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Evolution of decaying two-dimensional turbulence and self-similarity

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Abstract

We examine the consequences of self-similarity of the energy spectrum of two-dimensional decaying turbulence, and conclude that traditional closures (such as EDQNM and TFM) are consistent with this principle only if the regions of space contributing significantly to energy and enstrophy transfer comprise an ever diminishing region of space as time proceeds from the initial time of Gaussian chaos. Results of modifying the TFM according to this assumption are compared to the recent high resolution DNS of Chasnov.

1 Two-dimensional turbulence: Its scaling and self-similarity

Chasnov's (1996) 4096-resolution DNS suggests that decaying two-dimensional turbulence evolves *via* an approximate self-similar state. This surmise includes large scales, (for which the energy spectrum $E(k,t) \sim k^3$), the inertial range, ($E(k,t) \sim k^{-3}$), and a considerable portion of the dissipation range. We argue here that the constraint of high Reynolds number energy constancy and complete self-similarity implies that the energy (and enstrophy) transfer process must have an explicit time-dependence. We associate such time-dependence with the progressive spottiness of the energy transfer process: the turbulence lives on an ever-diminishing subset of the space available to it. The proposed modified transfer function, $T(k,t)$, ($T \ni (\dot{E} + 2\nu k^2 E) = T$), implied by complete self similarity together with energy constancy, is $T_{mod} = (t_0/t)^{1/2} T(k,t)$, as $t \to \infty$. This implies that the length scale increases as $t^{1/2}$ instead of t, *a la* Batchelor (1969). If the form of molecular dissipation changes, so must the power of the (t_0/t) factor in T_{mod}.

We write the spectral equations in a representation that is an adaptation of that of Linn (1961) to two-dimensions:

$$\partial_t U(k,t) = \mathcal{T}(k,t) - 2\nu k^2 U(k,t) \text{ with } E(k,t) \equiv 2\pi k U(k,t),\ T(k) \equiv 2\pi k \mathcal{T}(k) \quad (1)$$

Now put

$$U(k,t) = u_0^2 A(t) F(k\lambda(t), t),\ K \equiv k\lambda(t) \quad (2)$$

We search for functions $A(t)$ and $\lambda(t)$ for which $F(K,t)$ has no explicit dependence on t. Introducing (2) into (1) gives,

$$\dot{A}F + (A\dot{\lambda}/\lambda) K \partial F/\partial K + A\partial_t F = \mathcal{T}/(u_0^2) - 2\nu (K/\lambda)^2 AF \quad (3)$$

If (3) is to be satisfied with $\partial_t F(K,t) = 0$, the remaining terms must balance. Accordingly, we put

$$\dot{A}/A = C\dot{\lambda}/\lambda \Rightarrow A = \lambda^C \tag{4}$$

Note that the dimensions of $\mathcal{T} \rightarrow [U\sqrt{k^4U}]$, so that if $\mathcal{T}$ has the same time-dependence as the LHS of (3),

$$CA\dot{\lambda}/\lambda = \gamma A^{3/2}/\lambda^2, \ \lambda\dot{\lambda} = (\gamma/C)\lambda^{C/2} \tag{5}$$

If $\int_0^\infty dkE(k,t)$ is constant (as $t \rightarrow \infty$), (3) implies $C = 2$. Then

$$\lambda(t) = u_0 t + \lambda_0, \ A = \lambda^2, \ \dot{A} = 2\lambda u_0, \dot{A}/A = 2u_0/\lambda \tag{6}$$

If (6) is used, (3) becomes,

$$F + (1/2)KF_K + (\lambda/(2u_0)F_t = (1/(2\lambda u_0^3))\mathcal{T} - \nu K^2 F/(\dot{\lambda}\lambda) \tag{7}$$

However, without assuming any form for λ, we may use (4) to write (3) in a more convenient form for advancing $F(K,t)$ in time;

$$F_t = (\dot{\lambda}/\lambda)\{-(1/K)d(K^2F)/dK + (1/(u_0^2\lambda\dot{\lambda}))\mathcal{T}(k) - 2\nu K^2 F/(\dot{\lambda}\lambda)\} \tag{8}$$

Equation (8) cannot as yet be used to infer scaling of its various terms, because the argument of $\mathcal{T}$ is k, not $K \equiv \lambda k$. We make this conversion in the next section.

