DECAYING TURBULENCE IN TWO AND THREE DIMENSIONS

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A review of decaying isotropic turbulence in two and three dimensions is presented. In both dimensions, exact results can be obtained by assuming that for small Reynolds numbers the nonlinear convective terms in the equation of motion are negligible. These analytical results obtained from a linear equation demonstrate explicitly that the decay laws depend on the form of the energy spectrum near zero wavenumber. In three dimensions, high Reynolds number decay laws can be obtained by assuming that these laws continue to depend on invariant or near-invariant low wavenumber spectral coefficients but become independent of viscosity. Results of numerical simulations are presented which smoothly connect the asymptotically high and low Reynolds number solutions. In two dimensions however, high Reynolds number solutions can not be as easily obtained since the dominant nonlinear energy cascade is from small-to-large scales so that an assumption such as the invariance or near-invariance of low wavenumber spectral coefficients becomes untenable. However, exact analytical results in two dimensions can be determined at a transitional Reynolds number, below which final period of decay solutions result and above which the turbulence evolves with increasing Reynolds number.

I. Introduction

Decaying turbulent motions constitute an important class of turbulent flows. Although these flows never attain statistical stationarity, the turbulence decay may nevertheless be self-similar. The meaning of self-similarity in this context is that the turbulence spectrum at time t can be made to coincide with the spectrum at earlier times through a time-dependent scaling of the wavenumber and spectral amplitude. On a log-log plot, the spectrum would then appear to decay without change of shape. Typically, the desired axes scalings are power-laws in time and are related to the decay laws for the mean-square velocity and characteristic length scales of the turbulence. In this paper, we review known similarity solutions of decaying isotropic turbulence in two and three dimensions. We will see that the physics of these two flows is quite different as is the nature of the similarity states which may develop.

II. Decaying three-dimensional turbulence

That the decay laws of homogeneous isotropic turbulence depend on the form of the energy spectrum at low wavenumbers can be shown explicitly from the governing equations for the final period of decay[1]. Defining the energy spectrum $E(k,t)$ in the usual way to be the spherically integrated three-dimensional Fourier transform of the covariance

$\frac{1}{2}\langle u_i(\mathbf{x},t)u_i(\mathbf{x}+\mathbf{r},t)\rangle$, the equation governing its time-evolution can be written in the form

$$\frac{\partial E(k,t)}{\partial t} + 2\nu k^2 E(k,t) = T(k,t)\,, \tag{1}$$

where $T(k,t)$ represents the nonlinear transfer. During the final period of decay, one assumes that $T(k,t)$ is negligible so that the analytical solution for the spectrum is given trivially by

$$E(k,t) = E(k,0)\exp\left(-2\nu k^2 t\right). \tag{2}$$

The decay law of the mean-square velocity of the turbulence, or equivalently, (twice) the kinetic energy of the turbulence per unit mass, is obtained upon integration of (2) over k. As time advances, it is clear that the support for this integral moves to smaller and smaller wavenumbers. Changing variables to $\eta = k\sqrt{\nu t}$, the integral of (twice) Eq. (2) becomes

$$\langle \mathbf{u}^2\rangle = 2(\nu t)^{-1/2}\int_0^\infty E(\eta/\sqrt{\nu t},0)\exp\left(-2\eta^2\right)d\eta\,. \tag{3}$$

To obtain the asymptotic solution of (3) as $t\to\infty$, the form of the energy spectrum near zero wavenumber is required. Following the work of Batchelor & Proudman[2] and Saffman[3], we assume that the asymptotic expansion of the energy spectrum near $k=0$ can be written as one of

$$E(k,t) = 2\pi B_0 k^2 + \mathrm{o}(k^2)\,, \quad \text{or} \quad E(k,t) = 2\pi B_2(t)k^4 + \mathrm{o}(k^4)\,. \tag{4}$$

Saffman[3] showed that B_0 is an invariant during the entire history of the turbulence decay as a consequence of the transfer spectrum $T(k,t)$ in (1) being of order k^4. When B_0 is zero initially, it remains so and the expansion of $E(k,t)$ follows the second of (4) with B_2 in general a function of time except during the final period[2].

