the connected component Γ_σ, there is a directed path $V \overset{a_1}{\to} V_1 \overset{a_2}{\to} \cdots \overset{a_n}{\to} V_n = W$; we define $\mu(W) = \mu(V) + \sum_{i=1}^{n} a_i$. This is well-defined only if the total weight of any directed cycle is zero. Since any cycle is generated by the cycles of the form $(V \cap W) \hookrightarrow V \hookrightarrow (V + W) \hookleftarrow W \hookleftarrow (V \cap W)$, the total weight of such cycles is obviously zero; it follows that μ is well-defined on $L_0(\Omega)$. Extend μ to $L(\Omega)$ by setting $\mu(P) = \mu(P - v)$ for $P \in L(\Omega)$, where $v \in P$ and $P - v = \{p - v \mid x \in P\}$ is a vector space.

(c) The uniqueness in the sense is obvious. □

Corollary 2.6 *There exists a valuation $\chi : B(\mathbb{P}\Omega) \to \mathbb{Z}$, called the* Euler characteristic*, such that $\chi(\mathbb{P}W - \mathbb{P}V) = \dim W/V$ for projective subspaces $\mathbb{P}V$ and $\mathbb{P}W$ satisfying $\mathbb{P}V \subset \mathbb{P}W$ and $\dim W/V < \infty$.*

Proof. The Boolean algebra $B_0(\Omega)$ is isomorphic to its projectivization $B(\mathbb{P}\Omega)$. □

The hierarchical relation between the connected components of Γ is complicated when Ω is infinite dimensional. However, there are two connected components Γ_0 and Γ_1 which are important to us; that is, the component Γ_0 that contains the zero vector space $\{0\}$ and the component Γ_1 that contains the whole space Ω. We have two special valuations μ_0 and μ_1, where $\mu_0 = \mu$ on Γ_0 and $\mu_0 = 0$ on all other components; $\mu_1 = \mu$ on Γ_1 and $\mu_1 = 0$ on all other components. We assume that $\mu(\{0\}) = \mu(\Omega) = 0$. The valuation μ_1 is closely related to the motivic measures [9, 14]. In fact, let $\{\Omega_n \mid n \ge 0\}$ be a descending sequence of finite codimensional vector subspaces of Ω, such that $\bigcap_{k \ge 0} \Omega_k = \{0\}$; let $\pi_n : \Omega \to \Omega/\Omega_n$ and $\pi_{m,n} : \Omega/\Omega_n \to \Omega/\Omega_m$ be the obvious projections, $n > m$. Clearly, $\pi_m = \pi_{m,n} \circ \pi_n$. A subset $X \subset \Omega$ is called *cylindric* with respect to the sequence $\{\Omega_n\}$ if there exists a subset $B_k \subset \Omega/\Omega_k$ such that $X = \pi_k^{-1}(B_k)$; the cylinder set X is called *constructible* if B_k is constructible in Ω/Ω_k. Let $B(\Gamma_1)$ be the Boolean algebra generated by the subspaces in Γ_1. It is easy to see that Γ_1 is the same as the class of subspaces that contain some Ω_k. Obviously, μ_1 can be extended to the class of constructible sets of Ω.

Given an arbitrary subset $X \subset \Omega$. A point $x \in X$ is called Ω_k-constructible if $x + \Omega_k \subset X$; x is called *constructible* if it is Ω_k-constructible for at least one k; otherwise it is called *non-constructible*. Let X_k be the subset of X whose elements are Ω_k-constructible; and let X_∞ be the subset whose elements are non-constructible; $X_k^0 = X_k - X_{k-1}$. Obviously, $X = \bigsqcup_{k\le\infty} X_k^0$. A subset $X \subset \Omega$ is called *σ-constructible* if each X_k is Ω_k-constructible, or equivalently if all X_k^0 are constructible. One can construct a σ-ring $\mathbf{S}$ that contains all constructible sets, and a measure μ on $\mathbf{S}$ such that for any $X \in \mathbf{S}$,

$$\mu(X) = \sum_{k=0}^{\infty} \mu(X_k^0) \quad \text{where} \quad \mu(X_k^0) = \frac{[\pi_k(X_k^0)]}{[\Omega/\Omega_k]}.$$

The measure μ is the extension of the valuation μ_1 on the Boolean algebra $B(\Gamma_1)$. Let $K_0(\mathcal{V})$ be the Grothendieck ring of reducible algebraic varieties. The infinite sum is taken into account in the completion of the ring $K_0(\mathcal{V})[q^{-1}]$ with respect to the filtration

$$\cdots \supset F^{n-1}K_0(\mathcal{V})[q^{-1}] \supset F^n K_0(\mathcal{V})[q^{-1}] \supset \cdots, \qquad n \in \mathbb{Z}$$

where each $F^n K_0(\mathcal{V})[q^{-1}]$ is the subring of $K_0(\mathcal{V})[q^{-1}]$ generated by the elements of the form $[X]q^{-k}$ satisfying the condition $k - \dim X \ge n$. The construction is similar to that of motivic integration; see [9]. However, this construction is not needed for us to study curve singularities. We have a simple approach of completion when integrating with respect to a valuation; that is, the completion of the module M, not the ring $K_0(\mathcal{V})[q^{-1}]$.

3 Properties of the q-Series

Let $\Phi = \{\varphi_i\}_{i=1}^r$ be a family of holomorphic functions from $\mathbb{C}$ to $\mathbb{C}^n$ such that $\varphi_i(0) = 0$. Let $C_i = \operatorname{Im}\varphi_i$, $\mathbf{d} = (d_1, \ldots, d_r)$, where d_i is the GCD of the powers in the power series expansion of the coordinate components in the parameterization φ_i. Let $\mathbb{L}$ be the sublattice of $\mathbb{Z}^r$ generated by the vectors $\mathbf{d}_i$. We want to find the relationship between the q-series $L_\Phi(q;\mathbf{t})$, $L_\Phi^0(q;\mathbf{t})$, $M_\Phi(q;\mathbf{t})$, $M_\Phi^0(q;\mathbf{t})$ and the q-series $PL_\Phi(q;\mathbf{t})$, $PM_\Phi(q;\mathbf{t})$, $PL_\Phi^0(q;\mathbf{t})$, $PM_\Phi^0(q;\mathbf{t})$. From the geometric viewpoint, the q-series $PL_\Phi(q;\mathbf{t})$, $PM_\Phi(q;\mathbf{t})$, $PL_\Phi^0(q;\mathbf{t})$, and $PM_\Phi^0(q;\mathbf{t})$ are the "projectivization" of the q-series $L_\Phi(q;\mathbf{t})$, $L_\Phi^0(q;\mathbf{t})$, $M_\Phi(q;\mathbf{t})$, and $M_\Phi^0(q;\mathbf{t})$, respectively. In fact, any vector space V is a fiber bundle over its projectivization $\mathbb{P}V$ with the fiber $\mathbb{C}^*$, whose equivalence class in $K_0(\mathbb{C})$ is $[\mathbb{C}] - 1$.

Proposition 3.1 *Taking projectivization of spaces corresponds to acting the homomorphism map ϱ to the q-series of the corresponding spaces; that is,*

$$PL_\Phi(q;\mathbf{t}) = \varrho L_\Phi(q;\mathbf{t}) = \frac{L_\Phi(q;\mathbf{t})}{q-1}, \tag{34}$$

$$PL_\Phi^0(q;\mathbf{t}) = \varrho L_\Phi^0(q;\mathbf{t}) = \frac{L_\Phi^0(q;\mathbf{t})}{q-1}, \tag{35}$$

$$PM_\Phi(q;\mathbf{t}) = \varrho M_\Phi(q;\mathbf{t}) = \frac{M_\Phi(q;\mathbf{t})}{q-1}, \tag{36}$$

$$PM_\Phi^0(q;\mathbf{t}) = \varrho M_\Phi^0(q;\mathbf{t}) = \frac{M_\Phi^0(q;\mathbf{t})}{q-1}. \tag{37}$$

In particular, $L_\Phi(1;\mathbf{t}) = L_\Phi^0(1;\mathbf{t}) = M_\Phi(1;\mathbf{t}) = M_\Phi^0(1;\mathbf{t}) = 0$.