2 TFM analysis of $\mathcal{T}$ and its ingredients

For the sake of brevity, we state this conversion without any derivation:

$$\mathcal{T}(k) = \lambda u_0^3 \hat{\mathcal{T}}(K), \ \hat{\mathcal{T}}(K) \equiv \int_\triangle B_{KPQ}\hat{\Theta}_{KPQ}F(Q)(F(P) - F(K)) \tag{9}$$

$$d\{[\lambda/u_0]\hat{\Theta}_{KPQ}\}/dt = 1 - \{\hat{\eta}(K) + \hat{\eta}(P) + \hat{\eta}(Q)\}\hat{\Theta}_{KPQ} \tag{10}$$

and

$$\hat{\eta}(K) = \int_\triangle dPdQC_{KPQ}\hat{\Theta}_{KPQ}F(Q) + \nu K^2/(u_0\lambda) \tag{11}$$

So that (8) may be rewritten as,

$$F_t = (\dot{\lambda}/\lambda)\{-(1/K)d(K^2F)/dK + (u_0/\dot{\lambda})\hat{\mathcal{T}}(K) - 2\nu K^2 F/(\dot{\lambda}\lambda)\} \tag{12}$$

In order for $\partial_t F(K,t) = 0$ at small K, the viscous term may be neglected, and where we may use a $K \rightarrow 0$ expansion to $T(K)$ (Lesieur (1990)), we must have

$$(1/K)\partial(K^2F(K))/\partial K = \hat{\mathcal{T}}(K) \rightarrow \int_0^\infty dP\Theta_{0PP}F^2(P)/P^2 \tag{13}$$

This imposes an overall constraint on $F(K)$ if self-similar decay is to prevail over all K.

3 Scaling of transfer needed for self-similarity

Consider (8) which steps $F(K,t)$ forward in time, and let us take (6) for $\lambda(t)$. The first factor on the *rhs* simply changes dependent variable $[t]$ to something else $[ln(u_0 t + \lambda_0)]$, and this is of no real consequence. But within the $\{\cdot\}$, the effect of ν is progressively discounted (as compared to the first two terms) as $t \to \infty$. Hence, the impossibility, according to TFM, for a completely self-similar evolution of $E(k) = 2\pi k U(k,t)$.

On the other hand, we may ask what is necessary for $E(k,t)$ to be completely self-similar. According to (3), in order for the dissipation term (the last term on the *rhs*) to be $\sim$ the kinematic terms (the first two *lhs* terms) is:

$$A\dot{\lambda}/\lambda \sim 1, \quad \Rightarrow \lambda \sim t^{1/2} \tag{14}$$

where we use $C = 2$, which follows if at large R_λ, $E(t)$ is constant. If we also insist that $\mathcal{T}$ scales similarly with the kinematic and viscous terms, the non-dimensional factor, γ, in (5) must be some power of t, say,

$$\gamma(t) \sim (t/t_0)^p \tag{15}$$

Here, t_0 is a value of t less than which universal self-similarity is not obtained. Equation (5), with $C = 2$, $\lambda \sim t^{1/2}$ imply

$$p = -1/2 \tag{16}$$

Note that $\lambda \sim t^{1/2}$ assumes viscous losses $\equiv L \sim k^2 E(k)$. For $L \sim k^n E(k)$, we would obtain $\lambda \sim t^{1/n}$, with $-p = 1 - 1/n$.

