Simulation of the Kolmogorov inertial subrange using an improved subgrid model

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A subgrid model is developed and applied to a large-eddy simulation of the Kolmogorov inertial subrange. Currently popular subgrid models are derived from models of the turbulent energy equation, resulting in a significant loss of information as a consequence of the statistical averaging performed in going from the Navier–Stokes equation to the energy equation. The subgrid model developed here is based directly on a model of the Navier–Stokes equation. The improved subgrid model contains two terms: an eddy viscosity and a stochastic force. These terms are computed from the EDQNM stochastic model representation of the momentum equation, and from a fully resolved direct numerical simulation. Use of the subgrid model in a forced large-eddy simulation results in an energy spectrum that exhibits a clear $k^{-5/3}$ power-law subrange with an approximate value Ko = 2.1 of the Kolmogorov constant.

I. INTRODUCTION

Exact numerical simulation of turbulent flows is presently restricted to lower Reynolds numbers than are of interest to many researchers. Since higher Reynolds number flows contain a wider range of interacting scales than can be computationally resolved, to simulate these flows it is necessary to model the effect of the small unresolved scales on the large resolved scales. Such a model is called a subgrid model, and the resulting numerical simulation is referred to as a large-eddy simulation (LES).[1] The philosophy behind an LES is that the large-scale motions of turbulence are the scales most directly affected by the source of instability, vary from flow to flow, and must be simulated explicitly. On the other hand, the small scales of turbulence contain features that are more universal in nature, and their effect on the large scales may be represented by comparatively fewer parameters.

In fact, there is considerable experimental evidence that many different types of instabilities result in the generation of small-scale turbulence that is common to all highly turbulent flows. The most prominent characteristic of these universal small scales is their inertial subrange energy spectrum. This spectrum was first proposed by Kolmogorov,[2] who considered the inertial subrange of wave numbers to be defined by the following two conditions. *First*, there is effectively no energy input into the turbulent velocity fluctuations directly from the mechanism of instability (all of the energy input is to scales having wave numbers smaller than those considered), and; *second*, direct dissipation due to viscosity is negligible (it effectively acts on scales having wave numbers larger than those considered). In the inertial subrange, the energy which is supplied to the large scales progressively cascades to smaller and smaller scales having larger and larger wave numbers until dissipative effects become significant. The inertial subrange of wave numbers is assumed to be characterized only by ϵ, the flux of turbulent kinetic energy across wave number k, and k. Accordingly, the three-dimensional energy spectrum $E(k)$ in the inertial subrange must have the following universal form:

$$E(k) = \mathrm{Ko}\,\epsilon^{2/3} k^{-5/3}, \tag{1}$$

where Ko is the Kolmogorov constant. The integral of $E(k)$ over all wave numbers k yields the total turbulent kinetic energy of the fluid.

Beginning with the tidal channel experiment of Grant *et al.*[3] there is now ample experimental evidence for the existence of an inertial subrange in highly turbulent flows. As a consequence of the experimental evidence, as well as the early theortical argument advanced by Kolmogorov, one of the major goals of turbulence theorists and simulators has been the attainment of the inertial subrange spectrum, Eq. (1), as well as a determination of Ko. Some of the published values for Ko obtained from experimental data, numerical simulation data, and theoretical models are shown in Table I.

The methods used to simulate an inertial subrange are of four types. *First*, researchers have resolved all relevant scales of turbulent motion in a direct numerical simulation (DNS).[11] Such a simulation may be considered the most physical, but is currently limited to moderate Reynolds numbers. *Second*, researchers have artificially forced the large scales of the turbulence to raise the Reynolds number of the flow above that attainable in an unforced DNS.[9,12,14] If the arguments of Kolmogorov presented above are correct, a forcing concentrated at the lowest wave numbers of the simulation should only modify the inertial subrange form of $E(k)$ through the value of ϵ. *Third*, researchers have exploited symmetries which may be present in an initial flow field by artificially requiring that the turbulent flow remain symmetrical for all later times.[8,10] Such simulations substantially widen the range of scale sizes that can be computationally resolved, and hence makes possible the simulation of larger Reynolds numbers. And, *fourth*, researchers have used a subgrid model to represent the effect of unresolved dissipative scales on resolved inertial range scales. This method may be combined with an artificial force,[26] or applied to freely decaying turbulence.[13,27,28] In principle, the infinite Reynolds number limit may be approached by this procedure.

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TABLE I. Some of the published values for Ko.

Theoretical		
Kraichnan[4]	ALHDI	1.77
Herring and Kraichnan[5]	ALHDI	1.78
	SBALHDI	~2.0
Qian[6]	LFP	1.2
Yakhot and Orszag[7]	RNG	1.617
Computational		
Brachet *et al.*[8]	Symmetric DNS	4
Kerr[9]	Forced DNS	2.45
Kida and Murakami[10]	Symmetric DNS	1.6–1.9
Yamamoto and Hosokawa[11]	Unforced DNS	2.1
Panda *et al.*[12]	Forced DNS	1.6
Lesieur and Rogallo[13]	Unforced LES	1.5–1.8
Kerr[14]	Forced DNS	2
Experimental[a]		
Grant *et al.*[3]	Tidal channel	1.44 ± 0.06
M. M. Gibson[15]	Round jet	1.57, 1.62
C. H. Gibson and Schwartz[16]	Grid	1.34 ± 0.06
Pond *et al.*[17]	Atmosphere	1.50 ± 0.12
C. H. Gibson *et al.*[18]	Atmosphere	2.1
Paquin and Pond[19]	Atmosphere	1.74 ± 0.31
Shieh *et al.*[20]	Atmosphere	2.0
Wyngaard and Coté[21]	Atmosphere	1.59 ± 0.12
Boston and Burling[22]	Atmosphere	1.56 ± 0.06
Schedvin *et al.*[23]	Grid	1.47 ± 0.18
Williams and Paulson[24]	Atmosphere	1.65 ± 0.03
Champagne *et al.*[25]	Atmosphere	1.53 ± 0.06

[a] Translated from one-dimensional spectra.

The uncertainty in the value for Ko obtained from numerical simulations (see Table I) is a consequence of either an unreliable, or a too narrow inertial subrange. In this paper, we develop a subgrid model which, in combination with an artificial force, is used to simulate a clear $k^{-5/3}$ energy spectrum.

Currently popular subgrid models for homogeneous turbulence use a wave-number-dependent eddy viscosity to represent the effect of the subgrid scales on the numerically resolved scales.[27,29] The expression for the eddy viscosity is derived from an energy equation; however, in a numerical simulation, the eddy viscosity is used to solve a momentum equation. In contrast, it is possible to develop a subgrid model directly from a momentum equation.[30,31] We will show that the subgrid model thus developed provides two conceptual improvements over the standard eddy viscosity model. *First*, the energy transfer from the subgrid scales to the large scales is modeled as a random force acting on the large scales, whereas the drain of energy from the large scales to the subgrid scales is modeled as an eddy viscosity. In the eddy viscosity model, these two effects are not separated; instead, they are combined to construct a net eddy viscosity. *Second*, a new effect is modeled: the random sweeping of small scales by large scales. Since such a physical effect does not cause a net energy transfer, and thus is not present in an energy equation, it is omitted in the subgrid eddy viscosity.

In Sec. II we review the eddy viscosity model. We then rectify some of the deficiencies of this model by developing a subgrid model directly from the momentum equation representation of the eddy-damped quasinormal Markovian (EDQNM) model. In Sec. III, we show how the terms of this subgrid model may also be computed independently of the EDQNM model, using the results of a DNS. Such a computation is necessary to determine that the proposed model is consistent with the Navier–Stokes equation. Finally, in Sec. IV, the subgrid model, together with an artificial force, is applied to an LES of the inertial subrange.

II. ANALYTICAL DEVELOPMENT OF THE SUBGRID MODEL

A. Subgrid eddy viscosity

In a procedure developed by Kraichnan,[29] an effective eddy (turbulent) viscosity $\nu_t(k\,|k_m,t)$ acting at time t on scales of wave number k due to the effect of scales with wave numbers greater than k_m, is defined from the turbulence energy equation. In an LES in which not all of the relevant scales of motion are resolved, k_m is taken to be the wave number of the smallest scale that is resolved. The eddy viscosity $\nu_t(k\,|k_m,t)$ is defined as a part of the energy transfer $T(k,t)$ as follows. One can write the energy equation for isotropic turbulence as

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)E(k,t)=T(k,t), \tag{2}$$

where $E(k,t)$ is the kinetic energy spectrum, ν is the kinematic viscosity of the fluid, and $T(k,t)$ is the energy transfer. The energy transfer may be written as

$$T(k,t)=\int\int_\Delta dp\,dq\,S(k,p,q,t), \tag{3}$$

where the double integral is evaluated on the domain Δ of the p-q plane such that $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$ form a triangle (i.e., $\mathbf{k}=\mathbf{p}+\mathbf{q}$), $S(k,p,q,t)=S(k,q,p,t)$, and where $S(k,p,q,t)$ contains products of three velocity Fourier components. Kraichnan then defined an effective subgrid eddy viscosity acting on scales with wave number k by

$$\nu_t(k\,|k_m,t)=-T(k\,|k_m,t)/2k^2E(k,t), \tag{4}$$

where $T(k\,|k_m,t)$ is the part of the total energy transfer calculated by restricting the double integral in Eq. (3) to wave numbers p and q such that p or $q>k_m$. With this definition of $\nu_t(k\,|k_m,t)$, the energy equation, Eq. (2), may be rewritten for wave numbers $k\leqslant k_m$ as

$$\frac{\partial E(k,t)}{\partial t}+2[\nu+\nu_t(k\,|k_m,t)]k^2E(k,t)=T_R(k,t), \tag{5}$$

where $T_R(k,t)$ refers to the part of the integral in Eq. (3) where p and q are in the resolved scales, i.e., $p,q<k_m$. The usefulness of Eq. (5) is that $T_R(k,t)$ may be calculated explicitly using the numerically resolved scales.

