SELF-SIMILAR DECAY OF TWO-DIMENSIONAL TURBULENCE

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Recent numerical simulations [2] of decaying two-dimensional homogeneous turbulence at high Reynolds numbers have exhibited an approximate self-similar evolution of the energy spectrum. We analyze here the theoretical implications of self-similarity.

We consider a two-dimensional velocity field $\mathbf{u} = (u_1, u_2, 0)$ with vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u} = (0, 0, \omega)$. The equations for the mean-square velocity ($2\times$ energy) and mean-square vorticity ($2\times$ enstrophy) are given by [1]

$$\frac{d}{dt}\langle \mathbf{u}^2 \rangle = -2\nu \langle \omega^2 \rangle, \quad \frac{d}{dt}\langle \omega^2 \rangle = -2\nu \langle (\nabla\omega)^2 \rangle, \tag{1}$$

where ν is the kinematic viscosity of the fluid. Our main objective is to determine the long-time decay laws of the energy and enstrophy.

We begin with some relevant definitions. In two-dimensional turbulence, the following characteristic length scales λ and μ are of some importance:

$$\lambda = \langle \mathbf{u}^2 \rangle^{1/2} / \langle \omega^2 \rangle^{1/2}, \qquad \mu = \langle \omega^2 \rangle^{1/2} / \langle (\nabla\omega)^2 \rangle^{1/2}. \tag{2}$$

The Reynolds numbers R_λ and R_μ constructed from these length scales are defined as

$$R_\lambda = \frac{\langle \mathbf{u}^2 \rangle^{1/2} \lambda}{\nu}, \qquad R_\mu = \frac{\langle \mathbf{u}^2 \rangle^{1/2} \mu}{\nu}, \tag{3}$$

and their ratio ρ will play a pivotal role in our analysis:

$$\rho = R_\lambda / R_\mu = \lambda / \mu \,. \tag{4}$$

We will consider separately two distinct kinds of self-similar decay. First, we consider *complete self-similarity* for which the energy spectrum during the decay maintains its shape on log-log axes over all wave numbers. Second, we consider *partial self-similarity* for which the spectral shape is maintained only over scales directly unaffected by viscosity.

Using the length scale λ, we look for an energy spectrum of self-similar form

$$E(k,t) = \langle \mathbf{u}^2 \rangle \lambda \widehat{E}(\widehat{k}) \,, \qquad \widehat{k} = k\lambda. \tag{5}$$

For complete self-similarity, the length scale ratio ρ is necessarily constant during the decay: physically, all length scales must grow at the same rate. It is also simple to determine that ρ is directly related to the time-independent self-similar spectrum via

$$\rho = \frac{\left[\int_0^\infty \widehat{E}(\widehat{k}) d\widehat{k}\right]^{\frac{1}{2}} \left[\int_0^\infty \widehat{k}^4 \widehat{E}(\widehat{k}) d\widehat{k}\right]^{\frac{1}{2}}}{\int_0^\infty \widehat{k}^2 \widehat{E}(\widehat{k}) d\widehat{k}}. \tag{6}$$

We assume that for times $t \geq t_*$ following an initial transient, the spectrum undergoes complete self-similar decay with $\rho = \rho_*$ constant. Assuming constant ρ during the decay results in closure of (1), and an analytical solution for the decay laws is most easily determined by first obtaining an equation for λ:

$$\frac{d}{dt}\lambda^2 = 2\nu(\rho^2 - 1); \tag{7}$$

which may be integrated immediately from t_* to t:

$$\lambda^2 = 2\nu(\rho_*^2 - 1)(t - t_*) + u_*^2/\omega_*^2, \tag{8}$$

where u_* and ω_* are the root-mean-square values of the velocity and vorticity at $t = t_*$. The energy and enstrophy equations may then be subsequently integrated to obtain

$$\langle \mathbf{u}^2 \rangle = u_*^2 \left[1 + 2(\rho_*^2 - 1) R_{\lambda *}^{-1} \omega_* (t - t_*)\right]^{-1/(\rho_*^2 - 1)} , \tag{9}$$

$$\langle \omega^2 \rangle = \omega_*^2 \left[1 + 2(\rho_*^2 - 1) R_{\lambda *}^{-1} \omega_* (t - t_*)\right]^{-\rho_*^2/(\rho_*^2 - 1)} , \tag{10}$$

where $R_{\lambda *}$ is the value of R_λ at $t = t_*$. Fully-developed turbulence corresponds to large values of ρ_* signifying a wide separation of scales between λ and μ. For asymptotically large ρ_*, we see from (9) and (10) that the energy is conserved for finite times, (though for fixed ρ_* as $t \to \infty$, the energy decays to zero), and that the enstrophy decays as

$$\langle \omega^2 \rangle = \frac{u_*^2}{2\nu\rho_*^2} t^{-1} \,. \tag{11}$$

Previously [2], we have shown that complete self-similarity also occurs for decaying turbulence at constant R_λ. The Reynolds number R_λ can be shown to satisfy the equation

$$\frac{d}{dt}R_\lambda = (\rho^2 - 2)\langle\omega^2\rangle^{1/2}, \tag{12}$$

so that decay with constant R_λ corresponds to $\rho_*^2 = 2$. It can be further shown that the analytical results found in [2] can be recovered directly from (9) and (10).