To summarize, the requirement that the evolving energy spectrum be completely self-similar (as found by Chasnov (1996)), and that the energy transfer function, $\mathcal{T}$, have the functional form as given by TFM ((9)–(11)), requires $\mathcal{T}(k,t)$ to be re-scaled as:

$$\mathcal{T}(k,t) \to\sim (\lambda_0/\lambda(t))\mathcal{T}(k,t) \tag{17}$$

with

$$\lambda(t) = \sqrt{\lambda_0^2 + 2u_0\lambda_0 t} \tag{18}$$

With this re-scaling, each term within $\{\}$ in (8) has no explicit dependence on time. Hence, we may expect that as $t \to \infty$, this term will be driven to zero. If the molecular dissipation law is changed from νk^2 to a hyper-viscosity (i.e.: viscous dissipation $\sim \nu_{hyper} k^4$), the re-scaling to give self-similarity must be changed.

There remains the question of why such re-scaling is a plausible expectation for two-dimensional turbulence. The basic point is that (17) implies a continuous decrease of the relative importance of the inertial transfer with time[1]. We suggest that such may be expected if the turbulence is progressively confined to smaller regions of available space, as time proceeds. A model of how this may happen is sketched in Sec. 5. Before that, we show some numerical results for the TFM, integrated according to the formulas of Sec. 2 and 3.

[1] McWilliams (1990) demonstrated through an analysis of his DNS that the presence of intense vortices suppresses transfer.

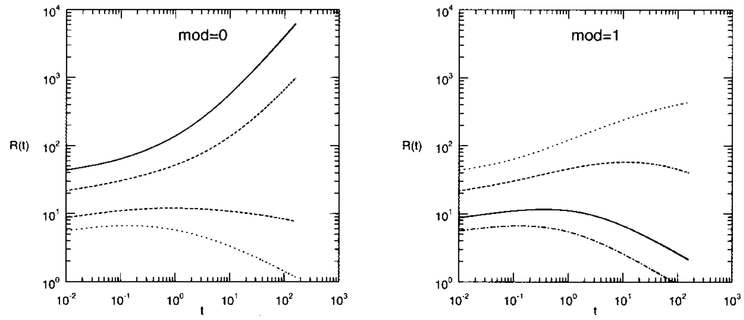

Fig. 1 $R_\lambda(t)$ for TFM (left panel) and TFM′ (right panel) and various values of ν: solid line, $\nu = .228$; short-dashed $\nu = .5000$; dot-dashed, $\nu = 1.142$; long-dashed, $\nu = 2.28$.

4 Numerical results

We present next some numerical results illustrating the issues discussed in the previous sections. First we explore how TFM and TFM′ behave for small R_λ, where

$$R_\lambda = \sqrt{E(t)}\ell/\nu$$

here

$$E(t) = \pi \int_0^\infty dKKF(K,t) \equiv \int_0^\infty dKE(K,t)$$

and

$$\ell^2 \equiv E(t)/\int_0^\infty dKK^2E(K,t)$$

We note that Chasnov (1997) found from his DNS that, below a critical value of initial R_λ, the flow reverted to a final period of decay in which $T(k,t)$ played no role. For $E(k,t=0) \sim k^3$, $k \to 0$, this condition implies $E(t) \sim t^{-2}$, and $R_\lambda(t) \sim t^{-1}$. The issue is explored in Fig. 1 (a,b) for an initial $E(K) \sim K^3/(1+K^6)$. Fig. (1a) shows results for TFM, while (1b) pertains to TFM′.

The amplitude of E is such that (13) is satisfied at small K. The value of $R_\lambda(0)$ is controlled by assigning various values of ν. We note that for TFM if $R_\lambda(0) \geq 20$, $R_\lambda(t) \sim t$ (in accord with Batchelor), whereas for initial values below this the final period of decay is recovered ($R_\lambda(t) \sim 1/t$). The TFM′ results appear to have a critical R_λ somewhat larger (~ 30), with the same time-dependence in the final period of decay. Above this value, $R_\lambda \sim t^{1/2}$, as expected. The short-time behavior of R_λ shown here differs from that of Chasnov in that our $R_\lambda(t)$ do not have an initial decrease. This is explained by the fact that Chasnov's simulations have a peak energy at much higher wave numbers than ours, so that viscous effects – which act immediately after the initial time, as contrasted to the more slowly