Two methods have been successfully used to compute $\nu_t(k\,|k_m,t)$. *First*, one may use analytical models of turbulence.[27,29] For homogeneous turbulence the preferred models are of the two-point closure variety, since these models contain sufficient information about the energy transfer among different scales. *Second*, $\nu_t(k\,|k_m,t)$ has been computed using a DNS.[32] In this method, a fully resolved numerical simulation is performed for scales k such that $k_0\leqslant k\leqslant k_m$, where k_0 is the minimum, and k_m is the maximum wave number of the simulation. An artificial cut is then made at a wave number $k_c < k_m$. One further labels the scales k such that $k_c < k \leqslant k_m$ as fictitious "subgrid" scales, and the scales k such that $k_0 \leqslant k \leqslant k_c$ as "resolved" scales, and then proceeds to calculate the subgrid energy transfer and $\nu_t(k|k_c,t)$. However, this method is of limited use for the calculation of a subgrid eddy viscosity associated with a high Reynolds number flow, since a fully resolved numerical simulation can be performed only for low Reynolds numbers. To calculate a subgrid eddy viscosity for high Reynolds number flows, one would have to perform a numerical simulation using an existing subgrid model, and then measure a new subgrid eddy viscosity using the results of the LES.[13]

However, it is important to note that in a numerical simulation one does not solve the turbulence energy equation; rather, one solves the Navier–Stokes equation directly. For incompressible turbulence the Fourier transformed Navier–Stokes equation may be written, in component form, as

$$\frac{\partial u_i(\mathbf{k},t)}{\partial t} + \nu k^2 u_i(\mathbf{k},t) = -iP_{ij}(\mathbf{k})k_n \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p},t)u_n(\mathbf{q},t), \tag{6}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$. Since in an LES the terms in the summation on the right-hand side of Eq. (6) with p or $q > k_m$ are not resolved, one actually solves

$$\frac{\partial u_i(\mathbf{k},t)}{\partial t} + [\nu + \nu_t(k|k_m,t)]k^2 u_i(\mathbf{k},t) = -iP_{ij}(\mathbf{k})k_n \sum_{\substack{\mathbf{p}+\mathbf{q}=\mathbf{k} \\ p \text{ and } q \leqslant k_m}} u_j(\mathbf{p},t)u_n(\mathbf{q},t) \tag{7}$$

using the expression for the eddy viscosity defined by Eq. (4).

The eddy viscosity model has met with some success. Because of its definition from the turbulence energy equation, the use of an eddy viscosity model results in the correct net energy transfer between the numerically resolved scales and the unresolved subgrid scales. Furthermore, as noted by Kraichnan,[29] in an inertial subrange and for $k \ll k_m$, $\nu_t(k|k_m,t)$ is positive and independent of k, *a posteriori* justifying the modeling of the subgrid scales as an enhancement of the kinematic viscosity.

However, there are theoretical difficulties associated with the eddy viscosity subgrid model defined by Eq. (4). Although the eddy viscosity model is plausible for $k \ll k_m$ (where there exists a clear separation in sizes between the resolved scales and the subgrid scales), when $k \to k_m$ this scale separation does not exist and an eddy viscosity model is no longer physically justified. In fact, $\nu_t(k|k_m,t)$ becomes k dependent and rises to a cusp as $k \to k_m$.[29]

In addition, the eddy viscosity subgrid term $2\nu_t(k|k_m,t)k^2E(k,t)$ in Eq. (5) models the net statistically averaged energy transfer from the resolved scales to the subgrid scales. In a given flow realization, this net energy transfer is expected to fluctuate about its statistical mean. Furthermore, the net energy transfer is the result of a two-way exchange of energy across k_m. Although the transfer of energy from the large scales to the small scales across k_m is by far the largest in three-dimensional turbulence (assuming that k_m is in the inertial subrange), there is, nevertheless, a substantial backscatter of energy from the small scales to the large scale across k_m. This backscatter of energy has been referred to as an eddy noise,[33] or as a stochastic backscatter,[34] and Leslie and Quarini[35] have suggested that it should be modeled distinctly from the forward cascade, since the physics in these two types of energy transfers is quite different. Furthermore, this backscatter of energy, after a sufficiently long time, will render the resolved large-scale velocity field unpredictable in anything but a statistical sense. This unpredictable nature of the large scales in an LES is, however, not respected by the eddy viscosity subgrid model, which for a given initial large-scale velocity field, will always yield the same large-scale velocity field at a later time.

The difficulties associated with an eddy viscosity subgrid model are predominantly a consequence of a simple fact: an energy equation is used to construct an eddy viscosity term which is then used in the numerical solution of a momentum equation.[36] The information lost in the statistical averaging performed to obtain an energy equation is irrecoverable. What one actually needs is a subgrid model that is based directly on a momentum equation. Fortunately, models of the momentum equation do exist in the two-point closure theories. These are the so-called stochastic model equations,[37] from which one can derive two-point closure energy equations. The stochastic model equations are constructed by neglecting the phase correlations in the nonlinear term of the corresponding momentum equation, while simultaneously requiring the resulting equation to conserve energy and satisfy other fundamental consistency requirements.

In the following section, we will show how one can use the stochastic equation for the EDQNM model to formulate a subgrid model which corrects some of the deficiencies of the eddy viscosity model.

B. Subgrid eddy damping and random force

1. EDQNM stochastic model equation

Use of an analytical closure approximation, such as the EDQNM or one of its related closures, typically results in an energy equation for turbulence, i.e., the closure approximation yields an analytical form for the energy transfer $T(k,t)$ of Eq. (2). However, there is another perspective on these closures.[30,37,38] It has been noted that the model energy equations derived using these approximations are the *exact* solutions of particular models of the momentum equation. That is, the model energy equations are exact solutions of equations which are direct models of the Navier–Stokes equation itself. In particular, the EDQNM representation of the Navier–Stokes equation may be a more suitable starting point in the development of a subgrid model than the EDQNM energy equation. Use of the stochastic model equations for the development of a subgrid model was originally suggested by Kraichnan[30] and has been pursued by Bertoglio[31] for homogeneous shear turbulence.

The EDQNM stochastic model equation may be written as[37]

$$\frac{\partial u_i(\mathbf{k},t)}{\partial t}+[\nu+\eta(k,t)]k^2u_i(\mathbf{k},t)=f_i(\mathbf{k},t), \quad (8)$$

where $f_i(\mathbf{k},t)$ is assumed to be uncorrelated in time. The EDQNM eddy-damping term $\eta(\mathbf{k},t)$ and the spectrum of the stochastic force $F(k,t)$, defined by

$$F(k,t)=4\pi k^2\int_0^t\langle f_i(\mathbf{k},t)f_i(\mathbf{k},t')^*\rangle dt', \quad (9)$$

where $\langle\cdots\rangle$ denotes an average over spherical shells in k space, can be related to the energy spectrum $E(k,t)$ as[39]

$$\eta(k,t)=\frac{1}{2}\int\int_\Delta dp\,dq\,\theta_{kpq}(t)\frac{p}{kq}b(k,p,q)E(q,t), \quad (10)$$

and

$$F(k,t)=\int\int_\Delta dp\,dq\,\theta_{kpq}(t)\frac{k^3}{pq}a(k,p,q)E(p,t)E(q,t), \quad (11)$$

where the geometrical coefficients, $a(k,p,q)$ and $b(k,p,q)$ may be written as

$$a(k,p,q)=\tfrac{1}{2}(1-2y^2z^2-xyz) \quad (12)$$

and

$$b(k,p,q)=(p/k)(xy+z^3), \quad (13)$$

and where x, y, and z are the cosines of the interior angles of the triangle formed by **k**, **p**, and **q** facing, respectively, the sides **k**, **p**, and **q**. For fully developed flows, the function $\theta_{kpq}(t)$ becomes