Proof. The formulas (34) and (36) follow from the fact that for any finite dimensional vector space V and finite codimensional vector subspace W of $\mathcal{O}_{\mathbb{C}^n,0}$,

$$\begin{aligned}
[\mathbb{P}V] &= 1+q+\cdots+q^{\dim \mathbb{P}V} = (q^{\dim V}-1)/(q-1),\\
[\mathbb{P}W] &= -(q^{-1}+\cdots+q^{-\dim \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}/W}) = (q^{-\dim \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}/W}-1)/(q-1),
\end{aligned}$$

and from the fact that $\sum_{\alpha\in\mathbb{L}} \mathbf{t}^\alpha = 0$ as a rational function. As for (35) and (37), they follow from the fact that

$$\begin{aligned}
\chi(\mathbb{P}\mathcal{I}^\alpha, q) &= \sum_{I\in L(\mathcal{I}^\alpha)} \mu(I, I^\alpha)(q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I}-1)/(q-1)\\
&= (\chi(\mathcal{I}^\alpha, q)-\chi(\mathcal{I}^\alpha, 1))/(q-1),\\
\chi(\mathbb{P}\mathcal{H}_\alpha, q) &= \sum_{\mathbb{P}H\in L(\mathbb{P}\mathcal{H}_\alpha)} \mu(\mathbb{P}H, \mathbb{P}\mathrm{Im}\,\ell_\alpha)(q^{\dim H}-1)/(q-1)\\
&= (\chi(\mathcal{H}_\alpha, q)-\chi(\mathcal{H}_\alpha, 1))/(q-1),
\end{aligned}$$

and from the fact that $\chi(\mathcal{I}^\alpha, 1) = \chi(\mathcal{H}_\alpha, 1) = \sum_{I\subset[r]}(-1)^{\#I} = 0$. □

Let us consider the infinite-dimensional vector space $\mathcal{O}_{\mathbb{C}^n,0}$ of the germs and let R be a commutative ring with unity $1\neq 0$. Because of Proposition 2.4, any set-function $\mu : L_0(\mathcal{O}_{\mathbb{C}^n,0}) \to R$ can be automatically viewed as a translation invariant valuation $\mu : B(\mathcal{O}_{\mathbb{C}^n,0}) \to R$. Note that the ring $R[[t_1,\ldots,t_r]]$ of the formal power series is complete. Assuming no topology on $R[[\mathbf{t}]]$, a sequence $F_n = \sum a_\alpha(n)\mathbf{t}^\alpha$ in $R[[\mathbf{t}]]$ is called *convergent* to $F = \sum a_\alpha \mathbf{t}^\alpha$ if for any $\alpha\in\mathbb{N}^r$ there is a number $N(\alpha)$ such that $a_\alpha(n) = a_\alpha$ for all $n\geq N(\alpha)$; see [17], page 6. So any formal power series is the limit of its truncation sequence of polynomials.

Lemma 3.2 *(a) The ideals I_i^∞ are not finite codimensional; that is, $I_i^\infty \notin \Gamma_1$. So $\mu_1(I_i^\infty) = 0$ for all $1\leq i\leq r$.*

(b) The function $F : \mathcal{O}_{\mathbb{C}^n,0} - \bigcup_{i=1}^r I_i^\infty \to R[[\mathbf{t}]]$ by $F(g) = \mathbf{t}^{\omega(g)}$ is measurable with respect to the intersectional class $L_0(\mathcal{O}_{\mathbb{C}^n,0})$ and is integrable with respect to any valuation $\mu : L_0(\mathcal{O}_{\mathbb{C}^n,0}) \to R$ such that $\mu(I_i^\infty) = 0$.

Proof. (a) For each fixed i it is clear that for any k there is a germ $g\in\mathfrak{m}^k$ such that $g\circ\phi_i \not\equiv 0$; that is, $\mathfrak{m}^k$ is not contained in the ideal I_i^∞ for all k. Select a sequence $g_{k_n}\in\mathfrak{m}^{k_n}$ such that $g_{k_n}\notin I_i^\infty$ and $g_{k_n}\notin\mathfrak{m}^{k_{n+1}}$. Then $g_{k_n}+I_i^\infty$ are independent cosets in $\mathcal{O}_{\mathbb{C}^n,0}/I_i^\infty$. Hence I_i^∞ is not finite codimensional.

(b) It is obvious that $\mathrm{Im}\,F$ is contained in $\{\mathbf{t}^\alpha \mid \alpha\in\mathbb{N}^r\}$. For any $\alpha\in\mathbb{N}^r$ the inverse image $F^{-1}(\mathbf{t}^\alpha)$ is the linear set I_0^α. So F is measurable with respect to $L_0(\mathcal{O}_{\mathbb{C}^n,0})$. Since the ring $R[[\mathbf{t}]]$ is complete, any truncated sequence of the formal power series $f(\mathbf{t}) = \sum_{\alpha\in\mathbb{N}^r}\mu(I_0^\alpha)\mathbf{t}^\alpha$ is convergent to $f(\mathbf{t})$. So F is integrable with respect to μ. □

Theorem 3.3 *Let $\mu : L_0(\mathcal{O}_{\mathbb{C}^n,0}) \to R$ be a valuation such that $\mu(I_i^\infty) = 0$, $1\leq i\leq r$. Then*

$$\begin{aligned}
\int_{\mathcal{O}_{\mathbb{C}^n,0}} \mathbf{t}^{\omega(g)}\mathrm{d}\mu(g) &= \sum_{\alpha\in\mathbb{L}\cap\mathbb{N}^r}\mu(I_0^\alpha)\mathbf{t}^\alpha\\
&= \frac{\prod_{i=1}^r(t_i^{d_i}-1)}{t_1^{d_1}\cdots t_r^{d_r}}\sum_{\alpha\in\mathbb{L}}\mu(I^\alpha)\mathbf{t}^\alpha, \qquad (38)\\
&= \frac{\prod_{i=1}^r(t_i-1)}{t_1\cdots t_r}\sum_{\alpha\in\mathbb{Z}^r}\mu(I^\alpha)\mathbf{t}^\alpha.
\end{aligned}$$

Proof. Let $[r] = \{1, 2, \dots, r\}$. For $I \subset [r]$, we denote by $\mathbf{d}_I$ the vector in $\mathbb{Z}^r$ whose ith entry is d_i for $i \in I$ and zero for $i \notin I$. When $I = \{i\}$ we write $\mathbf{d}_i$ instead of $\mathbf{d}_{\{i\}}$. The lattice $\mathbb{L}$ is generated by the vectors $\mathbf{d}_i$. Then

$$\begin{aligned}
\sum_{\alpha \in \mathbb{L} \cap \mathbb{N}^r} \mu(I_0^\alpha)\mathbf{t}^\alpha &= \sum_{\alpha \in \mathbb{L}} \mathbf{t}^\alpha \mu\left(I^\alpha - \bigcup_{i=1}^r I^{\alpha+\mathbf{d}_i}\right) \\
&= \sum_{\alpha \in \mathbb{L}} \mathbf{t}^\alpha \sum_{I \subset [r]} (-1)^{\#I} \mu(I^{\alpha+\mathbf{d}_I}) \\
&= \sum_{I \subset [r]} (-1)^{\#I} \mathbf{t}^{-\mathbf{d}_I} \sum_{\alpha \in \mathbb{L}} \mu(I^{\alpha+\mathbf{d}_I})\mathbf{t}^{\alpha+\mathbf{d}_I} \\
&= \prod_{i=1}^r \left(1 - t_i^{-d_i}\right) \sum_{\alpha \in \mathbb{L}} \mu(I^\alpha)\mathbf{t}^\alpha \\
&= \frac{\prod_{i=1}^r (t_i^{d_i} - 1)}{t_1^{d_1} \cdots t_r^{d_r}} \sum_{\alpha \in \mathbb{L}} \mu(I^\alpha)\mathbf{t}^\alpha.
\end{aligned}$$

Let $\mathbb{D} = \prod_{i=1}^r [0, d_i) \cap \mathbb{Z}$. Then $\mathbb{Z}^r = \bigsqcup_{\gamma \in \mathbb{D}} (\mathbb{L} - \gamma)$ and $I^{\alpha-\gamma} = I^\alpha$. We have

$$\begin{aligned}
\sum_{\alpha \in \mathbb{Z}^r} \mu(I^\alpha)\mathbf{t}^\alpha &= \sum_{\alpha \in \mathbb{L}} \mu(I^\alpha)\mathbf{t}^\alpha \sum_{\gamma \in D} \mathbf{t}^{-\gamma} \\
&= \sum_{\alpha \in \mathbb{L}} \mu(I^\alpha)\mathbf{t}^\alpha \prod_{i=1}^r \frac{t_i^{-d_i} - 1}{t_i^{-1} - 1} \\
&= \frac{\prod_{i=1}^r (t_i^{d_i} - 1)}{t_1^{d_1-1} \cdots t_r^{d_r-1} \prod_{i=1}^r (t_i - 1)} \sum_{\alpha \in \mathbb{L}} \mu(I^\alpha)\mathbf{t}^\alpha.
\end{aligned}$$

The formula (38) follows by substitution. □

Some special cases of Theorem 3.3 are particularly interesting. Let $R = \mathbb{Z}[q^{-1}]$, where $q = [\mathbb{C}]$. We have the valuation $\mu_1 : L_0(\mathcal{O}_{\mathbb{C}^n,0}) \to \mathbb{Z}[q^{-1}]$, defined by

$$\mu_1(V) = \begin{cases} 1/[\mathcal{O}_{\mathbb{C}^n,0}/V] & V \text{ is finite codimensional} \\ 0 & \text{otherwise} \end{cases},$$

where $L_0(\mathcal{O}_{\mathbb{C}^n,0})$ is the lattice of all vector subspaces of $\mathcal{O}_{\mathbb{C}^n,0}$; and the valuation $\tilde{\mu}_1 : L(\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}) \to \mathbb{Z}[q^{-1}]$, defined by

$$\tilde{\mu}_1(\mathbb{P}V) = \begin{cases} -[\mathbb{P}(\mathcal{O}_{\mathbb{C}^n,0}/V)]/[\mathcal{O}_{\mathbb{C}^n,0}/V] & V \text{ is finite codimensional in } \mathcal{O}_{\mathbb{C}^n,0} \\ 0 & \text{otherwise} \end{cases},$$

where $L(\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0})$ is the lattice of all projective subspaces of $\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}$. The motivation to define μ_1 on $L_0(\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0})$ is explained in Section 2. The order function $\omega(g)$ is well-defined on $\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}$ because $\omega(cg) = \omega(g)$ for any $c \neq 0$.

