# LETTERS

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# Computation of the Loitsianski integral in decaying isotropic turbulence

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The time evolution of the Loitsianski integral at high-Reynolds numbers is determined by computing an ensemble average of a large number of independent large-eddy simulations of decaying isotropic turbulence. It is found that the Loitsianski integral becomes proportional to $t^\gamma$ at large times and that $\gamma \approx 0.25$. The present simulations illustrate the efficient use of massively parallel computers for simulating large ensembles of turbulent flows.

It was originally thought[1] that the integral

$$B_2 = -\frac{1}{48\pi^3} \lim_{V\to\infty} \int_V \langle u_i(\mathbf{x}) u_i(\mathbf{x}+\mathbf{r})\rangle r^2 \, d\mathbf{r} \qquad (1)$$

was invariant during the decay of an isotropic turbulence, and that this invariance was a consequence of the more general law of conservation of angular momentum.[2] However, it was subsequently demonstrated that $B_2$ is in fact not invariant,[3,4] and under certain conditions of turbulence generation may diverge.[5]

When $B_2$ is finite it can be shown to be directly related to the form of the energy spectrum $E(k)$ at low wavenumber magnitudes.[6,7] An asymptotic expansion of the trace of the spectrum tensor $\Phi_{ii}(\mathbf{k},t)$ in isotropic turbulence near $k=0$ yields to leading order $\Phi_{ii}(\mathbf{k},t) \sim B_2 k^2$ so that the spherically integrated energy spectrum near $k=0$ follows

$$E(k,t) \sim 2\pi B_2 k^4. \qquad (2)$$

Studies of decaying isotropic turbulence using the eddy-damped quasinormal Markovian (EDQNM) approximation[8,9] shows that $B_2 = B_2(t)$ is a monotonically increasing function of time and in the limit of long times and high-Reynolds numbers,

$$B_2(t) = \beta t^\gamma. \qquad (3)$$

The EDQNM results indicate that an asymptotic similarity state develops during the turbulence decay which depends only on the value of this new "invariant" $\beta$, which arises from the nonlinear dynamics of the turbulence. Although the precise value of the exponent $\gamma$ depends on the choice of free parameters within the EDQNM model, it was estimated to be approximately $\gamma = 0.16$. Further support for the existence of this similarity state was obtained from recent large-eddy numerical simulations.[10]

The coefficient $B_2(t)$ defined by (1) can be computed directly by means of a numerical simulation. In the usual way, we assume that the velocity field is periodic in three directions with periodicity length $L = 2\pi$. The velocity field may then be expanded in a Fourier series as

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}), \qquad (4)$$

where the components of $k$ in the sum span the set of integers. A good approximation to homogeneous turbulence is thus obtained when the integral scale of the turbulence is much less than $\pi$. Under this condition, the coefficient $B_2$ given by (1) can be computed accurately within the finite volume $V = (2\pi)^3$ of the periodic domain. Treating the average in (1) as a volume average, and substituting the Fourier expansion (4) into (1), we obtain after one integration over the volume

$$B_2 = -\frac{1}{48\pi^3} \sum_{\mathbf{k}} \hat{u}_i(\mathbf{k}) \hat{u}_i(-\mathbf{k}) \int_V \exp(i\mathbf{k}\cdot\mathbf{r}) r^2 \, d\mathbf{r}. \qquad (5)$$

The remaining volume integral in (5) may be evaluated analytically, and making use of $\hat{u}_i(-\mathbf{k}) = \hat{u}_i(\mathbf{k})^*$, where $*$ denotes the complex conjugate, and $\hat{u}_i(0,0,0) = 0$ we obtain

$$B_2 = -\frac{2}{3} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} [|\hat{\mathbf{u}}(k,0,0)|^2 + |\hat{\mathbf{u}}(0,k,0)|^2 + |\hat{\mathbf{u}}(0,0,k)|^2]. \qquad (6)$$

There are two main difficulties in the direct use of (6) to compute $B_2$ in a numerical simulation. First, the correlation $\langle u_i(\mathbf{x}) u_i(\mathbf{x}+\mathbf{r})\rangle$ in (1) decreases in general as $O(r^{-5})$ in homogeneous turbulence[4]—although it decreases faster as $o(r^{-6})$ in an isotropic turbulence—so that the integral scale of the turbulence must be small enough for the integral in (1) to converge within the computational domain. Second, as the value of $r$ in (1) becomes comparable to $\pi$, the replacement of the ensemble average in (1) by a volume average becomes inaccurate because of a lack of sample of the largest computed scales. Explicit