$$\theta_{kpq}(t)=[\mu_{kpq}(t)+\nu(k^2+p^2+q^2)]^{-1}, \quad (14)$$

where μ_{kpq} is an "eddy-damping rate" of the third-order moments associated with the wave vectors **k**, **p**, and **q**. In the EDQNM model, μ_{kpq} is assumed to be of the form

$$\mu_{kpq}=\mu_k+\mu_p+\mu_q. \quad (15)$$

In the inertial subrange, all acceptable expressions for μ_k reduce to the simple form first proposed by Orszag,[40]

$$\mu_k(t)=0.19\,\mathrm{Ko}^{3/2}[k^3E(k,t)]^{1/2}. \quad (16)$$

The stochastic equation for the EDQNM model, Eq. (8), may be used to construct a subgrid model for the Navier–Stokes equation. One can define an effective eddy-damping term $\eta(k\,|k_m,t)$ and a stochastic force $f_i(\mathbf{k}|k_m,t)$ with spectrum $F(k\,|k_m,t)$ by restricting the double integrals in Eqs. (10) and (11) to wave numbers k,p, and q such that $k<k_m$ and p or $q>k_m$. The functions $\eta(k\,|k_m,t)$ and $f_i(\mathbf{k}|k_m,t)$ may then be used as a subgrid model in an LES; the eddy-damping term augments the kinematic viscosity, whereas $f_i(\mathbf{k}|k_m,t)$ appears as a random force in the Navier–Stokes equation. It should be noted that $f_i(\mathbf{k}|k_m,t)$ is chosen to be a particular realization of $F(k\,|k_m,t)$, and to make $f_i(\mathbf{k}|k_m,t)$ uncorrelated in time we choose realizations of $F(k\,|k_m,t)$ such that $f_i(\mathbf{k}|k_m,t')$ is uncorrelated with $f_i(\mathbf{k}|k_m,t)$ when $t'\neq t$. Furthermore, for the random force to supply a finite amount of energy to the turbulence, we must also require $f_i(\mathbf{k}|k_m,t)\propto 1/(\Delta t)^{1/2}$, where Δt is the time step in the simulation. Note that the arbitrary random phases in $f_i(\mathbf{k}|k_m,t)$ will result in a nondeterministic evolution of the resolved scales, since every simulation performed will use different phases, although the spectrum of $f_i(\mathbf{k}|k_m,t)$ may be unchanged.

2. Nonlocal expansions

It is possible to determine the analytical form for $\eta(k\,|k_m,t)$ and $F(k\,|k_m,t)$ in the limiting case $k\ll k_m$. In this limit, the integral over q in Eqs. (10) and (11) may be evaluated.[41] To lowest order in k/k_m,

$$\eta(k\,|k_m,t)=\frac{1}{15}\int_{k_m}^\infty dp\,\theta_{kpp}(t)\left(5E(p,t)+p\frac{\partial E(p,t)}{\partial p}\right) \quad (17)$$

and

$$F(k\,|k_m,t)=\frac{14}{15}k^4\int_{k_m}^\infty dp\,\theta_{kpp}(t)\frac{E(p,t)^2}{p^2}. \quad (18)$$

Equation (17) is the well-known form of the turbulent viscosity for $k\ll k_m$.[39] Since $\theta_{kpp}(t)\cong\theta_{0pp}(t)$ when $k\ll k_m$, $\eta(k\,|k_m,t)$ becomes independent of k for $k\ll k_m$, hence effectively renormalizing the kinematic viscosity. Equation (18) represents the energy transfer from the subgrid scales to the resolved scales. This backscatter of energy is the origin of the characteristic k^4 spectrum observed at small wave numbers in a numerical simulation of freely decaying turbulence when the initial energy spectrum is chosen to be steeper than k^4 at small wave numbers.[39]

It is current practice in large-eddy simulations of homogeneous turbulence[13] to write the effective eddy viscosity as in Eq. (4), i.e.,

$$\nu_t(k\,|k_m,t)=\eta(k\,|k_m,t)-F(k\,|k_m,t)/2k^2E(k,t). \quad (19)$$

From Eqs. (17) and (18), one can show that for k in the inertial subrange and $k\ll k_m$, the ratio of the second to the first term on the right-hand side of Eq. (19) is equal to $\frac{14}{15}(k/k_m)^{11/3}$. In this limit, it may be reasonable to neglect the contribution of $f_i(\mathbf{k}|k_m,t)$ to the subgrid model. However, for $k\sim k_m$, $\eta(k\,|k_m,t)$ and $F(k\,|k_m,t)/[2k^2E(k,t)]$ are of the same order of magnitude. In fact, as noted by Kraichnan,[29] for an inertial subrange extending to wave number zero, both contributions to $\nu_t(k\,|k_m,t)$ diverge as $k\to k_m$, but the net result is finite due to an exact cancellation of infinities. Hence the use of an η-F model at wave numbers near k_m may yield very different results than the use of a ν_t model.

The interactions which contribute the most to the subgrid eddy-damping term and to the stochastic force near $k=k_m$ arise from interactions in which the length of one of the legs **p** or **q** of the triangle formed by **k**, **p**, and **q** is much less than k. The clearest way to examine these nonlocal interactions is to consider the effect of scales with wave numbers less than some minimum wave number k_0 on scales with wave numbers k, where $k\gg k_0$. The scales with wave number less than k_0 may be considered "supergrid" scales. We define the corresponding supergrid eddy-damping term $\eta^{\mathrm{sup}}(k\,|k_0,t)$ and supergrid spectrum of the stochastic force $F^{\mathrm{sup}}(k\,|k_0,t)$ by restricting the double integrals in Eqs. (10)

and (11) to wave numbers k, p, and q such that $k \geqslant k_0$ and p or $q < k_0$.

An analytical form for $\eta^{\text{sup}}(k|k_0,t)$ and $F^{\text{sup}}(k|k_0,t)$ may be determined in the limiting case $k \gg k_0$. Since either p or q must be less than k_0, the condition $k \gg k_0$ implies that the other must be of order k. This interaction contains the same physics as the subgrid interactions in which $k \sim k_m$ and either p or q is much less than k. In the limit $k \gg k_0$, the integral over p in Eqs. (10) and (11) can be evaluated.[42] To lowest order in k_0/k

$$\eta^{\text{sup}}(k|k_0,t) = \frac{2}{3}\int_0^{k_0} dq\, \theta_{kkq}(t)E(q,t) \tag{20}$$

and

$$F^{\text{sup}}(k|k_0,t) = \frac{4}{3}k^2E(k,t)\int_0^{k_0} dq\, \theta_{kkq}(t)E(q,t). \tag{21}$$

Equations (20) and (21) separately affect the time evolution of the resolved scales. However, the net effect on the energy of the resolved scales is determined by calculating the supergrid energy transfer. It can be easily seen that the terms of Eqs. (20) and (21) cancel exactly in computing the transfer. This implies that although these terms affect the evolution of the resolved scales in the numerical simulation, they do not directly change the energy of these scales. When there exists an inertial subrange extending to wave number zero, η^{sup} and F^{sup} diverge as $q \to 0$. This divergence is a consequence of the same physics as the divergence of $\eta(k|k_m,t)$ and $F(k|k_m,t)$ as $k \to k_m$. These divergences are the result of the random uniform convection of the small scales by the large scales.[42] The next higher-order terms which enter in Eqs. (20) and (21) contribute to the energy equation and result in a net flux of energy from the supergrid scales to the resolved scales when $E(k,t) \propto k^{-5/3}$.

3. Numerical solution of the EDQNM model

To calculate the terms of the η-F subgrid model for all k between the minimum wave number k_0 and the maximum wave number k_m of the simulation, the EDQNM equations must be solved numerically for $\eta(k|k_m,t)$ and $F(k|k_m,t)$. Since we are interested in using the subgrid model to perform high Reynolds number numerical simulations, we will assume that the energy spectrum $E(k,t)$ is well represented by its inertial subrange form, Eq. (1), for k between k_0 and k_m. We further assume that the inertial form for $E(k,t)$ extends appreciably into the subgrid scales, so that it becomes reasonable to neglect the kinematic viscosity in Eq. (14) and to suppose that the energy is dissipated at infinite wave number. For wave numbers less than k_0 we assume, somewhat artificially, that $E(k,t) = 0$. In solving the EDQNM model, we use Eq. (16) for the phenomenological time scale with a value of the Kolmogorov constant taken to be 1.8.