Theorem 3.4 *The q-series $M_\Phi(q; \mathbf{t}) \prod_{i=1}^r (q^{-1}t_i^{d_i} - 1)(t_i^{d_i} - 1)$ and $PM_\Phi(q; \mathbf{t}) \prod_{i=1}^r (q^{-1}t_i^{d_i} - 1)$ are polynomials in q^{-1} and $\mathbf{t}$; $L_\Phi(q; \mathbf{t})$, $L_\Phi^0(q; \mathbf{t})$, $PM_\Phi(q; \mathbf{t})$, and $PL_\Phi^0(q; \mathbf{t})$ are rational* *functions in q and $\mathbf{t}$ with the same denominator $\prod_{i=1}^{r}(t_i^{d_i}-1)$. Moreover,*

$$M_\Phi^0(q;\mathbf{t}) = \int_{\mathcal{O}_{\mathbb{C}^n,0}} \mathbf{t}^{\omega(g)} \mathrm{d}\mu_1(g) = \frac{M_\Phi(q;\mathbf{t})\prod_{i=1}^{r}(t_i^{d_i}-1)}{t_1^{d_1}\cdots t_r^{d_r}}, \tag{39}$$

$$PM_\Phi^0(q;\mathbf{t}) = \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}} \mathbf{t}^{\omega(g)} \mathrm{d}\tilde{\mu}_1(g) = \frac{PM_\Phi(q;\mathbf{t})\prod_{i=1}^{r}(t_i^{d_i}-1)}{t_1^{d_1}\cdots t_r^{d_r}}, \tag{40}$$

$$L_\Phi^0(1/q;\mathbf{t}) = \int_{\mathcal{O}_{\mathbb{C}^n,0}} q^{\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\omega(g)+\mathbf{d}}} \mathbf{t}^{\omega(g)} \mathrm{d}\mu_1(g), \tag{41}$$

$$PL_\Phi^0(1/q;\mathbf{t}) = \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}} q^{\dim \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}/I^{\omega(g)+\mathbf{d}}} \mathbf{t}^{\omega(g)} \mathrm{d}\tilde{\mu}_1(g). \tag{42}$$

Proof. We first prove the formulas (39-42), then we prove the rationality of the q-series.

The formula (39) is a direct consequence of (38). Formula (40) follows from (36) and (39). Let H_i be the subspace of $\mathbb{C}^r$ whose ith coordinate is zero. For $I \subset [r]$, let $H_I = \bigcap_{i\in I} H_i$. The inverse image $\ell_\alpha^{-1}(\mathrm{Im}\,\ell_\alpha \cap H_I)$ is the ideal $I^{\alpha+\mathbf{d}_I}$. Then $\mathrm{Im}\,\ell_\alpha \cap H_I$ is isomorphic to $I^{\alpha+\mathbf{d}_I}/I^{\alpha+\mathbf{d}}$. Note that the quotient $\frac{\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}}{I^{\alpha+\mathbf{d}_I}/I^{\alpha+\mathbf{d}}}$ is isomorphic to $\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}_I}$. This implies that the dimension of $I^{\alpha+\mathbf{d}_I}/I^{\alpha+\mathbf{d}}$ is the dimension of $\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}$ minus the dimension of $\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}_I}$. Thus

$$\begin{aligned}\chi(\mathcal{H}_\alpha, q) &= \sum_{I\subset[r]} (-1)^{\#I} q^{\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}} q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}_I}} \\ &= q^{\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}} \chi(\mathcal{I}^\alpha, q).\end{aligned}$$

Take summation over $\mathbb{L}\cap\mathbb{N}^r$; the formula (41) follows.

Recall $\chi(\mathcal{H}_\alpha, q) = (q-1)\chi(\mathbb{P}\mathcal{H}_\alpha, q)$ and $\chi(\mathcal{I}_\alpha, q) = (q-1)\chi(\mathbb{P}\mathcal{I}_\alpha, q)$. Then

$$\begin{aligned}\chi(\mathbb{P}\mathcal{H}_\alpha, q) &= (q-1)^{-1} q^{\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}} \chi(\mathcal{I}^\alpha, q) \\ &= (q-1)^{-1} q^{\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}} (q-1)\chi(\mathbb{P}\mathcal{I}^\alpha, q) \\ &= q^{\dim \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}} \chi(\mathbb{P}\mathcal{I}^\alpha, q).\end{aligned}$$

Take summation over $\mathbb{L}\cap\mathbb{N}^r$; the formula (42) follows.

To prove the rationality for the q-series, we only need to show that $M_\Phi(q;\mathbf{t})$, $L_\Phi(q;\mathbf{t})$, and $L_\Phi^0(q;\mathbf{t})$ are rational functions. The rationality of the other q-series follows from Proposition 3.1 and formulas (39) and (40).

Let $\alpha = (a_1, \ldots, a_i, \ldots, a_r)$, $\alpha' = (a_1, \ldots, a_i', \ldots, a_r)$. Note that $I^{\alpha'} \subset I^\alpha$ if $\alpha \le \alpha'$. If both $a_i \le 0$ and $a_i' \le 0$, then $I^\alpha = I^{\alpha'}$. We fix a vector $\alpha \in \mathbb{L}$ and consider the vectors $\alpha_k = \alpha + k\mathbf{d}_i$, $k \in \mathbb{Z}$. Obviously, I^{α_k} is a sequence of descending chains of ideals in $\mathcal{O}_{\mathbb{C}^n,0}$. The ideals I^{α_k} for $k \le 0$ are constant. So

$$\sum_{k\le 0} [\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+k\mathbf{d}_i}]^{-1} \mathbf{t}^{\alpha+k\mathbf{d}_i} = \frac{[\mathcal{O}_{\mathbb{C}^n,0}/I^\alpha]^{-1} t_i^{d_i} \mathbf{t}^\alpha}{t_i^{d_i}-1}. \tag{43}$$

The sequence I^{α_k} for $k \ge 1$ can not be stable; otherwise, we must have $I^{\alpha_k} = I^{\alpha+\infty\mathbf{d}_i}$ for large enough k, but the ideal $I^{\alpha+\infty\mathbf{d}_i}$ is not finite codimensional, a contradiction. We claim that $[\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha_{k+1}}] = [\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha_k}]q$ for large enough k.

Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be an arbitrary germ, written as $g = \sum_{j,\dots,k\geq 0} c_{j,\dots,k} z_1^j \cdots z_n^k$. Let the parameterization $\varphi_i(z) = (z_1(z), \dots, z_n(z))$ be written as

$$\begin{cases} z_1 &= c_{1,j_1} z^{j_1} + c_{1,j_2} z^{j_2} + \cdots, \\ & \vdots \\ z_n &= c_{n,k_1} z^{k_1} + c_{n,k_2} z^{k_2} + \cdots, \end{cases}$$

where the powers of z are in ascending order, all the constants are nonzero, and each equation may have only finitely many terms. Then

$$g \circ \varphi_i(z) = \sum_{j,\dots,k\geq 0} c_{j,\dots,k} \left(c_{1,j_1} z^{j_1} + \cdots\right)^j \cdots \left(c_{n,k_1} z^{k_1} + \cdots\right)^k.$$

The terms $(c_{1,j_1} z^{j_1} + \cdots)^j, \dots, (c_{n,k_1} z^{k_1} + \cdots)^k$ may be written as the binomial expansions

$$\begin{gathered} \textstyle\sum_{j_1'+j_2'+\cdots=j} \binom{j}{j_1',j_2',\dots} (c_{1,j_1} z^{j_1})^{j_1'} (c_{1,j_2} z^{j_2})^{j_2'} \cdots, \\ \cdots \\ \textstyle\sum_{k_1'+k_2'+\cdots=k} \binom{k}{k_1',k_2',\dots} (c_{n,k_1} z^{k_1})^{k_1'} (c_{n,k_2} z^{k_2})^{k_2'} \cdots. \end{gathered}$$

Thus

$$g \circ \varphi_i(z) = \sum_{l\geq 0} z^l \sum \binom{j}{j_1',j_2',\dots} \cdots \binom{k}{k_1',k_2',\dots} \left(c_{1,j_1}^{j_1'} \cdots c_{n,k_1}^{k_1'} c_{1,j_2}^{j_2'} \cdots c_{n,k_2}^{k_2'} \cdots\right) c_{j,\dots,k},$$

where the second sum is extended over all indices $(j_1', j_2', \dots), \cdots, (k_1', k_2', \dots)$ such that

$$\begin{cases} j_1' + j_2' + \cdots = j \\ \qquad \cdots \\ k_1' + k_2' + \cdots = k \\ (j_1 j_1' + \cdots + k_1 k_1') + (j_2 j_2' + \cdots + k_2 k_2') + \cdots = l. \end{cases} \tag{44}$$

Note that $d_i = \gcd(j_1, \dots, k_1; j_2, \dots, k_2; \dots)$. The ideal $I^{\alpha_k} = I^{\alpha + k\mathbf{d}_i}$ is defined by the system of the linear equations

$$\begin{cases} \sum \binom{j}{j_1',j_2',\dots} \cdots \binom{k}{k_1',k_2',\dots} \left(c_{1,j_1}^{j_1'} \cdots c_{n,k_1}^{k_1'} \cdots\right) c_{j,\dots,k} = 0 \\ l < a_i + k d_i, \ d_i | l \end{cases} \tag{45}$$

in variables $c_{j,\dots,k}$, where the sum is extended over the indices $(j_1', j_2', \dots), \cdots, (k_1', k_2', \dots)$ such that (44) is satisfied, plus some other fixed similar systems of linear equations specified by the parameterizations $\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_r$ and integers $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r$, respectively.