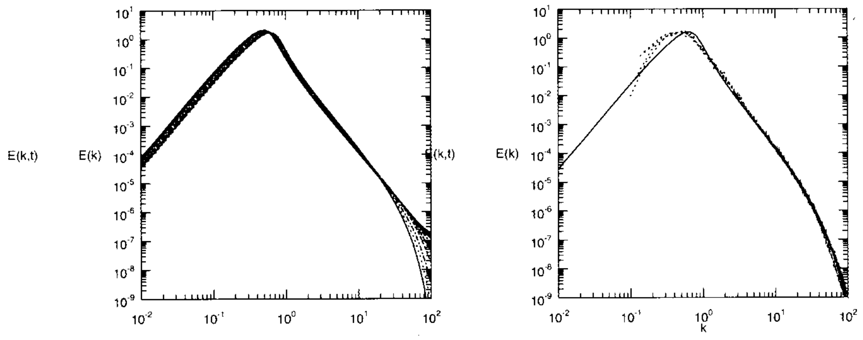

Fig. 2 $E(k,t)$ for TFM (left panel) and TFM$'$ (right panel) for $\nu = .010$ and for $t = .5,\ 1,\ 5,\ 10,\ 20,\ 30,\ 40,\ 50$.

developing energy transfer - are much more effective (near $t = 0$) in his case than ours.

We next discuss high R_λ spectra. For this purpose, it is convenient to have a measure of the extent of the inertial range. We do this by introducing another Reynolds number, ρ, which remains constant during the decay process. We first note that an integral measure of the extent of an inertial range that is $\sim k^{-3}$ would be indicated by:

$$\rho(t) \equiv \{\int E(K)dK \int E(K)K^4 dK\}^{1/2} / \int E(K)K^2 dK \tag{22}$$

In terms of the variable k, this is

$$\rho = u(k_s^2/k_\eta^3)/\nu \tag{23}$$

where

$$k_s^2 = \int k^4 dk E(k) / \int k^2 dk E(k), \text{ and } k_\eta = (\eta)^{1/6}/\sqrt{\nu}.$$

and η is the enstrophy dissipation rate, $2\nu \int dk E(k) k^4$. An interesting formula relating R_λ and ρ is

$$dR_\lambda/dt = -2\sqrt{\epsilon}(1 - \frac{1}{2}\rho^2) \tag{24}$$

Here, $\epsilon \equiv \int_0^\infty k^2 E(k) dk$. We note that Chasnov's highest R_λ DNS have $\rho \sim 8.5$, and we may match this value by the choice $\nu = 0.0100$, and an initial spectrum centered at $K = 1$.

Evolution of TFM and TFM' spectra, as evolved from (9) and (12), are shown in Fig. 2. The corresponding evolution of $R_\lambda(t)$ is shown in Fig. (3a), while $\rho(t)$ from Chasnov's DNS is depicted in Fig. (3b). The TFM spectrum shows a modest $E(K) \sim K^{-4}$ range just beyond the energy peak, as suggested by Lesieur

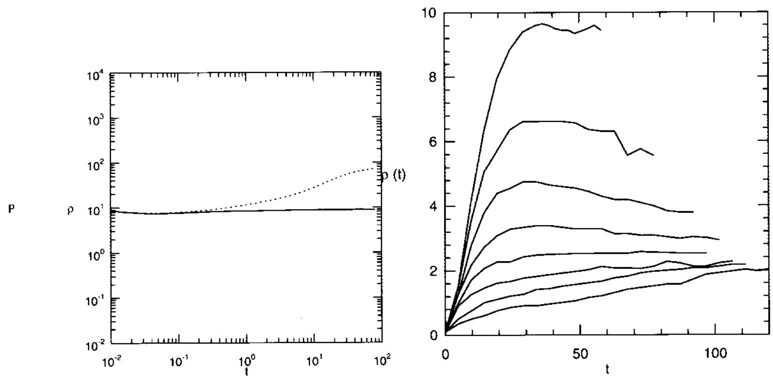

Fig. 3 Left panel: $\rho(t)$ for TFM$'$ (solid line) and TFM (dashed line) for the run described in Fig. 2. Right panel: $\rho(t)$ for Chasnov's simulation.