To evaluate the integrals in Eqs. (10) and (11), they may be written in the p-q symmetric form

$$\eta(k|k_m) = \frac{1}{2k^2}\int_{k_m}^{\infty} dp \int_{p-k}^{p} dq\, \theta_{kpq}\left(\frac{p^2}{q}(xy+z^3)E(q) + \frac{q^2}{p}(xz+y^3)E(p)\right), \tag{22}$$

$$F(k|k_m) = \int_{k_m}^{\infty} dp \int_{p-k}^{p} dq\, \theta_{kpq}\frac{k^3}{pq} \times (1-2y^2z^2-xyz)E(q)E(p), \tag{23}$$

and may be evaluated by Gaussian quadrature. Anticipating high Reynolds number numerical simulations, we solve the equations for $k_0 = 1$ and $k_m = 30.17$, as appropriate for a 64^3 numerical simulation.[43] The integrals over p in Eqs. (22) and (23) are computed numerically for $k_m \leqslant p \leqslant 3k_m$, whereas the contributions to the integrals from $p > 3k_m$ are obtained using the asymptotic analytical results presented in Eqs. (17) and (18). As in Chollet and Lesieur,[27] we have rendered $\eta(k|k_m,t)$ and $F(k|k_m,t)$ dimensionless using the maximum wave number k_m and the value of the energy spectrum at k_m, i.e., we define the dimensionless time-independent eddy-damping term $\eta^+(k/k_m)$ and the spectrum of the stochastic force $F^+(k/k_m)$ by

$$\eta(k|k_m,t) = \eta^+(k/k_m)E(k_m,t)^{1/2}k_m^{-1/2} \tag{24}$$

and

$$F(k|k_m,t) = F^+(k/k_m)[k_mE(k_m,t)]^{3/2}. \tag{25}$$

It should be noted that η^+ and F^+ depend on the Kolmogorov constant as $\text{Ko}^{-3/2}$ (which is a free parameter in the EDQNM model), so that if the numerical simulation yields a Kolmogorov constant significantly different than the chosen value of 1.8, then the numerical values of η^+ and F^+ need to be rescaled to coincide with the Kolmogorov constant emerging from the simulation.

The net eddy viscosity defined by Kraichnan[29] and used by Chollet and Lesieur[27] is given by

$$\nu_t^+\left(\frac{k}{k_m}\right) = \eta^+\left(\frac{k}{k_m}\right) - \frac{F^+(k/k_m)}{2(k/k_m)^2[E(k,t)/E(k_m)]}. \tag{26}$$

In Fig. 1 we plot the three terms in Eq. (26) appropriate for use in a 64^3 simulation. The asymptotic ($k \ll k_m$) eddy-damping term and the spectrum of the stochastic force in the EDQNM model may be calculated analytically from the nonlocal expansions, Eqs. (17) and (18). They are determined to be $\eta^+(k/k_m) = 0.44\ \text{Ko}^{-3/2}$ and $F^+(k/k_m) = 0.49\ \text{Ko}^{-3/2}(k/k_m)$.[4]

4. Summary

In the preceding subsections, a modification of the eddy viscosity subgrid model has been proposed. Having based the subgrid model on the EDQNM stochastic equation, the subgrid eddy viscosity $\nu_t(k|k_m,t)$ has been split into two terms—an eddy-damping term $\eta(k|k_m,t)$ and a stochastic force $f_i(\mathbf{k}|k_m,t)$. By computing nonlocal expansions of the EDQNM equations when $k \ll k_m$, clear interpretations for $\eta(k|k_m,t)$ and $f_i(\mathbf{k}|k_m,t)$ have been obtained: the eddy-damping term renormalizes the kinematic viscosity of the

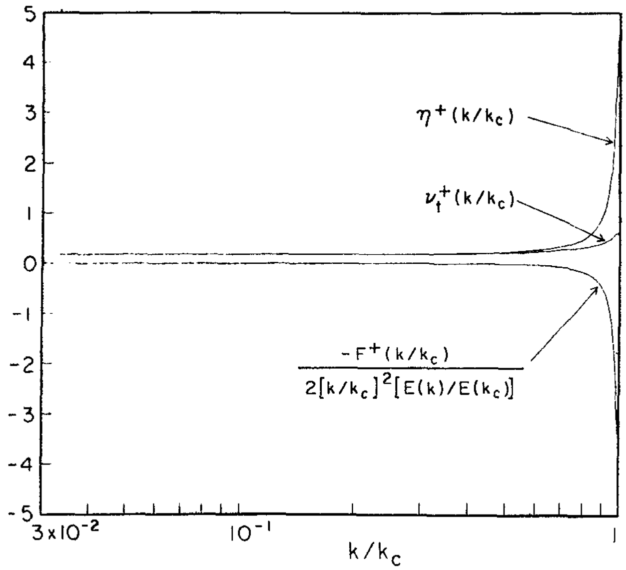


FIG. 1. Subgrid eddy viscosity, eddy-damping term, and spectrum of the stochastic force computed from the EDQNM model.

fluid, whereas the stochastic force models the backscatter of energy from the subgrid scales to the resolved scales.

However, although our discussion has focused on the EDQNM model, it is important to note that the form for the η-F subgrid model,

$$q_i(\mathbf{k}|k_m,t) = f_i(\mathbf{k}|k_m,t) - \eta(k|k_m,t)k^2 u_i(\mathbf{k},t), \quad (27)$$

is independent of any particular analytical closure approximation. In Eq. (27), $q_i(\mathbf{k}|k_m,t)$ is the subgrid nonlinear term in the Navier–Stokes equation that must be modeled in an LES, i.e.,

$$q_i(\mathbf{k}|k_m,t) = -iP_{ij}(\mathbf{k})k_n \sum_{\substack{\mathbf{p}+\mathbf{q}=\mathbf{k} \\ p \text{ or } q > k_m}} u_j(\mathbf{p},t)u_n(\mathbf{q},t), \quad (28)$$

and $f_i(\mathbf{k}|k_m,t)$ is assumed to be uncorrelated in time. One could use the EDQNM model, as we have in the last few subsections, to compute $\eta(k|k_m,t)$ and $F(k|k_m,t)$. However, in Sec. III, we shall also show that $\eta(k|k_m,t)$ and $F(k|k_m,t)$ may be computed independently of the EDQNM model by using the flow field calculated in a low Reynolds number DNS. Such a computation may then be used to corroborate the EDQNM model results. If these results are consistent with the DNS at low Reynolds numbers, then the subgrid model terms calculated using the EDQNM model may be used to perform an LES at Reynolds numbers unattainable in a DNS.

However, it is important to recognize that the computation of $\eta(k|k_m,t)$ and $F(k|k_m,t)$ in the numerical simulation by no means implies that the η-F subgrid model is a good physical model (in the same way that the calculation of ν_t in a DNS and an LES did not imply that ν_t was a good physical model). Rather, it only indicates what η and F must be to be consistent with the Navier–Stokes equation. Other means, such as a comparison of LES results with experimental data, are necessary to determine whether the η-F subgrid model is a good physical model for the omitted subgrid scale interactions.

III. COMPUTATION FROM A DIRECT NUMERICAL SIMULATION

In this section we will use the results of a DNS to compute the eddy-damping term and the stochastic force in the η-F subgrid model, Eq. (27). Such a computation will be performed independent of any particular analytical closure theory. To compute the subgrid terms of the η-F model we divide the Fourier space into "resolved" and "subgrid" scales—the Fourier components at wave numbers less than the wave number cut k_c being the "resolved" scales, and the Fourier components at wave numbers greater than k_c being the subgrid scales. Both the resolved and the subgrid scales will be fully resolved in the DNS.

A. Required statistics

To calculate both $\eta(k|k_c,t)$ and $F(k|k_c,t)$, we need to compute statistics over the subgrid scales in the numerical simulation. The first statistic we compute is the subgrid scale energy transfer. This statistic was previously computed to calculate a subgrid scale eddy viscosity.[32] The subgrid scale energy transfer to wave numbers greater than k_c may be determined in a numerical simulation by

$$T(k|k_c,t) = 4\pi k^2 \langle u_i(\mathbf{k},t) q_i(k|\mathbf{k}_c,t)^* \rangle, \quad (29)$$

using the masking method.[32] Here, $q_i(\mathbf{k}|k_c,t)$ is defined to be the subgrid nonlinear term in the Navier–Stokes equation [see Eq. (28), where now k_m is replaced by k_c]. The masking method consists of computing the complete Navier–Stokes nonlinear term $q_i(\mathbf{k},t)$ using fast Fourier transforms for the full velocity field, and for a velocity field which is masked for $k > k_c$, i.e., a velocity field in which all of the Fourier components with wave numbers greater than k_c have been set to zero. The latter nonlinear term is denoted by $q_i^<(\mathbf{k},t)$, where the superscript indicates that if one were to compute $q_i^<(\mathbf{k},t)$ by performing the required convolution sums both $\mathbf{p}$ and $\mathbf{q}$ would have lengths less than k_c. The subgrid scale energy transfer is then equal to the energy transfer calculated using $q_i(\mathbf{k},t)$ minus the transfer calculated using $q_i^<(\mathbf{k},t)$, i.e.,

$$T(k|k_c,t) = T(k,t) - T^<(k|k_c,t), \quad (30)$$

where

$$T(k,t) = 4\pi k^2 \langle u_i(\mathbf{k},t)^* q_i(\mathbf{k},t) \rangle \quad (31)$$

is the full energy transfer, and

$$T^<(k,t) = 4\pi k^2 \langle u_i(\mathbf{k},t)^* q_i^<(\mathbf{k},t) \rangle \quad (32)$$

is the masked energy transfer.