Let S_{C_i} be the semigroup of the irreducible curve C_i and let S_{φ_i} be the semigroup of the parameterization φ_i. Then $S_{\varphi_i} = d_i S_{C_i}$. Note that S_{C_i} contains any large enough integers; see [3, 18]. Hence S_{φ_i} contains any large enough integers in $d_i\mathbb{N}$. It is clear that for large enough integer kd_i, $\dim(I^{\alpha+k\mathbf{d}_i}/I^{\alpha+(k+1)\mathbf{d}_i}) = 1$ if and only if $kd_i \in S_{\varphi_i}$. It can be also directly shown that when k is large enough, the last equation in the system (45) with $a_i + (k+1)d_i$ is linearly independent of all other equations in the system with $a_i + kd_i$. Thus there exists a number $l \geq 1$ such that $I^{\alpha+(l+k)\mathbf{d}_i}/I^{\alpha+(l+k+1)\mathbf{d}_i}$ is one dimensional for all $k \geq 0$. This implies that $[\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+(l+k)\mathbf{d}_i}] = [\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+l\mathbf{d}_i}]q^k$. Therefore,

$$\begin{aligned}\sum_{k\geq 0}\left[\frac{\mathcal{O}_{\mathbb{C}^n,0}}{I^{\alpha+(l+k)\mathbf{d}_i}}\right]^{-1}\mathbf{t}^{\alpha+(l+k)\mathbf{d}_i} &= \sum_{k\geq 0}\left[\frac{\mathcal{O}_{\mathbb{C}^n,0}}{I^{\alpha+l\mathbf{d}_i}}\right]^{-1}\mathbf{t}^{\alpha+l\mathbf{d}_i}q^{-k}t_i^{kd_i} \\ &= \frac{\mathbf{t}^{\alpha+l\mathbf{d}_i}[\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+l\mathbf{d}_i}]^{-1}}{1-q^{-1}t_i^{d_i}}.\end{aligned}$$

This and (43) imply that the q-series $M_\Phi(q;\mathbf{t})\prod_{i=1}^r(q^{-1}t_i^{d_i}-1)(t_i^{d_i}-1)$ must be a polynomial in q^{-1} and $t_1,\ldots,t_r$.

Similarly, for the finitely many vectors $\alpha+\mathbf{d}_I$, $I\subset[r]$, there exists a common positive integer l such that $[\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}_I+(l+k)\mathbf{d}_i}] = [\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}_I+l\mathbf{d}_i}]q^k$ for $k\geq 0$. Then

$$\sum_{k\geq 0}\mathbf{t}^{\alpha+(l+k)\mathbf{d}_i}\sum_{I\subset[r]}(-1)^{\#I}\left[\frac{\mathcal{O}_{\mathbb{C}^n,0}}{I^{\alpha+(l+k)\mathbf{d}_i+\mathbf{d}_I}}\right]^{-1} = \frac{\mathbf{t}^{\alpha+l\mathbf{d}_i}}{1-q^{-1}t_i^{d_i}}\sum_{I\subset[r]}(-1)^{\#I}\left[\frac{\mathcal{O}_{\mathbb{C}^n,0}}{I^{\alpha+l\mathbf{d}_i+\mathbf{d}_I}}\right]^{-1}.$$

This shows that $PM_\Phi^0(q;\mathbf{t})\prod_{i=1}^r(q^{-1}t_i^{d_i}-1)$ must be a polynomial in q^{-1} and $t_1,\ldots,t_r$. Analogously,

$$\sum_{k\leq -1}\left[\frac{I^{\alpha+k\mathbf{d}_i}}{I^{\alpha+k\mathbf{d}_i+\mathbf{d}}}\right]\mathbf{t}^{\alpha+k\mathbf{d}_i} = \frac{[I^\alpha/I^{\alpha-\mathbf{d}_i+\mathbf{d}}]\mathbf{t}^\alpha}{t_i^{d_i}-1}.$$

Note that $\left[\frac{I^{\alpha+(l+k)\mathbf{d}_i}}{I^{\alpha+(l+k)\mathbf{d}_i+\mathbf{d}}}\right] = \frac{[\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+(l+k)\mathbf{d}_i}]}{[\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+(l+k)\mathbf{d}_i+\mathbf{d}}]}$. It follows that

$$\sum_{k\geq 0}\left[\frac{I^{\alpha+(l+k)\mathbf{d}_i}}{I^{\alpha+(l+k)\mathbf{d}_i+\mathbf{d}}}\right]\mathbf{t}^{\alpha+(l+k)\mathbf{d}_i} = \frac{\mathbf{t}^{\alpha+l\mathbf{d}_i}}{1-t_i^{d_i}}\left[\frac{I^{\alpha+l\mathbf{d}_i}}{I^{\alpha+l\mathbf{d}_i+\mathbf{d}}}\right].$$

This means that the q-series $L_\Phi(q;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)$ must be a polynomial in q and $t_1,\ldots,t_r$. The following

$$\sum_{k\geq 0}\sum_{I\subset[r]}(-1)^{\#I}\left[\frac{I^{\alpha+(l+k)\mathbf{d}_i+\mathbf{d}_I}}{I^{\alpha+(l+k)\mathbf{d}_i+\mathbf{d}}}\right]\mathbf{t}^{\alpha+(l+k)\mathbf{d}_i} = \frac{\mathbf{t}^{\alpha+l\mathbf{d}_i}}{t_i^{d_i}-1}\sum_{I\subset[r]}(-1)^{\#I}\left[\frac{I^{\alpha+l\mathbf{d}_i+\mathbf{d}_I}}{I^{\alpha+l\mathbf{d}_i+\mathbf{d}}}\right]$$

shows that $L_\Phi^0(q;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)$ is a polynomial in q and $t_1,\ldots,t_r$. □

Theorem 3.5 *The series $PL_\Phi^0(1;\mathbf{t})$ and $PM_\Phi^0(1;\mathbf{t})$ are polynomials for $r\geq 2$, but merely series for $r=1$. However, for all $r\geq 1$,*

$$PM_\Phi^0(1;\mathbf{t}) = \frac{PM_\Phi(1;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)}{t_1^{d_1}\cdots t_r^{d_r}} = PL^0(1;\mathbf{t}), \tag{46}$$

$$PL_\Phi(1;\mathbf{t}) = \frac{PM_\Phi(1;\mathbf{t})(t_1^{d_1}\cdots t_r^{d_r}-1)}{t_1^{d_1}\cdots t_r^{d_r}}, \tag{47}$$

$$PL_\Phi^0(1;\mathbf{t}) = \frac{PL_\Phi(1;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)}{t_1^{d_1}\cdots t_r^{d_r}-1}. \tag{48}$$

Proof. We show the identities first. The first half of (46) is a direct consequence of (40) by setting $q = 1$. Since $\chi(\mathbb{P}\mathcal{H}_\alpha, q) = q^{\dim \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}}\chi(\mathbb{P}\mathcal{I}^\alpha, q)$, we have $\chi(\mathbb{P}\mathcal{H}_\alpha, 1) = \chi(\mathbb{P}\mathcal{I}^\alpha, 1)$. This implies that $PL^0_\Phi(1;\mathbf{t}) = PM^0_\Phi(1;\mathbf{t})$, which is the second equality in (46).

Note that $\operatorname{Im}\ell_\alpha \simeq I^\alpha/I^{\alpha+\mathbf{d}}$ and $\frac{\mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}}}{I^\alpha/I^{\alpha+\mathbf{d}}} \simeq \mathcal{O}_{\mathbb{C}^n,0}/I^\alpha$. It follows that the dimension of $\operatorname{Im}\ell_\alpha$ equals the difference $\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}} - \dim \mathcal{O}_{\mathbb{C}^n,0}/I^\alpha$. Thus (47) is obtained as follows:

$$\begin{aligned}
PL_\Phi(1;\mathbf{t}) &= \sum_{\alpha\in\mathbb{N}^r}(\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{\alpha+\mathbf{d}} - \dim \mathcal{O}_{\mathbb{C}^n,0}/I^\alpha)\mathbf{t}^\alpha \\
&= (\mathbf{t}^{-\mathbf{d}} - 1)\sum_{\alpha\in\mathbb{Z}^r}\dim \mathcal{O}_{\mathbb{C}^n,0}/I^\alpha\mathbf{t}^\alpha \\
&= (1-\mathbf{t}^{-\mathbf{d}})PM_\Phi(1;\mathbf{t}) \\
&= PM_\Phi(1;\mathbf{t})(\mathbf{t}^{\mathbf{d}} - 1)/\mathbf{t}^{\mathbf{d}}.
\end{aligned}$$

The identity (48) follows from (47) and $PM^0_\Phi(1;\mathbf{t}) = PL^0_\Phi(1;\mathbf{t})$.