Taking the limit of (3) as $t\to\infty$, one uses (4) and obtains the exact solutions

$$\langle \mathbf{u}^2\rangle = a_0(4\pi B_0)(\nu t)^{-3/2}\,, \quad \text{or} \quad \langle \mathbf{u}^2\rangle = a_2(4\pi B_2)(\nu t)^{-5/2}\,, \tag{5}$$

where the integrals a_0 and a_2 are given explicitly by

$$a_0 = \int_0^\infty \eta^2\exp\left(-2\eta^2\right)d\eta = \frac{1}{8}\sqrt{\frac{\pi}{2}}\,, \tag{6a}$$

$$a_2 = \int_0^\infty \eta^4\exp\left(-2\eta^2\right)d\eta = \frac{3}{32}\sqrt{\frac{\pi}{2}}\,. \tag{6b}$$

Furthermore, the characteristic length scale of the turbulence increases as $\sqrt{\nu t}$ in the final period due to the erosion of the spectrum at large wavenumbers through viscous energy dissipation, so that a Reynolds number formed by $\langle \mathbf{u}^2\rangle^{1/2}l/\nu$ eventually decays to zero as the fluid returns to rest.

The $t^{-5/2}$ and $t^{-3/2}$ decay laws were first determined by Batchelor[4] and Saffman[3], respectively. Apart from the numerical constants, these decay laws may be determined also by dimensional analysis, with $[B_0] = l^5/t^2$ and $[B_2] = l^7/t^2$, if one assumes that the decay depends linearly on the low wavenumber coefficients, and on ν (with $[\nu] = l^2/t$) and t alone.

Exact closure of (1) is unknown at high Reynolds numbers. Nevertheless, asymptotic decay laws of the energy have been obtained by working in analogy to the final period of decay solutions. One assumes that the scaling of the energy now depends nonlinearly on the low wavenumber spectral coefficients and the time t alone. Viscosity no longer enters the scaling law as an independent parameter. This is in accordance with the usual turbulence phenomenology that the energy dissipation rate has a nonzero limit with vanishing viscosity. By dimensional analysis, one then obtains

$$\langle \mathbf{u}^2 \rangle = c_0 (4\pi B_0)^{2/5} t^{-6/5}\,, \quad l \propto B_0^{1/5} t^{2/5}\,, \tag{7a}$$

or

$$\langle \mathbf{u}^2 \rangle = c_2 (4\pi B_2)^{2/7} t^{-10/7}\,, \quad l \propto B_2^{1/7} t^{2/7}\,, \tag{7b}$$

where proportionality constants for the mean-square velocity have been inserted for later use. The time-dependence in (7a) is expected to be exact because of the invariance of B_0 whereas the explicit time-dependence in (7b) is modified by the time-dependence of B_2.

Equation (7a) was first obtained by Saffman[5], and (7b) much earlier by Kolmogorov[6], who assumed that B_2 was invariant as argued by Loitsianskii[7]. At very high Reynolds numbers, closure calculations and large-eddy simulations show that $B_2(t) \propto t^{\gamma}$, with $\gamma = 0.16$ from closure[8] and 0.25 from large-eddy simulation[9]. Use of the latter time-dependence of B_2 in (7b) results in a somewhat slower decay law for the mean-square velocity of $t^{-1.36}$ instead of the Kolmogorov law $t^{-1.43}$.

The high Reynolds number decay laws given by (7) have been confirmed by large-eddy simulations[10]. The decay of the energy spectrum with k^2 and k^4 low wavenumber asymptotics is shown in Figs. 1(a) and (b) and the computation of the decay exponents of the mean-square velocity is shown in Fig. 2. A self-similar decay of the energy spectrum has been observed by scaling the spectrum by $(\langle \mathbf{u}^2 \rangle(t) l(t))$ and the wavenumber by $l(t)^{-1}$. The Reynolds number of the flow field $\langle \mathbf{u}^2 \rangle^{1/2} l/\nu$ decays in time during the flow evolution. High Reynolds number turbulence will thus eventually decay to low Reynolds numbers and the final period of decay solutions given by (5) should eventually become valid.

Equations (5) and (7) give the asymptotic energy decay at the lowest and highest Reynolds numbers, respectively. Exact solution including the proportionality constant is know for the final period of decay while only the overall scaling is known at high Reynolds numbers. Numerical simulations can be used to connect the two asymptotic solutions for the decay. In particular, for nonzero initial B_0, the proportionality constant c_0 in (7a) should be independent of the history of the decay, and it becomes possible to construct a universal curve which follows the decay of the energy from the highest Reynolds numbers into the final period.