Once $T(k|k_c,t)$ and $E(k,t)$ have been computed, we can write an equation for the unknown subgrid model terms $\eta(k|k_c,t)$ and $F(k|k_c,t)$:

$$T(k|k_c,t) = -2\eta(k|k_c,t)k^2 E(k,t) + F(k|k_c,t). \quad (33)$$

To separate the effects of the eddy-damping term $\eta(k|k_c,t)$ and the spectrum of the stochastic force $F(k|k_c,t)$ we need to compute an additional subgrid scale statistic. A reasonable choice which accomplishes this separation is the statistic we will call $S(k|k_c,t)$, defined by

$$S(k|k_c,t) = \int_0^t dt' \, 4\pi k^2 \langle q_i(\mathbf{k}|k_c,t) \, q_i(\mathbf{k}|k_c,t')^* \rangle, \tag{34}$$

where $q_i(\mathbf{k}|k_c,t)$ is again the subgrid nonlinear term. In contrast with the eddy viscosity subgrid model, it is evident that the η-subgrid model will contain information about the time correlations in the turbulent field.

The function $S(k|k_c,t)$ may also be computed in a numerical simulation by use of the masking method. To compute this function, one must calculate

$$S(k|k_c,t) = \int_0^t dt' \, 4\pi k^2 \langle [q_i(\mathbf{k},t) - q_i^<(\mathbf{k},t)] \times [q_i(\mathbf{k},t') - q_i^<(\mathbf{k},t')]^* \rangle. \tag{35}$$

Again, $q_i(\mathbf{k},t)$ and $q_i^<(\mathbf{k},t)$ are calculated by fast Fourier transforms.

Using Eq. (27) and the definition of the subgrid scale statistic $S(k|k_c,t)$ we can derive another equation for $\eta(k|k_c,t)$ and $F(k|k_c,t)$:

$$S(k|k_c,t) = F(k|k_c,t) + 2k^4 \int_0^t dt' \, \eta(k|k_c,t) \times \eta(k|k_c,t') E(k,t,t'), \tag{36}$$

where we define the two-time energy spectrum $E(k,t,t')$ by

$$E(k,t,t') = 2\pi k^2 \langle u_i(\mathbf{k},t) u_i(\mathbf{k},t')^* \rangle. \tag{37}$$

As a consequence of the time integration in Eq. (36), to solve for $\eta(k|k_c,t)$ and $F(k|k_c,t)$ using Eqs. (33) and (36), we need $S(k|k_c,t')$, $T(k|k_c,t')$, and $E(k,t',t'')$ for all times $0 \leqslant t', t'' \leqslant t$. Assuming that the simulation is performed for times $t = 0, \Delta t, 2\Delta t, \ldots, N\Delta t$, Eqs. (33) and (36) yields $2N$ equations with $2N$ unknowns, which may then be solved for $\eta(k|k_c,t)$ and $F(k|k_c,t)$ at times $t = 0, \Delta t, 2\Delta t, \ldots, N\Delta t$.

It is instructive to solve Eqs. (33) and (36) analytically at the time Δt, assuming that the initial simulation velocity field has random phases. It can be shown[42] that the solution yields the EDQNM model results exactly provided that the EDQNM model reduces to the quasinormal approximation for small times, i.e., $\theta_{kpq}(\Delta t) = \Delta t$, as it should.

B. DNS results

To compute the terms of the η-F subgrid model from a numerical simulation, we perform a 64^3 DNS of freely decaying turbulence using the Rogallo code[43] for isotropic turbulence. Numerical simulations of freely decaying turbulence are characterized by the initial energy spectrum of a Gaussian velocity field and the value of the kinematic viscosity ν of the fluid. We take as our initial energy spectrum

$$E(k,0) = \left(\frac{256}{35}\right)\left(\frac{2}{\pi}\right)^{1/2} k_p^{-1} \left(\frac{k}{k_p}\right)^8 u_0^2 \times \exp\left[-2\left(\frac{k}{k_p}\right)^2\right], \tag{38}$$

normalized so that u_0 is the initial root-mean-square turbulent velocity. The use of a steep initial spectrum at small wave numbers permits the development of a k^4 power law as $k \to 0$, as a result of the backscatter of energy from scales at larger wave numbers. In our simulation, we take $u_0 = 1$, $k_p = 5$, and $\nu = 0.01$. The simulation is performed for 200 time steps and the velocity field is saved every ten time steps. Subgrid scale statistics are then computed for the stored velocity fields. The initial microscale Reynolds number of the flow is $R_\lambda = 30$, while the final value is determined to be $R_\lambda = 15$.

Figure 2 is a plot of the energy spectrum as it evolves in time; the cut is taken at $k_c = 16$. Energy is cascaded from smaller wave numbers to larger wave numbers as a result of the nonlinear interactions. Energy is also backscattered to smaller wave numbers, resulting in the formation of the characteristic k^4 spectrum at small k.

In Fig. 3, we plot the dimensionless eddy-damping term $\eta^+(k|k_c,t)$, the dimensionless spectrum of the stochastic force (written for comparison purposes as a viscosity $-F^+(k|k_c,t)/\{2(k/k_c)^2[E(k,t)/E(k_c,t)]\}$), as well as the dimensionless eddy viscosity $\nu_t^+(k/k_c,t)$, at two different times. Also shown in the plots are the values of the dimensionless kinematic viscosity ν^+. The characteristic cusps of the subgrid terms at $k = k_c$ are evident. At small wave numbers, the eddy-damping term and the eddy viscosity are negative. Although the stochastic force is expected to model the energy transfer that results in the formation of the k^4 spectrum at small wave numbers, one observes that for the subgrid scale interactions that we have measured, more energy is transferred to the large scales by the negative eddy-damping term than by the stochastic force. This is also predicted by the EDQNM model for an energy spectrum which is very steep for $k > k_c$. In fact, one notices that the kinematic viscosity of the fluid is typically larger than both the η and F terms for $k \ll k_c$.

According to the EDQNM model, most of the backscatter of energy arises from the most energetic scales, so

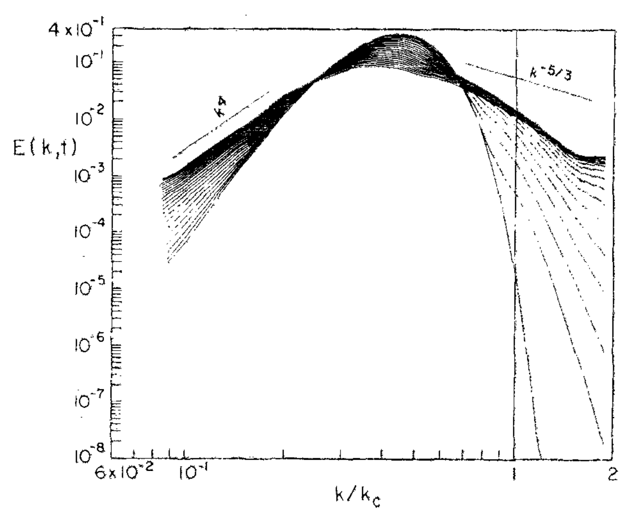


FIG. 2. Time evolution of the energy spectrum computed from the DNS of freely decaying turbulence. Time steps 0–200 by 10; $k_c = 16$.

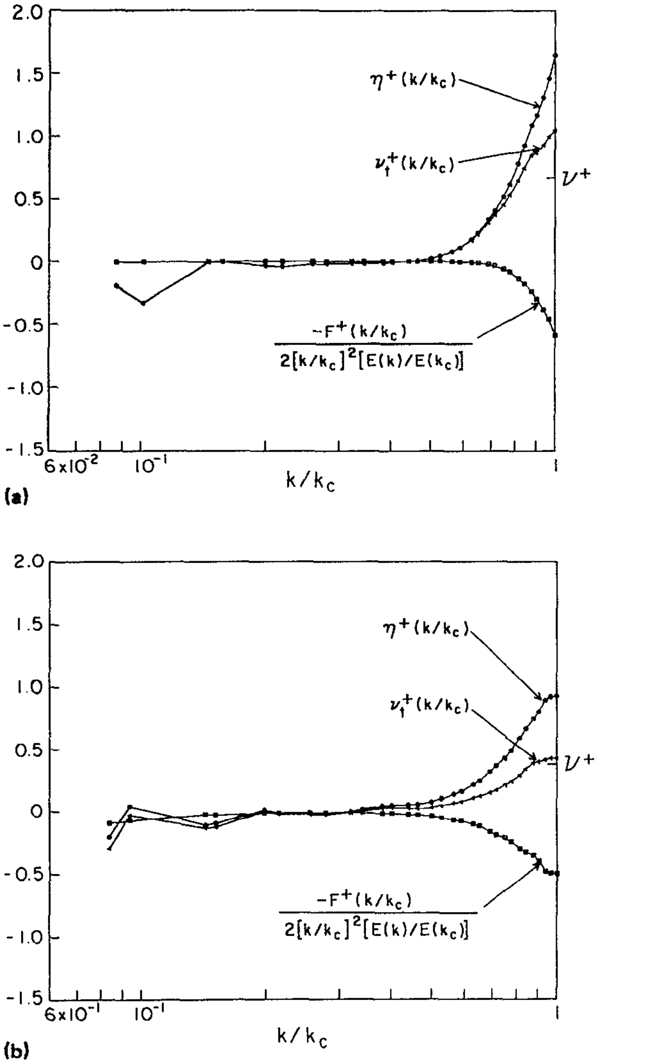


FIG. 3. Subgrid eddy viscosity, eddy-damping term, and spectrum of the stochastic force computed from the DNS of freely decaying turbulence; $k_c = 16$. (a) 30th time step; (b) 200th time step.