and Herring (1987). Note that the evolution of TFM is not self-similar at high K, where the viscous effects continue to weaken with time, according to the remarks of Sec. 3. The TFM' spectrum, on the other hand, quickly becomes self-similar in the variable K over the full range of K. Here, the energy transfer is much weaker. The evolution of R_λ (Fig. 3) shows late time behavior $R_\lambda \sim t$ for TFM, and $R_\lambda \sim \sqrt{t}$ for TFM'. The former is the classic Batchelor result, where the latter is that needed for complete self-similarity. The course of $E(t)$ for TFM' is $\sim \mathrm{C} - \ln(t)$, as predicted by Chasnov, and as it must be if $\epsilon(t) \sim 1/t$.

The comparison of TFM' with Chasnov's high R_λ case (Fig. (2b)) seems satisfactory for scales larger than the wave-number peak, but TFM' and the DNS differ for scales larger than the energy peak. The DNS seems a much more rounded spectrum at very large scales. Some of the difference may be attributable to a lack of large-scale resolution in the DNS.

If self-similarity is an approximate characterization of decay, it is of interest to explore the consequences of assuming $\rho(t) = \rho_0$, during the entire time of decay. In fact, this *ansatz* implies an equation for $E(t)$ through the use of $\rho(t) = \rho(0) \equiv \rho_0$, (1), (2) and the fact that $\int k^2 dk T(k,t) = 0$ for two-dimensional turbulence. Thus we have,

$$d\langle \mathbf{u}^2 \rangle / dt = -2\nu \langle \omega^2 \rangle \text{ and } d\langle \omega^2 \rangle / dt = -2\nu \langle (\nabla\omega)^2 \rangle \qquad (25\mathrm{a,b})$$

Using (22) for $\rho(t)$, and

$$\langle (\nabla\omega)^2 \rangle = 2 \int_0^\infty k^4 E(k,t) dk \qquad (26)$$

we find

$$\frac{d^2 \langle \mathbf{u}^2 \rangle}{dt^2} - \frac{\rho_0^2}{\langle \mathbf{u}^2 \rangle} \left(\frac{d\langle \mathbf{u}^2 \rangle}{dt} \right)^2 = 0 \qquad (27)$$

whose solution is:

$$\langle \mathbf{u}^2 \rangle = u_0^2 \left[1 + 2(\rho_0^2 - 1) R_0^{-1} \omega_0 t\right]^{-1/(\rho^2-1)} \tag{28}$$

An equivalent equation may be found for $\langle \omega^2 \rangle$:

$$\langle \omega^2 \rangle = \omega_0^2 \left[1 + 2(\rho_0^2 - 1) R_0^{-1} \omega_0 t\right]^{-\rho_0^2/(\rho_0^2-1)} , \tag{29}$$

with $R_0 = u_0^2/\omega_0 \nu$. The asymptotic solution for enstrophy ($t \to \infty$, $R_0, \rho_0 \gg 1$) is

$$\langle \omega^2 \rangle = \frac{R_0 \omega_0}{2\rho_0^2} t^{-1} \tag{30}$$

5 Dynamics of decay in terms of spectra

The underlying idea emerging from our discussion is that small scales of two-dimensional turbulence are parts of large scale structures. Hence, large and small scales decay together, with both length scales growing as $\sim \sqrt{t}$. (Such a growth is characteristic of pure viscous dissipation.) That large and small scales may be glued together is reasonable for two-dimensional flow, since the eddy turnover time for an enstrophy inertial-range eddy ($\sim \sqrt{k^3 E(k)}$) is roughly independent of scale (except for possible logarithmic corrections). Hence large and small scales may become coherent with impunity. In three dimensions, on the other hand, the inertial range eddy turn over time decreases with decreasing scale size, so that small scales may achieve a much higher degree of statistical independence from the large, energy-containing scales.