To compute this universal decay curve, it is necessary to first define nondimensional variables. According to the asymptotic decay laws (5) and (7a), the only two dimensional parameters (apart from time t) of the problem are B_0 and ν, and these parameters may

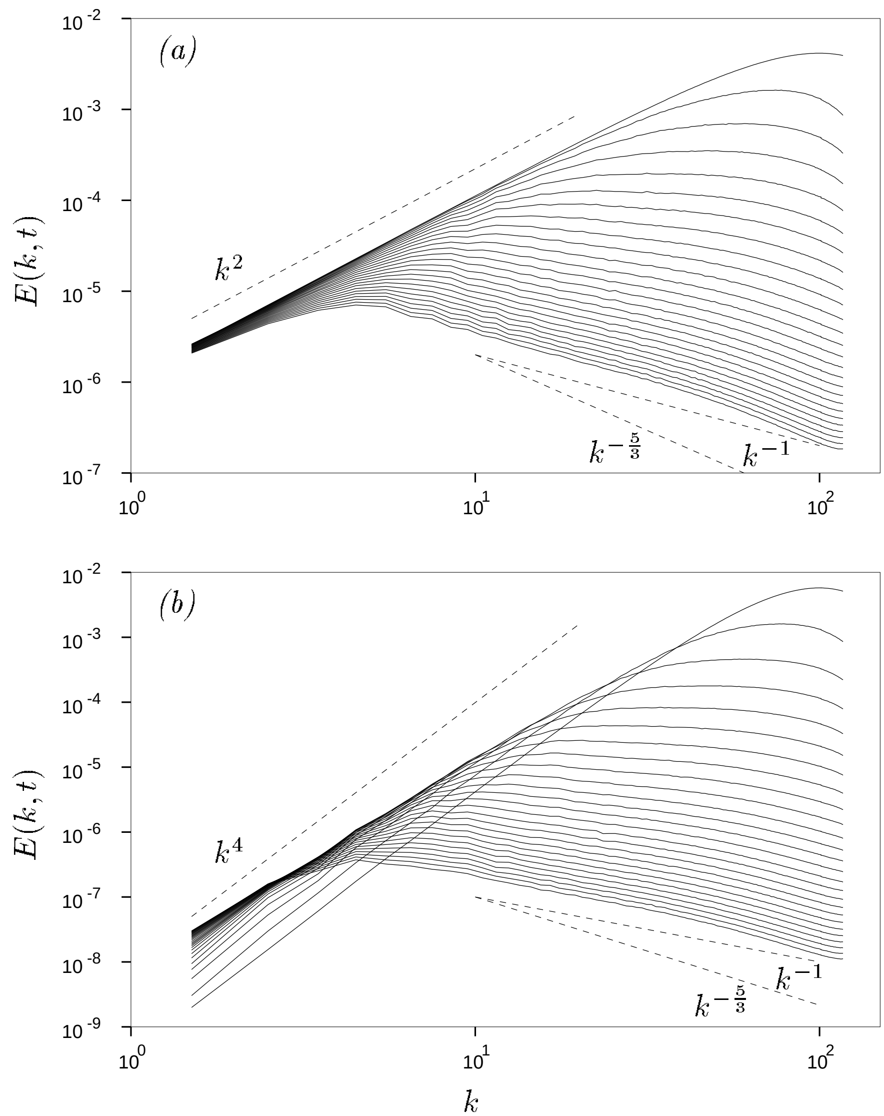

FIG. 1. Time-evolution of the energy spectrum in three-dimensional turbulence: (a) with leading-order coefficient B_0; (b) with leading-order coefficient B_2.

be used to nondimensionalize our variables. Accordingly, we define a nondimensional time T and a nondimensional mean-square velocity U^2 by

$$t = (4\pi B_0)^2 T/\nu^5 \,, \quad \langle \mathbf{u}^2 \rangle = \nu^6 U^2/(4\pi B_0)^2 \,. \tag{8}$$

The factor of 4π has been introduced for convenience. In these nondimensional variables, the final period and high Reynolds number decay laws become

$$U^2 = a_0 T^{-3/2} \,, \quad U^2 = c_0 T^{-6/5} \,, \tag{9}$$

with a_0 given by (6a) and c_0 the constant defined in (7a). The proportionality constant c_0 can be computed by a large-eddy simulation[10] and one determines that $c_0 \approx 4.0$. The