To show that $PL^0(1,\mathbf{t})$ is a polynomial, let us write the identity (48) as

$$(t_1^{d_1}\cdots t_r^{d_r} - 1)PL^0(1,\mathbf{t}) = PL_\Phi(1;\mathbf{t})\prod_{i=1}^r(t_i^{d_i} - 1). \tag{49}$$

On the one hand note that $L_\Phi(q;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)$ is a polynomial in q and $\mathbf{t}$. It follows that $PL_\Phi(q;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)$ is also a polynomial in q and $\mathbf{t}$ because $L_\Phi(q;\mathbf{t}) = (q-1)PL_\Phi(q;\mathbf{t})$. So $PL_\Phi(1;\mathbf{t})\prod_{i=1}^r(t_i^{d_i}-1)$ is a polynomial in $\mathbf{t}$. On the other hand, for each i there exists an integer N_i such that for $a_i \geq N_i$, there are germs g such that $g\circ\varphi_i = \varepsilon_i z^{a_i} + \cdots$ with $\varepsilon_i \neq 0$. This means that for $\alpha \in \mathbb{N}^r$ such that $\alpha \geq (N_1, \ldots, N_r)$, we have $\operatorname{Im}\ell_\alpha = \mathbb{C}^r$; it follows that the complement of the hyperplane arrangement $\mathcal{H}_\alpha$ is $(\mathbb{C}^*)^r$; thus the coefficient of $\mathbf{t}^\alpha$ in $PL^0_\phi(q;\mathbf{t})$ is $(q-1)^{r-1}$; so when $r \geq 2$ the coefficients of $\mathbf{t}^\alpha$ in $PL^0_\phi(1;\mathbf{t})$ are zero. However, the identity (49) and the polynomial property of its right side means that outside a bounded set in $\mathbb{N}^r$, the coefficients c_α of $PL^0(1;\mathbf{t})$ satisfy the relation: $c_\alpha = c_{\alpha+\mathbf{1}}$. Notice that the coefficients of $PL^0(1;\mathbf{t})$ are zero for large enough α. This forces that $PL^0(1;\mathbf{t})$ is actually a polynomial. □

4 Plane Curve Singularity

In this section we assume that $(C,0)$ is a plane curve singularity in $\mathbb{C}^2$ with irreducible components C_i, each passing through the origin; $C = \bigcup_{i=1}^r C_i$. Let C be defined by a germ $f(z_1,z_2) = 0$ and C_i by $f_i(z_1,z_2) = 0$, where $f = f_1\cdots f_r$. We further assume that $\Phi = \{\varphi_i\}_{i=1}^r$ are uniformizations of $\{C_i\}_{i=1}^r$, respectively; that is, each $\varphi_i : \mathbb{C} \to C_i$ is biholomorphic outside the origin; this can be done because every irreducible component is isomorphic to $\mathbb{C}$; see [2]. The powers of z in both coordinates $z_1(z)$ and $z_2(z)$ for $\varphi_i(z) = (z_1(z), z_2(z))$ must be coprime; that is, $\mathbf{d} = (1,\ldots,1) \in \mathbb{Z}^r$. Let S^3_ε be the 3-sphere of radius ε in $\mathbb{C}^2$ with the center at the origin. For small enough ε the intersection $S^3_\varepsilon \cap C$ defines a link L_r with r components. It is natural to ask the relationship between the polynomial $PL^0(1;\mathbf{t})$ and the Alexander polynomial $\Delta^{L_r}(\mathbf{t})$ of the link L_r because both are determined by the singularity. Recall that the Alexander polynomial is only well-defined up to multiplication by $\pm\mathbf{t}^\alpha$. Using the result of Eisenbud and Neumann

[11], Campillo, Delgado and Gusein-Zade [4] recently proved that the two polynomials are actually the same, provided that $\Delta^{L_r}(\mathbf{t})$ is normalized so that $\Delta^{L_r}(\mathbf{0}) = 1$. We generalize it to Theorem 4.3 with rigorous proof.

Let $\pi : (X, E) \to (\mathbb{C}^2, 0)$ be an emdedded resolution of the plane curve singularity $(C, 0)$ in [11] (page 142), where $E = \pi^{-1}(0)$ is the exceptional divisor; the components of E are listed by $E_1, \ldots, E_s$, where each is isomorphic to the complex projective line $\mathbb{CP}^1$; the components of the strict transform $\tilde{C}_i$ of the curve C_i intersect E transversely with the intersection multiplicity one at each intersection point. Let D_j be the set of intersection points of E_j and all other components in the total transform $(f \circ \pi)^{-1}(0)$. For each germ $g \in \mathcal{O}_{\mathbb{C}^2,0}$ such that $g(0) = 0$, let $m_{g,j}$ be the multiplicity of the lifting $\tilde{g} = g \circ \pi$ along the exceptional divisor E_j; for the germs f_i we write m_{ij} instead of $m_{f_i,j}$. The total transform $\tilde{g}^{-1}(0)$ always contains E; the strict transform of $C_g = \{g = 0\}$ is the closure of $\pi^{-1}(C_g - \{0\})$ in X, denoted $\tilde{C}_g$. The number of intersections of $\tilde{C}_g$ and E (counted by multiplicities) is a multiset of E.

Let $\mathcal{V}_{\mathbb{C}}$ be the category of reducible varieties and let $K_0(\mathcal{V}_{\mathbb{C}})$ be the Grothendieck ring generated by the isomorphism classes of objects in $\mathcal{V}_{\mathbb{C}}$, subject to the relations (19) and (20). Given a constructible set Y; its equivalence class in $K_0(\mathcal{V}_{\mathbb{C}})$ is denoted $[Y]$. By a *multiset* of Y we mean a collection of some elements in Y with *repetition* allowed. Equivalently, we consider a multiset as a function $v : Y \to \mathbb{N}$; a k-*multiset* is a function v such that $\sum_{y \in Y} v(y) = k$. We denote by $\langle {Y \atop k} \rangle$ the set of all k-multisets of Y. The set $\langle {Y \atop k} \rangle$ can be made into a topological space by identifying it with the symmetric power $S^k Y = Y^k / S_k$, where S_k is the symmetric group of $[k] = \{1, 2, \ldots, k\}$, acting on Y^k in an obvious way. An ordinary subset of Y can be viewed as a function $v : Y \to \mathbb{N}$ such that $v(y) \le 1$ for all $y \in Y$. The set $\binom{Y}{k}$ of all k-subsets of Y can also be made into a topological space by the identification

$$\binom{Y}{k} = \left(Y^k - \bigcup_{i \neq j} Y^k_{i,j} \right) \Big/ S_k,$$

where $Y^k_{i,j} = \{(y_1, \ldots, y_k) \in Y^k \,|\, y_i = y_j\}$. Note that $\langle {Y \atop 0} \rangle = \binom{Y}{0}$ is a singleton, consisting of the only empty set. Note that the equivalence class $[\binom{Y}{k}]$ is not an element in $K_0(\mathcal{V}_{\mathbb{C}})$, but an element in $\mathbb{Q} \otimes K_0(\mathcal{V})$, having the form

$$\left[\binom{Y}{k} \right] = \binom{[Y]}{k} =: \frac{[Y]([Y]-1)\cdots([Y]-k+1)}{k!}.$$

There is a projection $\pi : \langle {Y \atop k} \rangle \to \bigsqcup_{l=1}^k \binom{Y}{l}$, given by $\pi(v)(y) = 1$ if $v(y) \neq 0$ and $\pi(v)(y) = 0$ otherwise for $v \in \langle {Y \atop k} \rangle$. The fiber over the set $\binom{Y}{l}$ is a finite set whose cardinality is the number of positive integral solutions of the equation $y_1 + \cdots + y_l = k$, which turns out to be $\binom{k-1}{l-1}$. Note that $\binom{Y}{k}$ is open in $\langle {Y \atop k} \rangle$ and is stratified by $\binom{k-1}{l-1}$ many copies of $\binom{Y}{l}$, $1 \le l \le k$. We then have

$$\begin{aligned} [S^k Y] &= \left[\left\langle {Y \atop k} \right\rangle \right] = \sum_{l=1}^{k} \binom{k-1}{l-1} \left[\binom{Y}{l} \right] \\ &= \binom{[Y]+k-1}{k} = (-1)^k \binom{-[Y]}{k}. \end{aligned}$$

The symmetric space $S(Y) = \bigsqcup_{k\geq 0} S^k(Y)t^k$ can be considered as a graded space and its equivalence class can be considered as an element in $\mathbb{Q}\otimes K_0(\mathcal{V})[[t]]$; that is,

$$[S(Y)] = \sum_{k\geq 0}[S^k(Y)]t^k = (1-t)^{-[Y]}.$$

For disjoint constructible sets Y_1 and Y_2, the symmetric space $S(Y_1 \sqcup Y_2)$ is isomorphic to the product $S(Y_1)\times S(Y_2)$. Write $[S(Y_1)] = (1-t_1)^{-[Y_1]}$ and $[S(Y_2)] = (1-t_2)^{-[Y_2]}$; we have $[S(Y_1 \sqcup Y_2)] = [S(Y_1)][S(Y_2)]$.

Let us consider the symmetric space $S(E)$ of the exceptional divisor E in a resolution of the plane curve singularity $(C,0)$. Write $E^0 = \bigcup_{j=1}^s E_j^0$ and $D = \bigcup_{j=1}^s D_j$. Obviously, E is the disjoint union of E^0 and D, and

$$S(E) = S(E^0 \sqcup D) = \Big(S(E^0)\times\{1\}\Big)\bigsqcup\left(S(E^0)\times\bigcup_{k\geq 1} S^k D\right).$$

We define the projection $\phi : \mathcal{O}_{\mathbb{C}^2,0} \to S(E)$ as follows: For $g\in\mathcal{O}_{\mathbb{C}^2,0}$, the image $\phi(g)$ is the multiset of intersections of the strict transform $\tilde{C}_g$ and the exceptional divisor E (counted by multiplicities); that is, $\phi(g)$ is the function on E whose support is $\tilde{C}_g\cap E$ and the value of $\phi(g)$ at each point $p\in\tilde{C}_g\cap E$ equals its intersect multiplicity. For a germ $g\in\mathcal{O}_{\mathbb{C}^2,0}$, if the strict transform $\tilde{C}_g$ only intersects E in E^0, then the intersection set must be a finite set and the intersection multiplicity must be finite at each intersection point. Moreover, let $k_j = \sum_{p\in E_j^0}\phi(g)(p)$; it is not hard to see that $\omega_i(g) = \sum_{j=1}^s m_{ij}k_j$. On the other hand, if the strict transform $\tilde{C}_g$ intersects E at some singular point in D, then one can see that there is at leat one $\omega_i(g)=\infty$ because the curve C_g shares a tangent line with one of the irreducible curves C_i. Therefore, the map ϕ transforms the union $\bigcup_{i=1}^r I_i^\infty$ of the arrangement $\mathcal{I}^\infty$ to $S(E^0)\times\bigcup_{k\geq1}S^kD$ and the complement of $\mathcal{I}^\infty$ to $S(E^0)$.