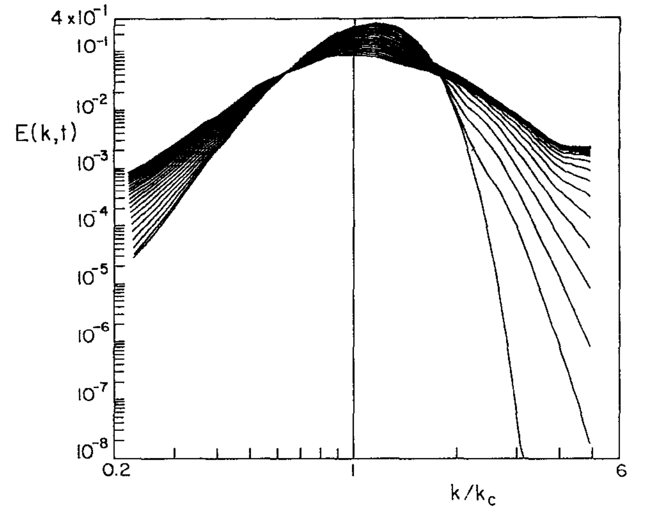


FIG. 4. Time evolution of the energy spectrum computed from the DNS of freely decaying turbulence. Time steps 0–200 by 10; $k_c = 6$.

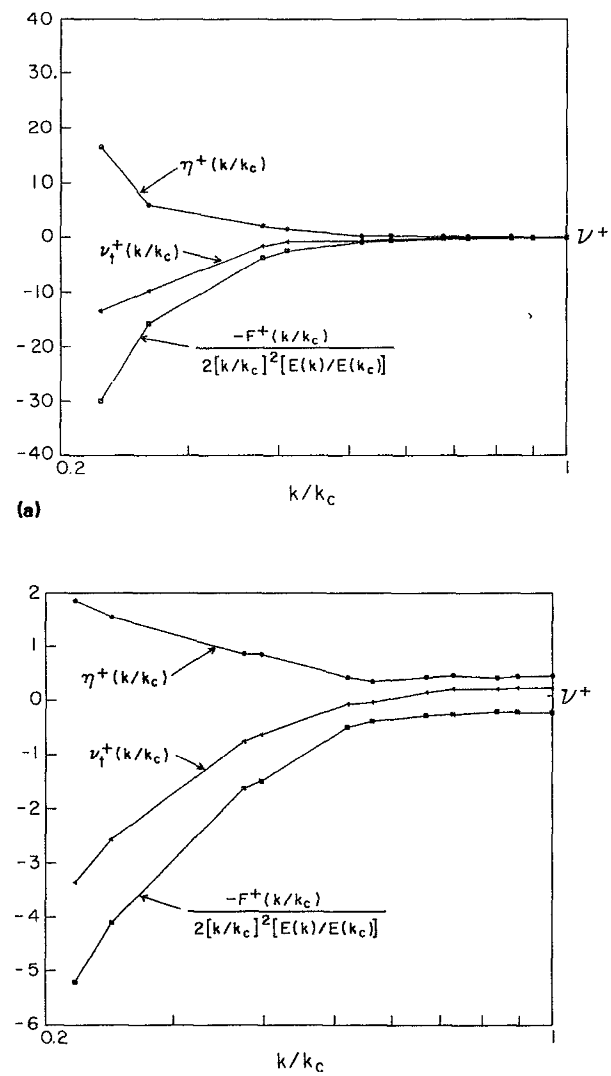
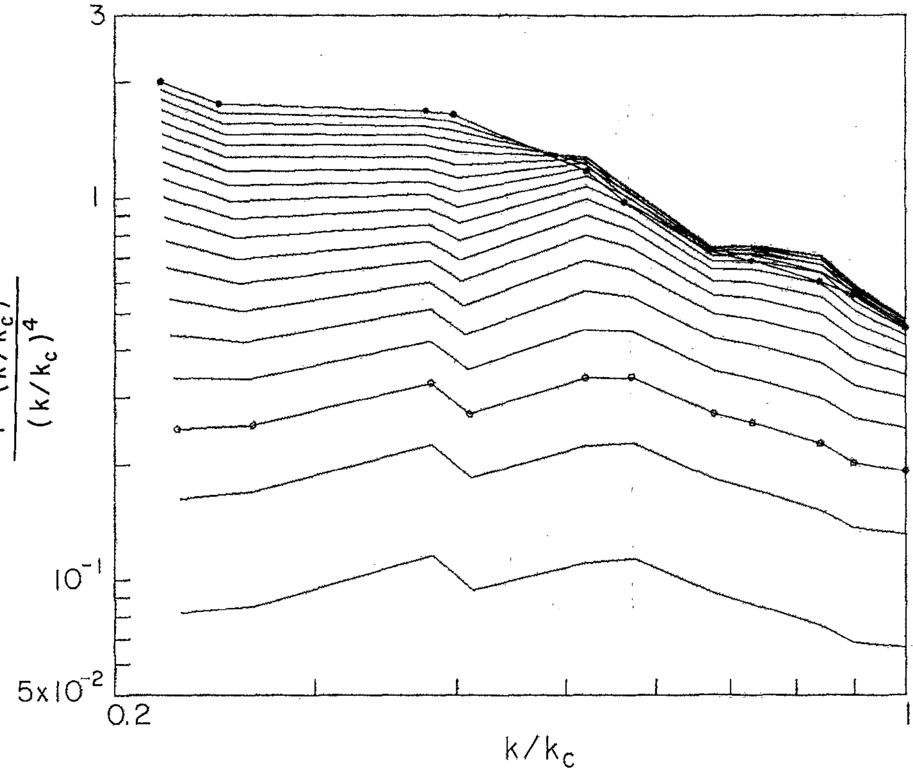


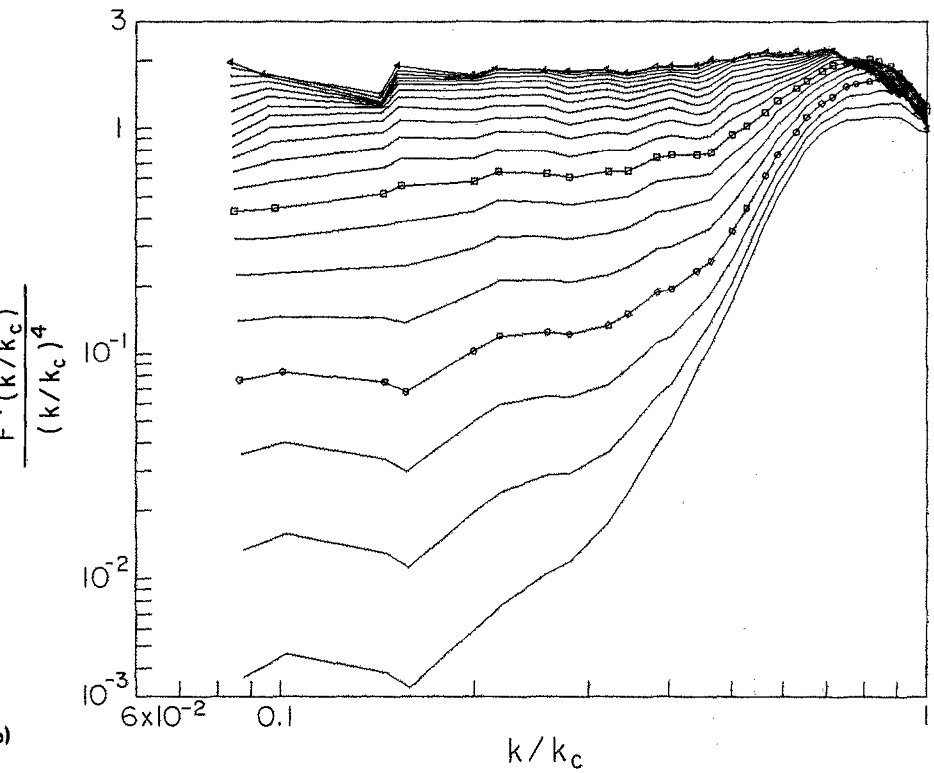
FIG. 5. Subgrid eddy viscosity, eddy-damping term, and spectrum of the stochastic force computed from the DNS of freely decaying turbulence; $k_c = 6$. (a) 30th time step; (b) 200th time step.

that if we want to measure the stochastic backscatter, it would be better to choose k_c nearer to the energy peak. In Fig. 4, we show the spectrum with a cut taken at $k_c = 6$. Here, k_c is slightly to the left of the maximum of $E(k,0)$ and slightly to the right of the maximum of $E(k,t)$ at the final time step. In Fig. 5 we plot the subgrid terms for this value of k_c at two different times. As expected, we now observe that the stochastic forcing term is substantially larger in magnitude than the eddy-damping term for $k \ll k_c$, and furthermore, the eddy-damping term $\eta(k\,|k_c,t)$ is now observed to be positive, indicating a substantial enhancement of the kinematic viscosity by the nonlinear interactions. The eddy viscosity model yields a negative $\nu_t(k\,|k_c,t)$ that, in this case, would result in an incorrect physical representation of the subgrid nonlinear interactions. On this basis, one would expect very different simulation results using the η-F subgrid model than using the ν_t subgrid model for a simulation having its maximum wave number in the most energetic scales.