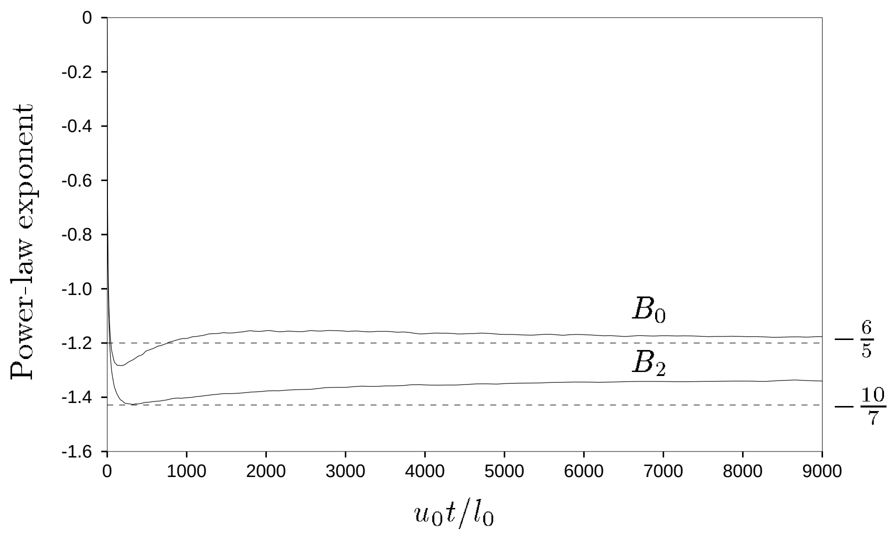

FIG. 2. Time evolution of the logarithmic derivative of $\langle \mathbf{u}^2 \rangle$ for decaying three dimensional turbulence at high Reynolds numbers. The solid lines are the results of the large-eddy simulations and the dashed lines are the analytical results assuming the invariance of the low-wavenumber coefficients.

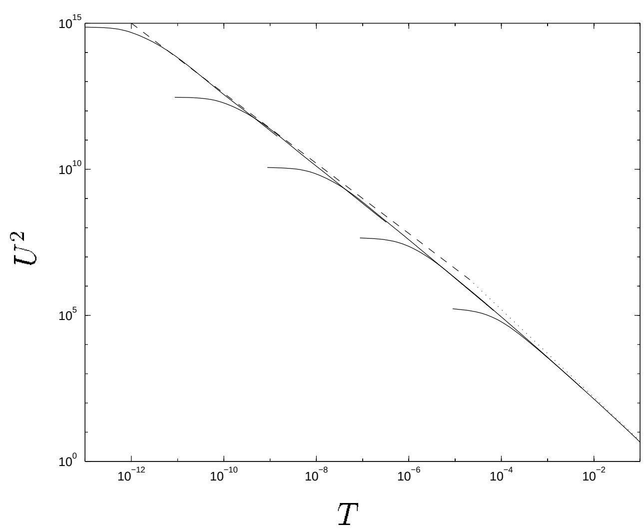

FIG. 3. The universal decay curve of U^2 versus T obtained from five numerical simulations over a range of viscosities. The dashed and dotted lines correspond to the asymptotic high and low Reynolds number solutions, respectively.

high Reynolds number and final period of decay solutions thus match at a transition time of $T_* = (a_0/c_0)^{10/3} \approx 2.04 \times 10^{-5}$.

The universal decay curve from small to large times, or equivalently large to small Reynolds numbers, has been computed by smoothly connecting the results of large-eddy and direct numerical simulations. Five numerical simulations of 256^3 resolution have been

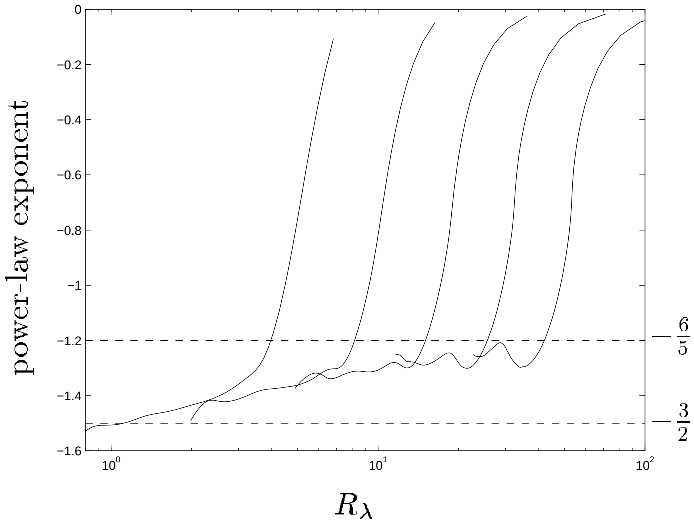

FIG. 4. The logarithmic derivative of $\langle \mathbf{u}^2 \rangle$ versus the microscale Reynolds number R_λ for decaying three-dimensional turbulence, obtained from the computations shown in Fig. 3. The dashed lines are the asymptotic high and low Reynolds number solutions.