Let $\lambda : \mathcal{O}_{\mathbb{C}^n,0}\to\mathbb{Z}^s$, $\lambda(g) = (m_{g,1},\ldots,m_{g,s})$. For each vector $\beta\in\mathbb{Z}^s$, we define

$$J^\beta = \{g\in\mathcal{O}_{\mathbb{C}^2,0} \mid \lambda(g)\geq\beta\} \quad\text{and}\quad J_0^\beta = \{g\in\mathcal{O}_{\mathbb{C}^2,0}\mid\lambda(g)=\beta\}.$$

It is clear that J^β is an ideal of $\mathcal{O}_{\mathbb{C}^2,0}$ and contains the ideals $\mathfrak{m}^k$ for large enough k; so J^β is finite codimensional. We have $J_0^\beta = J^\beta - \bigcup_{j=1}^s J^{\beta+\mathbf{1}_j}$, where $\mathbf{1}_j\in\mathbb{Z}^s$ is the vector whose jth coordinate is 1 and zero elsewhere. The collection $\mathcal{J}^\beta = \{J^{\beta+\mathbf{1}_j}\}_{j=1}^s$ is an arrangement of subspace in J^β and $\mathbb{P}\mathcal{J}^\beta = \{\mathbb{P}J^{\beta+\mathbf{1}_j}\}_{j=1}^s$ is an arrangement of some projective subspaces in $\mathbb{P}J^\beta$. We then have intersection semi-lattices $L(\mathcal{J}^\beta)$ and $L(\mathbb{P}\mathcal{J}^\beta)$. One can define the q-series

$$\begin{aligned}
S_C(q;\mathbf{u}) &= \sum_{\beta\in\mathbb{Z}^s} q^{-\dim\mathcal{O}_{\mathbb{C}^2,0}/J^\beta}\mathbf{u}^\beta,\\
S_C^0(q;\mathbf{u}) &= \sum_{\beta\in\mathbb{N}^s}\mathbf{u}^\beta\sum_{J\in L(\mathcal{J}^\beta)}\mu(J,J^\beta)[J],\\
PS_C(q;\mathbf{u}) &= \sum_{\beta\in\mathbb{Z}^s}[\mathbb{P}(\mathcal{O}_{\mathbb{C}^2,0}/J^\beta)]\mathbf{u}^\beta,\\
PS_C^0(q;\mathbf{u}) &= \sum_{\beta\in\mathbb{N}^s}\mathbf{u}^\beta\sum_{\mathbb{P}J\in L(\mathbb{P}\mathcal{J}^\beta)}\mu(\mathbb{P}J,\mathbb{P}J^\beta)[\mathbb{P}J].
\end{aligned}$$

For each $\beta = (b_1, \dots, b_s) \in \mathbb{Z}^s$, there is a linear map $\ell'_\beta : J^\beta \to \mathbb{C}$ by $\ell'_\beta(g) = (c_1, \dots, c_s)$, where c_j $(1 \le j \le s)$ are given as follows: Choose a coordinate (x, y) in X such that $E_j = \{x = 0\}$; the lifting $\tilde{g} = g \circ \pi$ can be written as

$$\tilde{g} = c_j x^{b_j} y^{d_j} + \text{higher order terms of } x,\ c_j \in \mathbb{C}.$$

Let H_j be the hyperplane of $\mathbb{C}^s$ whose jth coordinate is zero. Then the collection $\mathcal{H}_\beta = \{\operatorname{Im} \ell'_\beta \cap H_j\}_{j=1}^s$ is a hyperplane arrangement of $\operatorname{Im} \ell'_\beta$. One can similarly define the q-series:

$$\begin{aligned}
R_C(q; \mathbf{u}) &= \sum_{\beta \in \mathbb{Z}^s} q^{\dim \operatorname{Im} \ell'_\beta} \mathbf{u}^\beta, \\
R_C^0(q; \mathbf{u}) &= \sum_{\beta \in \mathbb{N}^s} \chi(\mathcal{H}_\beta, q) \mathbf{u}^\beta, \\
PR_C(q; \mathbf{u}) &= \sum_{\beta \in \mathbb{Z}^s} (1 + q + \cdots + q^{\dim \mathbb{P}\operatorname{Im} \ell'_\beta}) \mathbf{u}^\beta, \\
PR_C^0(q; \mathbf{u}) &= \sum_{\beta \in \mathbb{N}^s} \chi(\mathbb{P}\mathcal{H}_\beta, q) \mathbf{u}^\beta.
\end{aligned}$$

The following theorem is similar to Proposition 3.1, Theorems 3.3 and 3.4, except for the rationality; the proof are parallel and is omitted here.

Theorem 4.1 *For $Q = R_C, S_C, R_C^0, S_C^0$, $PQ(q; \mathbf{u}) = Q(q; \mathbf{u})/(q - 1)$. Moreover,*

$$\begin{aligned}
S_C^0(q; \mathbf{u}) &= \int_{\mathcal{O}_{\mathbb{C}^2,0}} \mathbf{u}^{\lambda(g)} \mathrm{d}\mu_1(g) = \frac{S_C(q; \mathbf{u}) \prod_{j=1}^s (u_j - 1)}{u_1 \cdots u_s}, \\
PS_C^0(q; \mathbf{u}) &= \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} \mathbf{u}^{\lambda(g)} \mathrm{d}\tilde{\mu}_1(g) = \frac{PS_C(q; \mathbf{u}) \prod_{j=1}^s (u_j - 1)}{u_1 \cdots u_s}, \\
R_C^0(q; \mathbf{u}) &= \int_{\mathcal{O}_{\mathbb{C}^2,0}} q^{\dim \mathcal{O}_{\mathbb{C}^2,0}/J^{\gamma(g)+\mathbf{1}}} \mathbf{u}^{\lambda(g)} \mathrm{d}\mu_1(g), \\
PR_C^0(q; \mathbf{u}) &= \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} q^{\dim \mathbb{P}(\mathcal{O}_{\mathbb{C}^2,0}/J^{\gamma(g)+\mathbf{1}})} \mathbf{u}^{\lambda(g)} \mathrm{d}\tilde{\mu}_1(g).
\end{aligned}$$

The rationality of the q-series can be similarly established. But it seems more interesting to integrade the function $\mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)}$ over $\mathcal{O}_{\mathbb{C}^2,0}$. To this end we need the following lemma which can be found in [6] and will be needed in proving Theorem 4.3.

Lemma 4.2 *For any element $v \in S(E^0)$, the inverse image $\phi^{-1}(v)$ is given by*

$$\phi^{-1}(v) = \operatorname{span}\{g, J^{\lambda(g)+\mathbf{1}}\} - J^{\lambda(g)+\mathbf{1}},$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^s$, g is any germ such that $\phi(g) = v$.

Proof. For germs $g_1, g_2 \in \mathcal{O}_{\mathbb{C}^2,0}$, it suffices to show that $\phi(g_1) = \phi(g_2)$ if and only if there exist $c \neq 0$ and $h \in \mathcal{O}_{\mathbb{C}^2,0}$ such that $g_2 = cg_1 + h$ and $m_{h,j} > m_{g_1,j} (= m_{g_2,j})$, $1 \le j \le s$. Consider the liftings $\tilde{g}_1 = g_1 \circ \pi$, $\tilde{g}_2 = g_2 \circ \pi$, and the meromorphic function $\tilde{g}_2/\tilde{g}_1$ on X. If $\phi(g_1) = \phi(g_2)$, the zeros and poles of $\tilde{g}_2/\tilde{g}_1$ cancel each other on the exceptional divisor E. Then $\tilde{g}_2/\tilde{g}_1$ is a nonzero constant on E, say, $\tilde{g}_2/\tilde{g}_1|_E = c \in \mathbb{C}^*$. Set $h = g_2 - cg_1$; we have $m_{g_1,j} = m_{g_2,j} < m_{h,j}$ for all $1 \le j \le s$. Conversely, if $g_2 = cg_1 + h$ and $m_{h,j} > m_{g_1,j}$

for all $1 \le j \le s$, $c \in \mathbb{C}^*$. Then $\tilde{g}_2/\tilde{g}_1 = c$ on the exceptional divisor E. It follows that the zeros and poles of $\tilde{g}_2/\tilde{g}_1$ cancel each other on E. This means that $\phi(g_1) = \phi(g_2)$. □

Let us consider the embedding $\psi : \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0} \to S(E) \times \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ by $\psi(g) = (\phi(g), g)$; this is well-defined because $\phi(g) = \phi(cg)$ for $c \in \mathbb{C}^*$. Note that $E_j \simeq \mathbb{CP}^1$ and E_j^0 is a punctured complex plane; hence $(E_j^0)^k$ is isomorphic to a linear set in $\mathbb{C}^k$ for any $k \ge 0$. The linear set $(E_j^0)^k$ can be divided into smaller linear sets so that $S^k(E_j^0)$ is still a linear set after identification. Thus $\prod_{j=1}^s S^{k_j}(E_j^0)$ is a linear set for any nonnegative integers k_j. By a *linear set* of $S(E_j^0)$ we mean a finite union of some linear sets in $\prod_{j=1}^s S^{k_j}(E_j^0)$ for various $k_j \ge 0$. Projective linear sets can be similarly constructed on $S(E)$. Let ν be the valuation on $S(E)$, defined on the class of projective linear sets in $S(E)$ such that $\nu(Y) = [Y]$ if $Y \subset S(E^0)$ and $\nu(Y) = 0$ if $Y \subset S(E^0) \times \bigcup_{k \ge 1} S^k(D)$. Let χ be the valuation on $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$, defined on the Boolean algebra $B(\mathbb{P}\Omega)$ of projective subspaces by Corollary 2.6; that is, $\chi(\mathbb{P}W - \mathbb{P}V) = \dim W/V$ for projective subspaces $\mathbb{P}V$ and $\mathbb{P}W$ satisfying $\mathbb{P}V \subset \mathbb{P}W$ and $\dim W/V < \infty$. We then have a product valuation $\nu \times \chi$ on $S(E) \times \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$. The pullback $\tilde{\nu} = \psi^*(\nu \times \chi)$ is a valuation on $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$.