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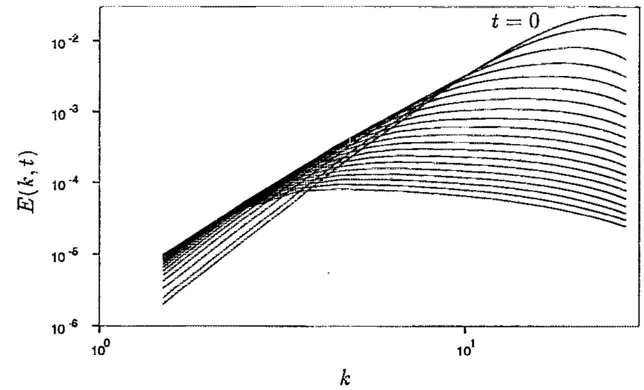


FIG. 1. Time evolution of the energy spectrum.

computation has shown that direct use of (6) to compute $B_2$ in a single realization of a turbulent flow is highly inaccurate. We are thus led to average $B_2$ over an ensemble of such flows. This is equivalent to treating the original average in (1) as a combination of a volume and ensemble average.

In this Letter, we report on a computation of $B_2(t)$ accomplished by performing 1024 independent simulations of resolution $64^3$. The size of this ensemble is sufficient to compute $B_2(t)$ to a statistical accuracy better than 5% over the entire time-evolution considered. The computations are performed on an Intel iPSC/860 hypercube machine containing 128 processors. The machine had 8 megabytes RAM per processor which allowed 64 realizations to be performed in parallel with each independent realization computed on 2 processors. Communication between processors computing different realizations is minimal so that the simulation of an ensemble of turbulent flows makes very efficient use of parallel computer architectures. Sixteen independent runs — each of 800 total time steps — were performed. With each time step taking approximately 10.6 sec of CPU time, a total of 38 h of dedicated machine use was required.

Our main goal in computing $B_2(t)$ is to determine its long time, high-Reynolds number behavior. Under the constraints imposed by $64^3$ resolution simulations, this necessitates the use of a large-eddy simulation with the initial peak of the energy spectrum placed at as large a value of $k$

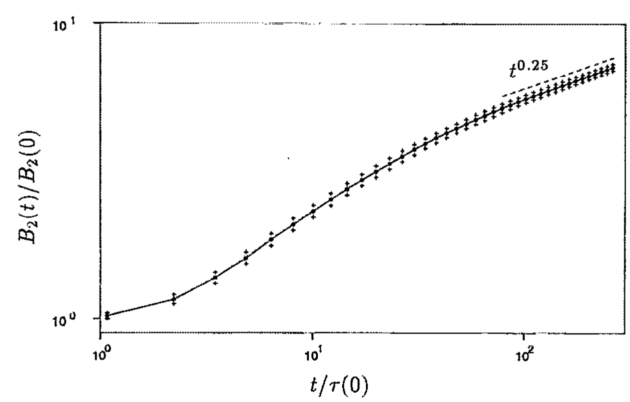


FIG. 2. Time evolution of the Loitsianski integral.

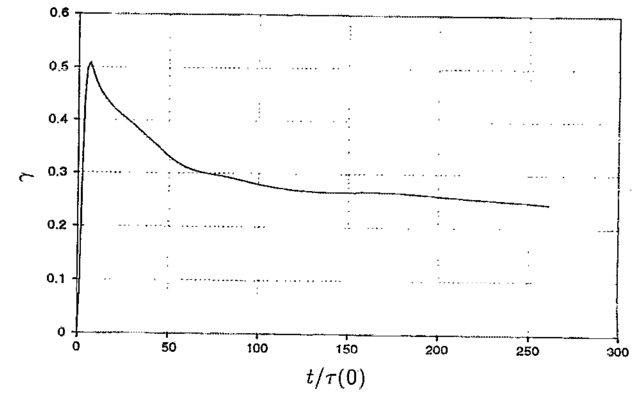


FIG. 3. Time evolution of the logarithmic derivative of the Loitsianski integral.

magnitude as possible.[10] Here, the initial energy spectrum is taken to be

$$E(k,0)=2\pi B_2(0)k^4 \exp\left[-2(k/k_p)^2\right], \qquad (7)$$

with $k_p=25$ and $B_2(0)=6.934\times 10^{-8}$, so that $\langle \mathbf{u}^2\rangle=1$. As we have done previously, an eddy-viscosity subgrid scale model[11,12] is used to model the unresolved small-scale turbulence. Although the inclusion of a stochastic backscatter term in the subgrid model[13] can directly affect the time variation of $B_2$, this effect is negligible at the later times of the turbulence evolution of interest to us here.