performed. An initial isotropic random phase velocity field with period 2π in each direction is constructed from the energy spectrum

$$E(k,0) = 2\pi B_0 k^2 \exp\left[-\left(k/k_p\right)^2\right], \tag{10}$$

with $B_0 = u_0^2/\pi^{3/2}k_p^3$, and $k_p = 100$, $u_0 = 1$. The kinematic viscosity ν is varied from 4.0×10^{-5} to 1.6×10^{-3}. In the calculation, the viscosity is augmented by an eddy viscosity of the Kraichnan[11], Chollet-Lesieur[12] form

$$\nu_e(k|k_m,t) = \left[0.145 + 5.01\exp\left(\frac{-3.03k_m}{k}\right)\right]\left[\frac{E(k_m,t)}{k_m}\right]^{1/2}, \tag{11}$$

where k_m is the maximum wavenumber of the simulation. For well-resolved simulations at low Reynolds number, the eddy-viscosity is negligible with respect to ν and the calculation is a direct numerical simulation.

A plot of U^2 versus T is shown in Fig. 3. The dashed line represents the asymptotic high Reynolds number result and the dotted line the exact final period of decay solution. Apart from initial transients which should be discarded, the five simulations taken together show the entire evolutionary history of a decaying isotropic turbulence with $B_0 \neq 0$.

It is of further interest to determine the power-law exponent of the energy decay (*i.e.*, its logarithmic derivative) as a function of an experimentally measurable quantity such as the microscale Reynolds number R_λ, defined as

$$R_\lambda = \sqrt{\frac{5}{3}}\frac{\langle \mathbf{u}^2 \rangle}{\sqrt{\nu\epsilon}}. \tag{12}$$

A plot of $d \log \langle \mathbf{u}^2 \rangle / d \log t$ versus R_λ for the five simulations is shown in Fig. 4. It appears that high Reynolds number scaling occurs when $R_\lambda > 30$, while the final period of decay solution requires $R_\lambda < 1$. High Reynolds number grid turbulence experiments[13] reached an R_λ of 70, which seems to be sufficiently large to attain asymptotic decay laws, while low Reynolds number experiments[14] have been performed down to $R_\lambda = 4$, which may be too large to be considered a true final period of decay.

III. Decaying two-dimensional turbulence

Decaying two-dimensional turbulence differs qualitatively from that of three-dimensional turbulence due to the absence of the vortex stretching term in the vorticity evolution equation. As in three-dimensional turbulence, the evolution equation for the mean-square velocity (2× energy) is given by

$$\frac{d}{dt} \langle \mathbf{u}^2 \rangle = -2\nu \langle \omega^2 \rangle \,, \tag{13}$$

where ω is the vorticity $\omega = \nabla \times \mathbf{u}$. In two-dimensional turbulence only the vorticity component perpendicular to the plane of the turbulent motion is nonzero. The equation for the mean-square vorticity (2× enstrophy) is

$$\frac{d}{dt} \langle \omega^2 \rangle = -2\nu \langle (\nabla \omega)^2 \rangle \,, \tag{14}$$

which differs from three-dimensional turbulence in that the enstrophy is now bounded by its initial value. In three-dimensions, it is suspected that the enstrophy diverges in finite time as $\nu \to 0$, leading to an energy cascade from large-to-small scales. In two-dimensions, an enstrophy cascade has been postulated instead of an energy cascade[15].

Since the enstrophy is bounded by its initial value as a consequence of (14), the energy goes to a constant in the limit $\nu \to 0$. Nonzero enstrophy dissipation at constant energy implies that energy is transfered predominantly from small-to-large scales. This is easily illustrated by considering the evolution of a model energy spectrum which remains sharply peaked about a single wavenumber $k_*(t)$ so that $2E(k,t) = u_0^2 \delta(k - k_*)$, where δ is the usual Dirac-delta function. The energy and enstrophy are related to the energy spectrum by means of

$$\langle \mathbf{u}^2 \rangle = \int_0^\infty 2E(k,t) dk \,, \quad \langle \omega^2 \rangle = \int_0^\infty 2k^2 E(k,t) dk \,, \tag{15}$$

so that the model energy spectrum yields $\langle \mathbf{u}^2 \rangle = u_0^2$ and $\langle \omega^2 \rangle = u_0^2 k_*(t)^2$. An enstrophy decay with constant energy thus implies a continual decrease in k_* signifying an inverse cascade of energy from small-to-large scales.