Theorem 4.3 *The pullback $\tilde{\nu}$ of the product valuation $\nu \times \chi$ is a valuation on a sub-class of projective linear sets in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$. Write $\mathbf{m}_j = (m_{1j}, \ldots, m_{rj}) \in \mathbb{Z}^r$, $1 \le j \le s$. Then*

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)} \mathrm{d}\tilde{\nu}(g) \;=\; \prod_{j=1}^{s} (1 - \mathbf{t}^{\mathbf{m}_j} u_j)^{-[E_j^0]}. \tag{50}$$

In particular, set $q = 1$; the formula (50) reduces to

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)} \mathrm{d}\chi(g) \;=\; \prod_{j=1}^{s} (1 - \mathbf{t}^{\mathbf{m}_j} u_j)^{-\chi(E_j^0)}. \tag{51}$$

Proof. The map ϕ is measurable because for any $\beta = (b_1, \ldots, b_s)$ the inverse image $\phi^{-1}(\prod_{j=1}^s S^{b_j}(E_j^0))$ is the linear set J_0^β in $\mathcal{O}_{\mathbb{C}^2,0}$. The function $\mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)}$ is obviously a measurable function from $\mathcal{O}_{\mathbb{C}^2,0}$ to the ring $\mathbb{Z}[q, q^{-1}][[\mathbf{t}; \mathbf{u}]]$. Consider the trivial vector bundle $S(E^0) \times \mathcal{O}_{\mathbb{C}^2,0}$ over $S(E^0)$ and its sub-bundle $E_C = \bigcup_{v \in S(E^0)} \mathrm{span}\,\{g, J^{\lambda(v)+\mathbf{1}}\}$ and sub-bundle $E_C^\beta = \bigcup_{v \in S(E^0)} J^{\lambda(v)+\beta}$, where $\lambda(v) = \lambda(g)$ for any germ g such that $\phi(g) = v$, $\mathbf{1} = (1, \ldots, 1), \beta \in \mathbb{Z}^s$. Lemma 4.2 implies that the germ space $\mathcal{O}_{\mathbb{C}^2,0}$, being a fiber bundle over $S(E^0)$, can be identified to the difference $E_C - V_E^{\mathbf{1}}$ of sub-bundles by the map ϕ. The integration of these bundle differences with respect to $\nu \times \chi$ is a q-series. In fact, the difference $E_C - E_C^\beta$ can be identified with the quotient bundle E_C/E_C^β, which can be further identified by translation to a finite dimensional vector bundle over $S(E^0)$. Now the valuation on the fiber is the ordinary Euler characteristic χ, which alway has value 1. Then

$$\begin{aligned} \int_{E_C - E_C^\beta} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)} \mathrm{d}(\nu \times \chi)(g) &= \int_{E_C - E_C^{\mathbf{1}}} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)} + \int_{E_C^{\mathbf{1}} - E_C^\beta} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)} \\ &= \int_{E_C/E_C^{\mathbf{1}}} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)} + \int_{E_C^{\mathbf{1}}/E_C^\beta} \mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)}. \end{aligned}$$

In particular, the quotient fiber $\mathrm{span}\,\{g, J^{\lambda(g)+\mathbf{1}}\}/J^{\lambda(g)+\mathbf{1}}$ in $E_C/E_C^{\mathbf{1}}$ over the point $\phi(g)$ is one-dimensional. However, the projectivization $\mathbb{P}(\mathrm{span}\,\{g, J^{\lambda(g)+\mathbf{1}}\} - J^{\lambda(g)+\mathbf{1}})$ is isomorphic to the affine space $g + J^{\lambda(g)+\mathbf{1}}$, which can be identified with $J^{\lambda(g)+\mathbf{1}}$ by translation.

Let $k_j(g)=\sum_{p\in E_j^0}\phi(g)(p)$. Then $\omega(g)=\sum_{j=1}^s k_j(g)\mathbf{m}_j$. We then have

$$\begin{aligned}\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}}\mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)}\mathrm{d}\tilde{\nu}(g) &= \int_{E_C/E_C^1}\mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)}\mathrm{d}(\nu\times\chi)\\ &= \sum_{k_1,\dots,k_s\ge 0}\int_{\prod_{j=1}^s S^{k_j}(E_j^0)}\mathbf{t}^{\sum_{j=1}^s k_j\mathbf{m}_j}\prod_{j=1}^s u_j^{k_j}\mathrm{d}\nu\\ &= \sum_{k_1,\dots,k_s\ge 0}\prod_{j=1}^s[S^{k_j}(E_j^0)]\mathbf{t}^{k_j\mathbf{m}_j}u_j^{k_j}\\ &= \prod_{j=1}^s(1-\mathbf{t}^{\mathbf{m}_j}u_j)^{-[E_j^0]}.\end{aligned}$$

Take $q=1$; the valuation ν becomes the Euler characteristic χ. Fix a vector $\alpha=(a_1,\dots,a_r)\in\mathbb{N}^r$; let $\mathcal{O}_{\mathbb{C}^2,0}(\alpha)$ be the set of germs g such that $\omega(g)\le\alpha$ and let

$$S(E^0,\alpha)=\bigsqcup_{k_1\mathbf{m}_1+\cdots+k_s\mathbf{m}_s\le\alpha}\prod_{j=1}^s S^{k_j}(E_j^0).$$

Clearly, $\phi(\mathcal{O}_{\mathbb{C}^2,0}(\alpha))=S(E^0,\alpha)$. Let $k\ge 1+\max\{a_1,\dots,a_r\}$ and let π_k be the projection from $\mathcal{O}_{\mathbb{C}^2,0}$ to the jet space $J^k_{\mathbb{C}^2,0}$. Then $\pi_k(\mathcal{O}_{\mathbb{C}^2,0}(\alpha))$ is a linear set of the finite dimensional vector space $J^k_{\mathbb{C}^2,0}$, and is also a fiber bundle over $S(E^0,\alpha)$ whose fiber is a difference $W-V$ of vector subspaces where V is codimension one in W. It follows that $\mathbb{P}\pi_k(\mathcal{O}_{\mathbb{C}^2,0}(\alpha))$ is a fiber bundle over $S(E^0,\alpha)$ whose fiber is $\mathbb{P}(W-V)$. The fiber can be identified as a coset of V and the coset can be identified as the vector space V as in Section 2. So $\mathbb{P}\pi_k(\mathcal{O}_{\mathbb{C}^2,0}(\alpha))$ can be identified as a vector bundle over $S(E^0)$. Now the Euler characteristic on $\mathcal{O}_{\mathbb{C}^2,0}(\alpha)$ induces the Euler characteristic on $\pi_k(\mathcal{O}_{\mathbb{C}^2,0}(\alpha))$, and further to the Euler characteristic on $\mathbb{P}\pi_k(\mathcal{O}_{\mathbb{C}^2,0}(\alpha))$, which can be split into integration over the base space and the fiber space, respectively (here we actually treated the constructible sets as real semi-algebraic sets and the Euler characteristic for such sets are well treated in [**?**, 7]). Therefore

$$\begin{aligned}\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}}\mathbf{t}^{\omega(g)}\mathbf{u}^{\lambda(g)}\mathrm{d}\chi(g) &= \lim_{\alpha\to\infty}\int_{\mathbb{P}\pi_k\mathcal{O}_{\mathbb{C}^2,0}(\alpha)}\mathbf{t}^{\omega(\pi_k(g))}\mathbf{u}^{\lambda(\pi_k(g))}\mathrm{d}\chi(\pi_k(g))\\ &= \lim_{\alpha\to\infty}\sum_{k_1\mathbf{m}_1+\cdots+k_s\mathbf{m}_s\le\alpha}\chi\left(\prod_{j=1}^s S^{k_j}(E_j^0)\right)\mathbf{t}^{k_j\mathbf{m}_j}u_j^{k_j}\\ &= \sum_{k_1,\dots,k_s=0}^{\infty}\prod_{j=1}^s\chi(S^{k_j}E_j^0)\mathbf{t}^{k_j\mathbf{m}_j}u_j^{k_j}\\ &= \prod_{j=1}^s(1-\mathbf{t}^{\mathbf{m}_j}u_j)^{-\chi(E_j^0)}.\end{aligned}$$