The finite resolution of the simulation results in a spherical truncation of the Fourier series in (4) at $k_m$, the maximum wave number of the simulation, so that the sum to $\infty$ in (6) is replaced by a sum to $k_m/\sqrt{3}$. At small times when the peak of the energy spectrum is near $k_m$, this sharp cutoff results in errors in the computed value of $B_2$. We have shown that these errors can be easily removed by applying an additional Gaussian filter of the form $\exp[-(k/k_f)^2]$ with $k_f=12$ to $\hat{\mathbf{u}}(\mathbf{k})$ before computing (6). At the later evolution times of interest to us here, the effect of this additional filter is negligible.

The results obtained from the simulations are shown in Figs. 1–3. In Fig. 1, we plot the time evolution of the ensemble-averaged energy spectrum obtained from the large-eddy simulations by summing the contributions of $|\hat{\mathbf{u}}(\mathbf{k})|^2$ into wave-number shells of thickness $\Delta k=1$ in the usual way, i.e.,

$$E(k,t)=\frac{2\pi k^2}{S_k}\sum_{k-(1/2)<|\mathbf{q}|<k+(1/2)} \hat{u}_i(\mathbf{q},t)\hat{u}_i(-\mathbf{q},t),$$

where $S_k$ is the number of Fourier modes in each wave-number shell and $k=1.5,2.5,\ldots,29.5$. A good approximation to the homogeneous turbulence energy spectrum is this obtained at high wave numbers, while the approximation is less accurate at low wave numbers. Nevertheless, the increase in time of the low wave number $k^4$ coefficient is clearly evident from the plot.

The coefficient $B_2(t)/B_2(0)$ versus time, in units of the initial large-eddy turnover time $\tau(0)$, where $\tau(0)=1.38/[k_p^7 B_2(0)]^{1/2}$, is plotted in Fig. 2. The points represent the statistical mean of the ensemble while the pluses represent one standard deviation from the mean. The standard deviation of the distribution of $B_2$ itself, which we have shown from the simulation data to be approximately Gaussian, varies somewhat over the course of the simulation but at the last time plotted is 80% of the mean. With 1024 realizations, the statistical uncertainty of the mean at the latest time is 2.5%.

In Fig. 3, we plot the logarithmic derivative of $B_2$, with respect to time in order to determine the validity of (3) and to compute a value of $\gamma$ from the simulation. In agreement with the EDQNM model,[8,9] we find that $B_2(t)$ follows an approximate power law at large times. From Fig. 3, we estimate the power law exponent to be $\gamma \approx 0.25$, with a statistical uncertainty of 6% at the latest time. The straight line drawn on the log–log plot of Fig. 2 represents this result. The value of $\gamma$ we obtain from the simulation is about 50% larger than that estimated previously.[8,9]

The statistical uncertainty of our asymptotic result for $\gamma$ can be reduced further by computing additional realizations. However, there may be other errors in our result associated with the deviation of "periodic turbulence" from homogeneous turbulence at the latest times of evolution, as well as the expected slow approach of the turbulence to asymptotics.[10] The evident trend of Fig. 3 is towards a somewhat smaller asymptotic value for $\gamma$ than we have estimated. It would be of interest to repeat the present computation at higher resolution with a larger ensemble after parallel machines have become substantially more powerful.

We also note here another approach to the current computation. Rather than simulate 1024 $64^3$ turbulent fields, we could have simulated 16 $256^3$ fields with slightly more computer time due to the need for interprocessor communications. To obtain similar statistics between these two simulations, we would have to increase the initial peak of the energy spectrum $k_p$ by a factor of 4 and truncate the volume integration in (5) to 1/64 the volume of the entire periodic domain. It is unclear which simulation would result in a more accurate computation of $B_2(t)$, but we chose the former mainly to illustrate the efficiency of performing realization averages of turbulent flows on parallel machines.


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