This inverse cascade is in marked contrast to decaying three-dimensional turbulence where the dominant energy cascade is from large-to-small scales. Although the characteristic length scale of three-dimensional turbulence increases during the decay this is due to the dissipation of small scale energy with a corresponding depletion of large-scale energy (the source of the cascade). The assumption that the decay laws in two-dimensional turbulence depend on invariant, or near-invariant low wavenumber spectral coefficients becomes untenable at higher Reynolds numbers than that of the final period of decay. High Reynolds number decay laws in two-dimensional turbulence are currently unknown except by numerical experiment[16], and an early hypothesis of Batchelor[15] – equivalent to $k_* \propto (u_0 t)^{-1}$ in the simplistic energy spectrum model above – does not agree with the results of the numerical simulations.

As in three-dimensional turbulence, at very low Reynolds numbers the equation for the energy spectrum is closed under the assumption of a final period of decay for which the nonlinear transfer may be neglected. Equation (3) holds for two-dimensional turbulence, but instead of (4) we have

$$E(k,t) = \pi B_2 k^3 + o(k^3)\,, \tag{16}$$

where the phase space factor is now $2\pi k$ instead of $4\pi k^2$ because of integration over a circle rather than a sphere. Here, we only consider the case $B_0 = 0$ in anticipation that the initial low wavenumber form of the spectrum is irrelevant at high Reynolds numbers, and that the nonlinear transfer results in a nonzero value for the coefficient B_2. The final period of decay solution can thus be found analytically to be

$$\langle \mathbf{u}^2 \rangle = d_2 (2\pi B_2)(\nu t)^{-2}\,, \tag{17}$$

where

$$d_2 = \int_0^\infty \eta^3 \exp(-2\eta^2) d\eta = \frac{1}{8}\,. \tag{18}$$

The final period solution for the enstrophy may be found from (17) and (13) to be

$$\langle \omega^2 \rangle = d_2 (2\pi B_2)(\nu t)^{-3}\,. \tag{19}$$

The (microscale) Reynolds number of the turbulence, for convenience in this section defined as

$$R(t) = \frac{\langle \mathbf{u}^2 \rangle^{1/2} l}{\nu}\,, \quad \text{with} \quad l = \left[\frac{\langle \mathbf{u}^2 \rangle}{\langle \omega^2 \rangle} \right]^{\frac{1}{2}}\,, \tag{20}$$

decreases in time as $t^{-1/2}$ so that the final period of decay solution is internally consistent with the eventual approach of the turbulence to zero Reynolds number. However, at high Reynolds numbers we have already stressed that $\langle \mathbf{u}^2 \rangle$ should remain close to its initial value while $\langle \omega^2 \rangle$ decays. From (20), this implies that $R(t)$ increases during the decay. In other words, although the velocity scale stays constant during the decay, the length scale grows because of the inverse energy cascade. An argument can then be advanced that a transitional, or critical, Reynolds number R_c must exist, below which the nonlinear transfer eventually becomes negligible and a final period of decay solution results, and above which the nonlinear transfer becomes more and more essential to the dynamics since the Reynolds number of the turbulence increases in time.

Exactly at $R(0) = R_c$, the flow field evolves with constant Reynolds number and an analytical closure of (13) is possible. If we assume that a flow with initial Reynolds number R_c evolves in time so that after an initial transient the decay proceeds at constant

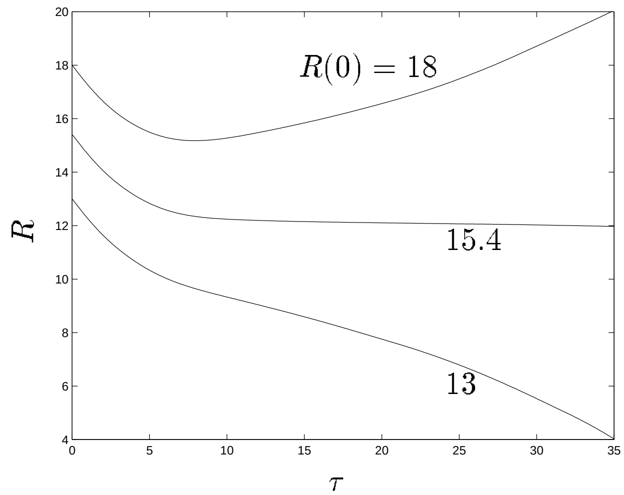

FIG. 5. Time-evolution of the Reynolds number $R(t)$ for initial values $R(0) = 13$, 15.4 and 18.