The time-evolution equation for the energy spectrum is written as

$$\frac{\partial}{\partial t}E(k,t) + 2\nu k^2 E(k,t) = T(k,t)\,, \tag{13}$$

where $T(k,t)$ is the nonlinear transfer spectrum. We now transform (13) into an equation for the self-similar spectrum $\widehat{E}(\widehat{k})$. Using (1), (5) and (7), and after some algebraic manipulations, we find

$$2(\widehat{k}^2 - 1)\widehat{E}(\widehat{k}) + \left(\rho^2 - 1\right)\left(\widehat{E}(\widehat{k}) + \widehat{k}\frac{d}{d\widehat{k}}\widehat{E}(\widehat{k})\right) = \langle\mathbf{u}^2\rangle^{-3/2}R_\lambda T(k,t)\,. \tag{14}$$

For complete self-similar decay with $\rho = \rho_*$ constant, the transfer spectrum must thus evolve with self-similar form

$$T(k,t) = \langle\mathbf{u}^2\rangle^{3/2}R_\lambda^{-1}\widehat{T}(\widehat{k})\,, \qquad \widehat{k} = k\lambda, \tag{15}$$

which depends explicitly on the viscosity ν through the Reynolds number R_λ. This differs from standard two-point closure theories [4], for which the factor R_λ^{-1} is absent.

On the other hand, partial self-similarity assumes that the self-similar form (5) is valid only over wave numbers for which viscosity is negligible. The transfer scaling given by (15) is thus unsuitable because of its direct dependence on viscosity. Partial self-similar decay solutions may be obtained from (14) under the assumption $\rho \to \infty$, asymptotically. Equation (14) then reduces at long-times to

$$\left(\widehat{E}(\widehat{k}) + \widehat{k}\frac{d}{d\widehat{k}}\widehat{E}(\widehat{k})\right) = \langle\mathbf{u}^2\rangle^{-3/2}\frac{R_\lambda}{\rho^2}T(k,t)\,. \tag{16}$$

In order for viscosity to cancel from the right-hand side of (16), ρ^2 must scale like

$$\rho^2 = cR_\lambda, \tag{17}$$

where c is a nondimensional proportionality constant. Together with (4), this implies $R_\lambda = cR_\mu^2$.

The relationship between ρ and R_λ given by (17) permits analytical closure of (1). We obtain for the enstrophy equation

$$\frac{d}{dt}\langle\omega^2\rangle = -2c\langle\omega^2\rangle^{3/2}, \tag{18}$$

which may be integrated from a time t_* after which partial self-similarity occurs:

$$\langle\omega^2\rangle = \omega_*^2\left[1 + c\omega_*(t - t_*)\right]^{-2}. \tag{19}$$

Further integration of the energy equation results in the solution

$$\langle\mathbf{u}^2\rangle = u_*^2\left[1 - \frac{2R_{\lambda_*}^{-1}\omega_*(t - t_*)}{1 + c\omega_*(t - t_*)}\right]. \tag{20}$$

The long-time asymptotic solutions of (19) and (20) are given by

$$\langle\mathbf{u}^2\rangle = u_*^2\left[1 - 2(cR_{\lambda_*})^{-1}\right], \quad \langle\omega^2\rangle = (ct)^{-2}. \tag{21}$$

The assumption of partial self-similarity thus results in the t^{-2} enstrophy decay law originally proposed by Batchelor [1], and found by standard two-point closures [4].

For partial self-similar decay the energy approaches a nonzero value asymptotically (apart from the special case $\rho_*^2 = 2$ discussed earlier, for which $\langle\mathbf{u}^2\rangle$ vanishes in (21)). This presents a significant physical difference between complete and partial self-similarity.

Our previous direct numerical simulations [2] seem to support complete self-similarity in decaying two-dimensional turbulence, where the asymptotic decay law of the enstrophy for high initial Reynolds numbers was found to be approximately $t^{-0.8}$. Numerical simulations currently in progress provide even stronger support in favor of complete self-similarity. Hence the partial self-similar decay obtained by standard two-point closure theories accounts for the disagreement found earlier between simulations and theory [3].

References

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