Please note that all the bundles mentioned above are not ordinary bundles; their fibers have no constant dimensions. □

5 q-Analog of Zeta-Functions

In this section we consider the case where there is only one parameterization φ_1, that is, $r=1$. We write $\varphi=\varphi_1$, $d=d_1$, and the semigroup S_Φ is denoted by S_φ. Let $C=\operatorname{Im}\varphi$

be given by a germ f; that is, $C = \{f = 0\}$. The local coordinate ring of C at 0 is $\mathcal{O}_{C,0} = \mathcal{O}_{\mathbb{C}^n,0}/(f)$ and there is a canonical linear map $\varphi^* : \mathcal{O}_{C,0} \to \mathcal{O}_{\mathbb{C},0}$. Let $\mathfrak{m}$ be the maximal ideal in $\mathcal{O}_{\mathbb{C},0}$. The filtration $\mathcal{O}_{\mathbb{C},0} = \mathfrak{m}^0 \supset \mathfrak{m}^1 \supset \mathfrak{m}^2 \supset \cdots$ induces a filtration $\mathcal{O}_{C,0} = I^0 \supset I^d \supset I^{2d} \supset \cdots$, where $I^{kd} = (\varphi^*)^{-1}(\mathfrak{m}^k)$. The Poincaré series is defined by

$$P_\varphi(t) = \sum_{k=0}^{\infty} \dim \left(I^{kd}/I^{(k+1)d} \right) t^{kd}.$$

Note that an integer $kd \in S_\varphi$ if and only if $\dim(I^{kd}/I^{(k+1)d}) = 1$; and that S_φ contains any large enough integers in $d\mathbb{N}$. The series $P_\varphi(t)$ is a rational function of t with denominator $1 - t^d$, and

$$P_\varphi(t) = \sum_{n \in S_\varphi} t^n.$$

Since $\operatorname{Im} \ell_k = \{0\}$ for $k < 0$ and $\operatorname{Im} \ell_0 = \mathbb{C}$, then we have

$$\begin{aligned} L_\varphi(q;t) &= \sum_{k \in d\mathbb{Z}} q^{\dim \operatorname{Im} \ell_k} t^k, \\ L^0_\varphi(q;t) &= \sum_{k \in d\mathbb{N}} (q^{\dim \operatorname{Im} \ell_k} - 1) t^k \\ &= \sum_{k \in S_\varphi} (q-1) t^k. \end{aligned}$$

Then

$$PL_\varphi(q;t) = \sum_{k \in d\mathbb{N}, \ell_k \neq 0} t^k = \sum_{k \in S_\varphi} t^k = PL^0_\varphi(q;t).$$

Thus $L_\varphi(q;t) = L^0_\varphi(q;t) = (q-1)PL\varphi(q;t)$; $PL\varphi(q;t)$ is independent of q and is the same as the Poincaré series $P_\varphi(t)$.

Let $\varphi : \mathbb{C} \to \mathbb{C}^2$ be a uniformization of an irreducible plane curve singularity $C = \{f = 0\}$, passing through the origin. Let V_f be the Milnor fiber of the singularity f; that is, $V_f = \{z \in \mathbb{C}^2 : ||z|| \leq \varepsilon, f(z) = \delta\}$, $0 < \delta \ll \varepsilon$. Let $h_f : V_f \to V_f$ be the classical monodromy transformation. The *zeta-function* is defined by

$$\zeta_f(t) = \prod_{k=0}^{\infty} \left[\det \left(\mathrm{id} - t \cdot h_*|_{H_k(V_f;\mathbb{R})} \right) \right]^{(-1)^{k+1}}.$$

Let $\pi : (X, E) \to (\mathbb{C}^2, 0)$ be the embedded resolution of $(C, 0)$ with the exceptional divisor $E = \bigcup_{j=1}^s E_j$, where E_j are the components of E. Let D_j be the set of intersection points of E_j and other components in the total transform of $(f \circ \pi)^{-1}(0)$; let E^0_j be the complement of D_j in E_j, see the previous section. It is known from [1] that $\zeta_f(t) = \prod_{j=1}^s (1 - t^{m_j})^{-\chi(E^0_j)}$, where m_j is the multiplicity of the lifting $f \circ \pi$ along the exceptional divisor E_j in the embedded resolution X of C. This is generalized by Campillo, Delgado and Gusein-Zade in [5] to define the zeta-function for a reducible curve C by

$$\zeta_C(t) = \sum_{\alpha \in S_C} \chi(\mathbb{P}\mathcal{I}^\alpha, 1) t^{||\alpha||},$$

where $\alpha = (a_1, \ldots, a_r)$ and $||\alpha|| = a_1 + \cdots + a_r$. Then

$$\zeta_C(t) = \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} t^{||\omega(g)||} \mathrm{d}\chi(g) = \Delta^{L_r}(t, \ldots, t).$$

We define the following *q-analog zeta-function* for any parameterization Φ by

$$\zeta_\Phi(q;t) = \sum_{\alpha\in S_C} \chi(\mathbb{P}\mathcal{I}^\alpha, q)t^{||\alpha||}.$$

It is clear that $\zeta_\Phi(q;t) = PM^0_\Phi(q;t,\dots,t)$. If Φ is a uniformization and C is a plane curve singularity, then $\zeta_\Phi(1;t) = \zeta_C(t)$. Another q-analog zeta-function can be defined by

$$Z_C(q;t) = \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} t^{||\omega(g)||}\mathrm{d}\tilde{\nu}(g).$$

Proposition 5.1 *Let $C = \bigcup_{i=1}^r C_i$ be a plane curve singularity passing through the origin. Let m_j be the multiplicity of $f = f_1\cdots f_r$ along the exceptional divisor E_j. Then*

$$\begin{aligned} Z_C(q;t) &= \prod_{j=1}^s (1-t^{m_j})^{-q+\chi(E_j^0)+1}, \\ \zeta_C(q;t) &= \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} t^{||\omega(g)||}\mathrm{d}\mu_1(g). \end{aligned}$$

In particular, $Z_C(1;t) = \zeta_C(1;t) = \zeta_C(t) = \Delta^{L_r}(t,\dots,t)$. For $r=1$,

$$\zeta_C(q;t) = q^{-1}\sum_{k\in S_\varphi} q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^k} t^k.$$

Proof. The first two formulas are straightforward consequences of Theorem 4.3. As for the last formula, we have $I^k = \mathcal{O}_{\mathbb{C}^n,0}$ for $k \le 0$ and I^k/I^{k+d} has dimension one if and only if $k \in S_\varphi$; that is, $\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{k+d} = 1 + \dim \mathcal{O}_{\mathbb{C}^n,0}/I^k$ for $k\in S_\varphi$.

$$\begin{aligned} M_\varphi(q;t) &= \sum_{k\in d\mathbb{Z}} q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^k} t^k, \\ M^0_\varphi(q;t) &= \sum_{k\in d\mathbb{Z}} \left(q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^k} - q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{k+d}}\right) t^k \\ &= \sum_{k\in d\mathbb{N}} \left(q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^k} - q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{k+d}}\right) t^k \\ &= \sum_{k\in S_\varphi} (q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^k} - q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^{k+d}}) t^k \\ &= M_\varphi(q;t)(t^d-1)/t^d, \\ PM^0_\varphi(q;t) &= q^{-1}\sum_{k\in S_\varphi} q^{-\dim \mathcal{O}_{\mathbb{C}^n,0}/I^k} t^k = \zeta_C(q,t). \end{aligned}$$

□

One can similarly define the q-analog zeta-function for a plane curve singularity at ∞. The germ to define the curve C is a polynomial and the space $\mathcal{O}_{\mathbb{C}^n,\infty}$ of germs at infinity is just the polynomial ring $\mathbb{C}[z_1,z_2]$. The detailed technical setting may be completed by mimicking the treatment in [5].

Remark: For an arbitrary link we mention the following observation. Let $L = \{C_i\}_{i=1}^r$ be a collection of r disjoint real curves in $\mathbb{R}^3$, each parameterized by some smooth functions $f_i : [-\pi,\pi] \to \mathbb{R}^3$, $f_i(-\pi) = f_i(\pi)$. Write f_i as the Fourier series

$$f_{ij}(\theta) = \sum_{n=-\infty}^{\infty} c_n(i,j)e^{\sqrt{-1}n\theta}, \qquad i=1,\dots,r;\ j=1,2,3.$$

We complexify the functions f_i by changing f_i into complex functions $F_i : \mathbb{C} \to \mathbb{C}^3$, where $F_i = (F_{i1}, F_{i2}, F_{i3})$ and $F_{ij}(z) = \sum_{n=-\infty}^{\infty} c_n(i,j) z^n$. Write $F_{ij} = F_{ij}^+ + F_{ij}^-$, where

$$F_{ij}^+ = \frac{c_0}{2} + \sum_{n=0}^{\infty} c_n(i,j) z^n \quad \text{and} \quad F_{ij}^- = \frac{c_0}{2} + \sum_{n=0}^{\infty} c_{-n}(i,j) z^n.$$

We then have the curve parameterizations $\Phi^+ = \{F_i^+\}_{i=1}^r$ and $\Phi^- = \{F_i^-\}_{i=1}^r$. Thus one can define Laurent polynomials $\Delta_\Phi(q; \mathbf{t}) = PL^0_{\Phi_+}(q; \mathbf{t}) + PL^0_{\Phi_-}(q; 1/\mathbf{t})$. The Laurent polynomials $\Delta_\Phi(1; \mathbf{t})$ may be related to the Alexander polynomial $\Delta^L(\mathbf{t})$ of the link L.

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