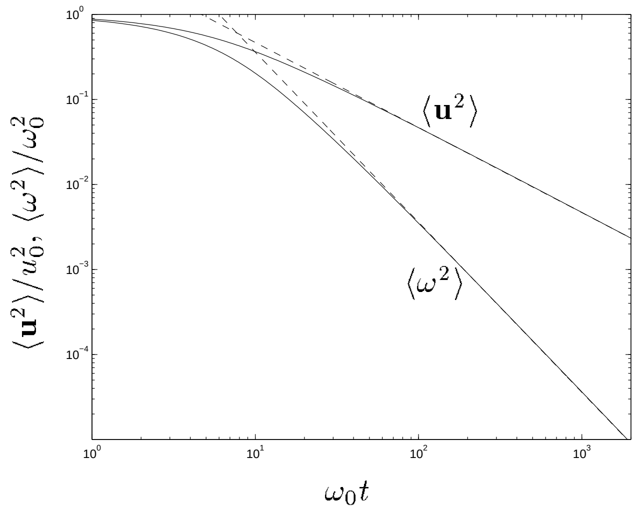

FIG. 6. Time-evolution of the energy and enstrophy for $R(0) = 15.4$ compared to the analytical results of (21).

Reynolds number R_c', then the enstrophy on the right-hand-side of (13) may be eliminated in favor of R_c'. Doing so, we obtain the closed evolution equation

$$\frac{d}{dt}\langle \mathbf{u}^2 \rangle = -\frac{2}{\nu R_c'^2} \langle \mathbf{u}^2 \rangle^2 \,, \tag{21}$$

which may be solved analytically. The asymptotic solution of (21) at large times is given by

$$\langle \mathbf{u}^2 \rangle = \frac{1}{2} \nu R_c'^2 t^{-1} \,, \quad \langle \omega^2 \rangle = \frac{1}{4} R_c'^2 t^{-2} \,, \tag{22}$$

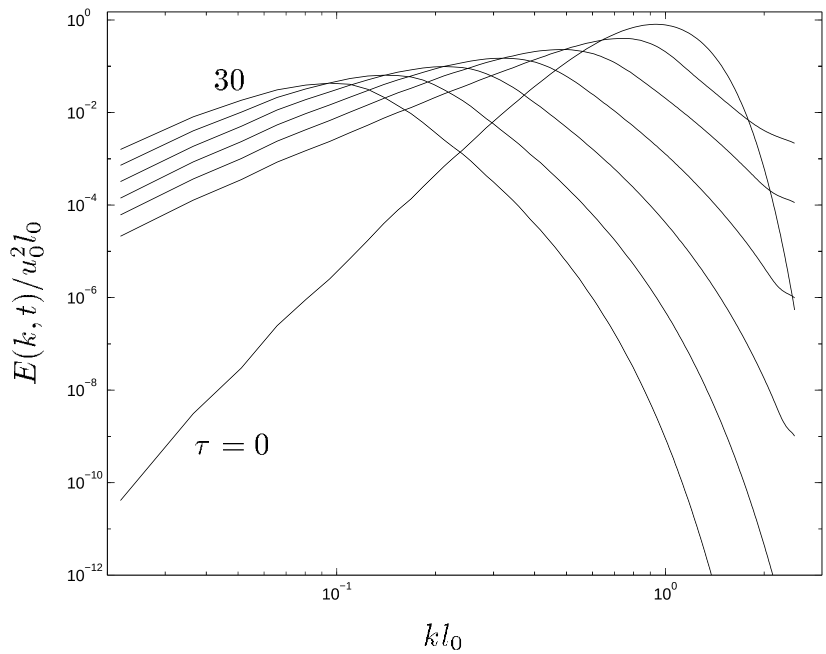

FIG. 7. Evolution of the energy spectrum in time with $R(0) = 15.4$. The times plotted correspond to $\tau = 0, 5, 10, \ldots, 30$.