In Fig. 6, we plot the time evolution of $F^+(k/k_c,t)/$



(a)



(b)

FIG. 6. Time evolution of the subgrid spectrum of the stochastic force divided by k^4 computed from the DNS of freely decaying turbulence. (a) $k_c = 6$; (b) $k_c = 16$.

$(k/k_c)^4$ for $k_c = 6$ and $k_c = 16$. The EDQNM model predicts that this function approaches a constant value for $k \ll k_c$ [i.e., $F^+(k/k_c,t) \propto k^4$]. The numerical simulations are observed to be in excellent agreement with this prediction.

The results of this section indicate that the η-F subgrid model derived from the EDQNM stochastic model equation is consistent with the Navier–Stokes equation. Assuming that the η-F subgrid model is also a good physical model of the subgrid scale interactions, in the following section we apply this subgrid model to an LES of the Kolmogorov inertial subrange.

IV. LARGE-EDDY SIMULATION OF THE KOLMOGOROV CONSTANT

The η-F subgrid model has been implemented in the Rogallo code.[42] Furthermore, to obtain an inertial subrange beginning at the earliest possible wave number of the simulation, we artificially force the turbulence. The forcing which we use is a modification of a forcing originally proposed by Siggia and Patterson.[26] In their forcing scheme, all velocity Fourier components with $\mathbf{k}$ satisfying $1 \leqslant |\mathbf{k}| < 2$ are frozen at their initial values. The net result of freezing these modes is to restore, at each time step of the simulation, the energy cascaded out of these modes during the previous time step. This is accomplished by keeping all of the amplitudes and all of the phase relations among these modes fixed. The forcing which we adopt is similar to that of Siggia and Patterson in that we keep the energy in each mode with $\mathbf{k}$ satisfying $1 \leqslant |\mathbf{k}| < 2$ constant, i.e., $|u_i(\mathbf{k},t)| = \text{const}$. However, we now allow the phases and the distribution of energy among the velocity components of the forced modes to evolve according to their interactions with all the other modes in the simulation. The advantage of this method over the Siggia and Patterson forcing is twofold. *First*, higher-order correlations among the Fourier components in the forced modes are allowed to develop, and *second*, the forcing we use results in more realistic two-time correlations among the forced Fourier modes. Simply freezing these Fourier modes results in an infinite correlation time among these modes. We have determined that this further results in unphysical two-time correlations among modes at slightly larger wave numbers.

We initially simulate a 64^3 stationary flow taking the values of the η-F subgrid model terms from the EDQNM model computations discussed in Sec. II. The resulting stationary energy spectrum determined from the LES is then reintroduced into the EDQNM subgrid model equations. Assuming that the $k^{-5/3}$ power law of the spectrum continues beyond k_m to very large wave numbers, the EDQNM subgrid model equations are then resolved. Furthermore, we adjust the value of the Kolmogorov constant assumed in the EDQNM model to be in agreement with the LES results. This procedure is iterated until a satisfactory $k^{-5/3}$ energy spectrum is obtained over a substantial range of wave numbers in the LES.

Figure 7 presents these final results. In Fig. 7(a) we plot the time average of $k^{5/3}E(k,t)/\epsilon(t)^{2/3}$, while in Fig. 7(b) we plot this function at a single instant of time. The cascade rate $\epsilon(t)$ is computed by calculating the energy flux across k_m, i.e.,

$$\epsilon(t) = \int_{k_0}^{k_m} 2\eta(k\,|k_m,t)k^2E(k,t)dk - \int_{k_0}^{k_m} F(k\,|k_m,t)dk. \tag{39}$$

If $E(k)$ has a Kolmogorov inertial subrange, Eq. (1), then the time average plotted in Fig. 7(a) should be a constant, with a value equal to the Kolmogorov constant Ko. Examining the time-averaged spectrum in Fig. 7(a), we observe that a $k^{-5/3}$ spectrum begins to develop a decade in wave number space from the minimum wave number of the simulation.

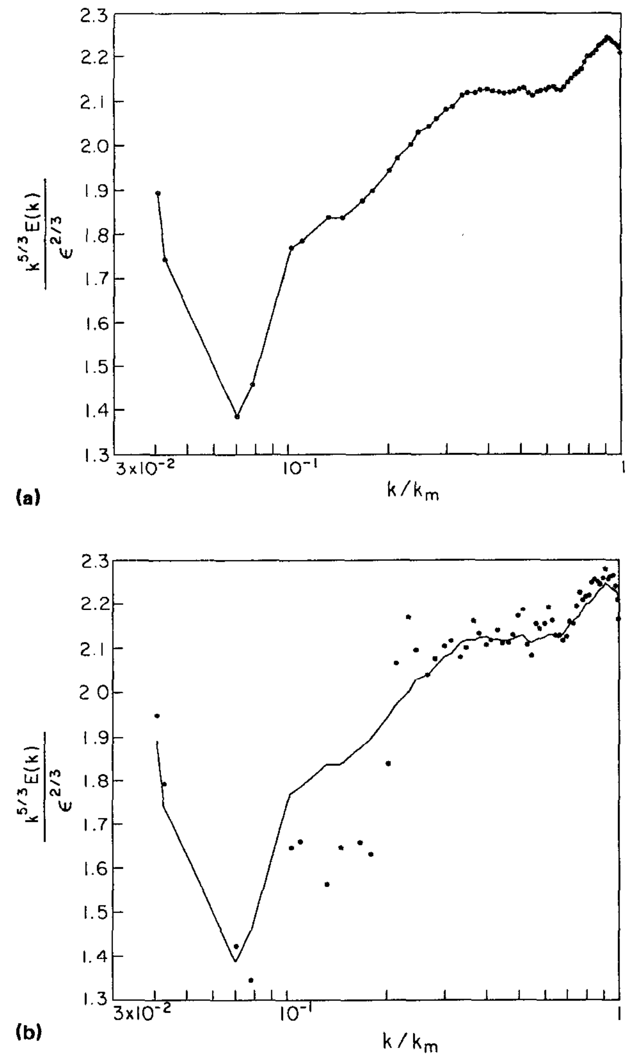


FIG. 7. Spectrum of $k^{5/3}E(k)/\epsilon^{2/3}$ computed from the LES of stationary turbulence. (a) Time average; (b) single instant of time.

For an inertial subrange to develop at a smaller wave number, we need to improve the artificial force we use so that it better represents the physical cascade of energy from scales with wave numbers smaller than k_0 into our resolved scales.

Examining Fig. 7(a), we observe that the $k^{-5/3}$ spectrum is best fitted for wave numbers $0.3 < k/k_m < 0.7$. For this range of wave numbers, the Kolmogorov constant is observed to be slightly greater than 2.1. The final value we used in the EDQNM model to compute the subgrid terms was $\text{Ko} = 2.1$. Use of a lesser or greater value of Ko in the EDQNM model resulted in substantially larger deviations from the $k^{-5/3}$ spectrum than that finally obtained.

Our simulation results indicate a Kolmogorov constant substantially larger than experimentally acceptable values (Table I). A plausible explanation given for this (Rogallo[44]) is that the discretization of wave numbers due to the imposition of periodic boundary conditions may inhibit the transfer, resulting in a larger value of Ko. To test this hypothesis, we have performed an LES using a 128^3 lattice, with all wave numbers less than two truncated (i.e., we take the minimum wave number $k_0 = 2$). The forcing and subgrid model we use is identical to that used in the 64^3 LES. This ensures that the 128^3 LES is identical to the 64^3 LES, except for a finer wave number discretization in the former. A plot of the time average of $k^{5/3}E(k,t)/\epsilon(t)^{2/3}$ is shown in Fig. 8(a). A better $k^{-5/3}$ energy spectrum appears to have developed. However, Ko is now even slightly larger, being approximately 2.15. Hence, we conclude that the periodic boundary conditions are not the cause of the large value of Ko observed in our simulation.

In Fig. 8(b), we plot the time average of the energy spectrum (prior to normalization) resulting from the 128^3 simulation. The appearance of a $k^{-5/3}$ power law over a wide range of wave numbers is quite remarkable.