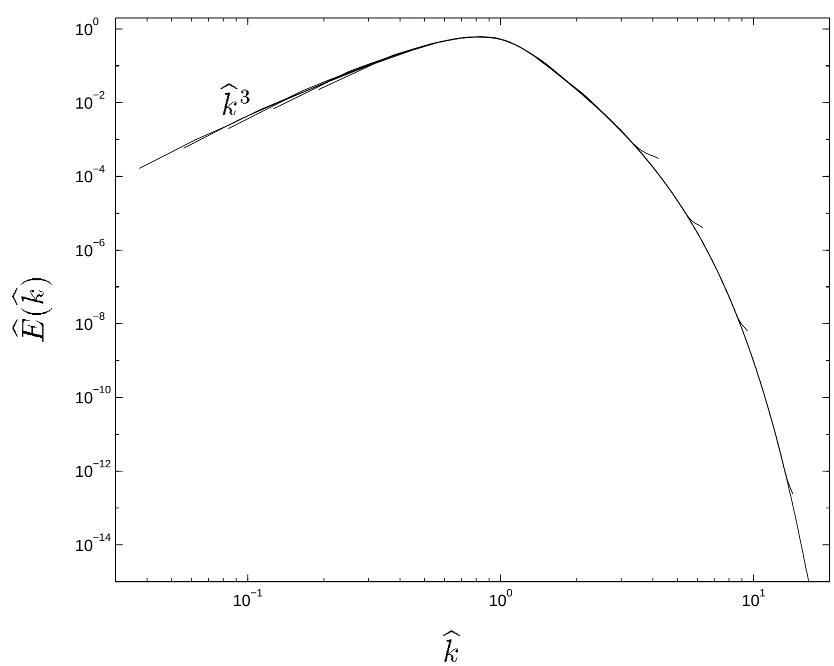

FIG. 8. Rescaling of the energy spectrum of Fig. 7. The times plotted correspond to $\tau = 10, 15, \ldots, 30$.

where the solution for the enstrophy has been determined using (13). The only unknown parameter in (22) is R_c' and its computation is easily obtained from relatively low resolution numerical simulations.

In Fig. 5, the evolution of the Reynolds number $R(t)$ verus $\tau(t)$ is shown for initial values above and below the critical value R_c, and at $R_c = 15.4$, where

$$\tau = \int_0^t dt \langle \omega^2 \rangle^{1/2} .$$

The value for R_c obtained here is slightly less than that reported earlier[16] due to the better resolution of the present simulations, though it is expected that R_c (and R_c') depends weakly on the initial conditions of the flow field. In Fig. 6, the decay of the energy and enstrophy are compared to the analytical results (22), with $R_c' = 12$ obtained from Fig. 5. Excellent agreement between the simulation and the theoretical scaling laws is observed.

As seen from Fig. 5, the analytical solution given by (22) is rather special, being unstable to perturbations in $R(0) = R_c$, with values slightly lower or higher resulting in asymptotically decreasing or increasing Reynolds numbers, respectively. Nevertheless, exact results in turbulence theory are rare, particularly when nonlinearity plays an essential role, and it is worthwhile to consider this solution in greater detail.

Exactly at $R(0) = R_c$, the energy spectrum $E(k,t)$ has been determined[16] to decay self-similarly over all wavenumbers with scaling

$$E(k,t) = \frac{\sqrt{2}}{2} \nu^{\frac{3}{2}} R_c'^2 t^{-\frac{1}{2}} \widehat{E}(\widehat{k}) ; \quad \widehat{k} = (2\nu t)^{\frac{1}{2}} k , \tag{23}$$

where the normalization of the self-similar spectrum $\widehat{E}(\widehat{k})$ is such that

$$\int_0^\infty \widehat{E}(\widehat{k}) d\widehat{k} = \int_0^\infty \widehat{k}^2 \widehat{E}(\widehat{k}) d\widehat{k} = \frac{1}{2} . \tag{24}$$

In Fig. 7, the evolution of the energy spectrum $E(k,t)$ at the times $\tau = 0, 5, 10, \ldots, 30$ is plotted. The spectra are smoother than those computed earlier[16] as a consequence of ensemble-averaging over a large number of independent realizations. In Fig. 8, the self-similar spectrum $\widehat{E}(\widehat{k})$ versus $\widehat{k}$ is plotted at the times $\tau = 10, 15, \ldots, 30$. A near-perfect collapse of the spectra is found indicating that the scaling given by (23) is exact.

With $E(k,t) = \pi k^3 B_2(t) + o(k^3)$ as $k \to 0$, the self-similar solution of (23) requires the scaling

$$B_2(t) \propto \nu^3 R_c'^2 t . \tag{25}$$

Apparently, nonlinearity plays an essential role in establishing this self-similar solution since $B_2(t)$ is independent of time during the final period when the nonlinear transfer is negligible. Also, an assumption such as the near-invariance of B_2 is clearly untenable, in contrast to three-dimensional decay where B_2 is weakly dependent on time.

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