We have also simulated a velocity field using an eddy viscosity model.[27,29] The eddy viscosity model we use results in the identical subgrid scale transfer as the model used to compute the energy spectrum displayed in Fig. 7. The time average of $k^{5/3}E(k,t)/\epsilon(t)^{2/3}$ is shown in Fig. 9. Although a log–log plot of $E(k)$ vs k would show a reasonable $k^{-5/3}$ power law, we observe that a computation of a reliable value for Ko from this simulation data is impossible.

We have remeasured the subgrid terms in our 64^3 stationary LES by taking a fictitious cut in wave number space at $k_c = 16$. The subgrid terms we have measured are actually only the effect of wave numbers between k_c and k_m on scales with wave numbers less than k_c. The results are shown in Fig. 10. The asymptotically constant eddy-damping term is observed to be approximately 0.08. If one considers the missing subgrid scale interactions (wave numbers greater than

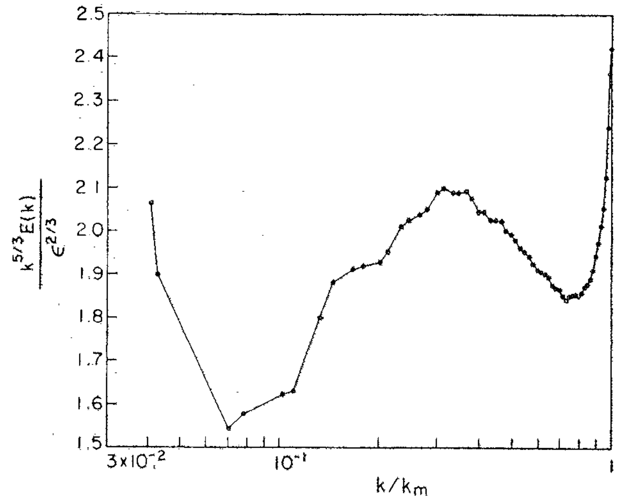


FIG. 9. Time-averaged spectrum of $k^{5/3}E(k)/\epsilon^{2/3}$ computed from the LES of stationary turbulence using an eddy viscosity subgrid model.

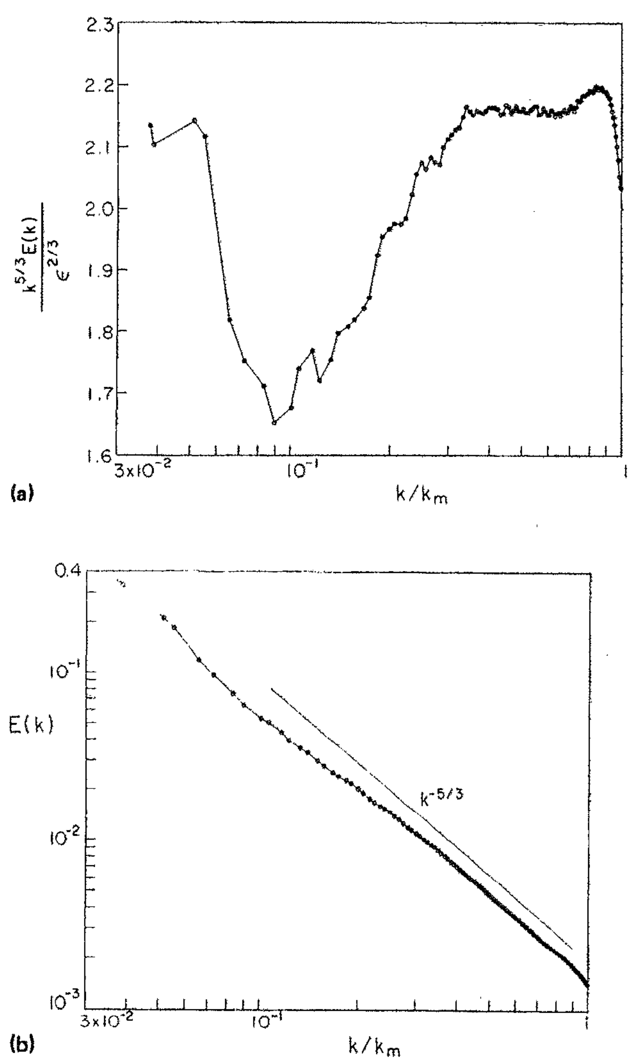


FIG. 8. Time-averaged spectra computed from the LES of stationary turbulence with 128^3 grid points and $k_0 = 2$. (a) $k^{5/3}E(k)/\epsilon^{2/3}$; (b) $E(k)$.

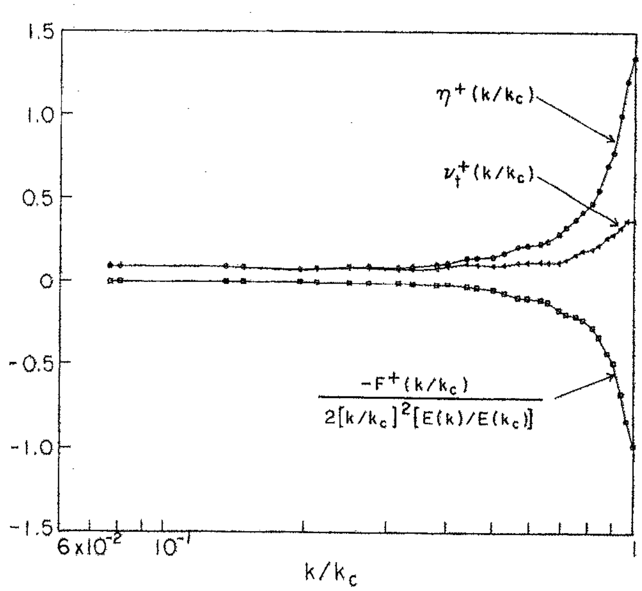


FIG. 10. Subgrid eddy viscosity, eddy-damping term, and spectrum of the stochastic force computed from the LES of stationary turbulence.

k_m), one can easily show[13,42] that for $k/k_c \ll 1$, $\eta^+(k/k_c) \to 0.14$. Since the asymptotic EDQNM result is $\eta^+(k/k_c) \to 0.44\ \mathrm{Ko}^{-3/2}$, one observes that for the EDQNM model result to be consistent with the simulation data, we must take Ko = 2.1. This further corroborates the value of Ko obtained from the energy spectrum computed in our LES.

V. DISCUSSION

Today, the most extensively used subgrid model for homogeneous turbulence is the eddy viscosity subgrid model. We have presented physical arguments which suggest that this model may be incomplete, especially for wave numbers near the maximum wave number k_m of the numerically resolved scales. We further showed how an improved model of the subgrid scales may be constructed by using two separate terms: an eddy viscosity (the eddy-damping term η), and a stochastic force that is uncorrelated in time (the stochastic force f_i with spectrum F). Both terms of this η-F subgrid model were computed in two different ways: from the EDQNM stochastic equation, and from the Navier–Stokes equation using the results of a DNS. When computing η and F in a DNS, two subgrid scale statistics are required. In addition to the subgrid scale energy transfer, a fourth-order two-time moment of the velocity field must also be computed. In contrast, the subgrid scale energy transfer—which is the only statistic required to compute a subgrid eddy viscosity—is a single-time third-order moment. Hence the η-F model contains substantially more statistical information concerning the subgrid scales than does the eddy viscosity model. This additional information results in a more complete physical model of the subgrid scales. In the η-F subgrid model, a two-way exchange of energy occurs between the numerically resolved scales and the modeled subgrid scales, whereas in the eddy viscosity model, energy is transferred one way from the resolved scales to the subgrid scales. Furthermore, the η-F subgrid model contains an additional effect that is completely absent in the eddy viscosity subgrid model: the random sweeping of small scales by large scales.

The computation of the subgrid model terms using a DNS showed that the results of the EDQNM model are consistent with the Navier–Stokes equation. We then applied the η-F subgrid model computed from the EDQNM equations to the large-eddy simulation of the Kolmogorov inertial subrange energy spectrum. We showed that the use of an η-F subgrid model results in a closer $k^{-5/3}$ energy spectrum than an identical simulation using an eddy viscosity subgrid model. The Kolmogorov constant of this inertial subrange was determined to be approximately 2.1.

The high Reynolds number atmospheric turbulence experiments[24,25] performed in 1977 yield a value $1.4 < \mathrm{Ko} < 1.7$ for the Kolmogorov constant. These values are obtained from measured one-dimensional energy spectra by assuming isotropy (although it has been suggested[45] that isotropy may not be satisfied in the inertial subrange of these high Reynolds number experiments). Although our simulation result Ko = 2.1 is internally self-consistent, it is also appreciably higher than the accepted experimental values. However, our result is in agreement with other numerical simulations[11,14] performed without a subgrid model.

In conclusion, it appears that the value of Ko obtained from numerical simulations is approximately 30% higher than that obtained in high Reynolds number atmospheric experiments. The origin of this discrepancy remains to be understood.

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