We first recall that the overall flow pattern revealed by DNS is a system of isolated vortices, surrounded by strain regions in which transfer is substantial (McWilliams, (1984), Benzi *et al.*, (1987), and Siegel and Weiss (1997)). Since the regions of strong strain are circumferential with respect to the isolated vortices, we would expect their number to be equal to the number of vortices. If we suppose the turbulence exist on independent patches, on each of which a statistical theory such as TFM holds, then the decay properties of the total system are the same as for each individual patch. If, however some of the patches – those corresponding to isolated vortices – have very little strain, and hence little transfer to small scales, the net transfer for the overall system is reduced. In this way, the net transfer, averaged across patches, decreases with time. The question of whether this simple picture is sensible may be answered by scanning a DNS to see if there are regions where some measure of enstrophy transfer is unusually small. A convenient measure would be the rate of production of mean-square vorticity gradients,

$$\frac{1}{2}\frac{d(\nabla\omega)^2}{dt} = (\nabla\omega)^T S(\nabla\omega) \tag{31}$$

where ω is the vorticity, and S the strain matrix,

$$= \begin{pmatrix} -\psi_{xy} & -\psi_{yy} \\ \psi_{xx} & \psi_{xy} \end{pmatrix} \tag{32}$$

Note that the normalized average value of the right hand side of (28) is, for isotropic turbulence, the two-dimensional skewness

$$\mathcal{S} \equiv -2\langle(\partial u/\partial x)(\partial\omega/\partial x)^2\rangle/\{\langle(\partial u/\partial x)^2\rangle^{1/2}\langle(\partial\omega/\partial x)^2\rangle^{1/2}\}$$

6 Summary and concluding comments

Classical closures such as TFM applied to high Reynolds number two-dimensional turbulence do not yield self-similar decay. The reason is traceable to (near) energy conservation at high Reynolds numbers. The two-dimensional DNS of Chasnov, on the other hand, exhibit a high degree of self-similarity, with a near constancy of total energy at the largest R_λ reported. The earlier DNS of McWilliams (1984), and the comparison of that DNS with TFM by Herring and McWilliams (1985) – if examined carefully – would have indicated the same result, although the resolution there was too limited to be conclusive. In three dimensions, self-similar decay is possible, and indeed was proposed many years ago as the universal mode of decay for high Reynolds number flows (see i.e. Batchelor, (1959)). But in that case energy decays as t^{-1}. The DNS and closure may be partially reconciled by the assumption that the energy (and enstrophy) transfer takes place in progressively smaller subregions of high strain, surrounding intense vortices. Such physics is indicated by a number of DNS (McWilliams (1984), Benzi *et al.* (1987)). If complete self-similarity is assumed to rule the decay process, the length scale grows as $\sqrt{t}$, and the energy transfer decreases as $1/\sqrt{t}$. The growth in length scale differs from Chasnov's (1996) numerical finding of $t^{.4}$ at the highest R_λ simulated.

Our discussion here has focused on the idea that progressive diminution of transfer is associated with a corresponding decrease of the space on which the turbulence lives. An alternate approach is to posit a Reynolds number dependence of the transfer process. This is explored elsewhere (Chasnov and Herring (1998)).

We have noted (in Sec. 3) that if the functional form of dissipation is modified, completely self-similar decay would dictate a modification in the equation that determines the length scale, $\lambda(t)$. Thus, if $\nu_{hyper}k^n$ represents dissipation, $\lambda \sim t^{1/n}$, if the statistics of second moments are self-similar. Whether the slower λ-growth rates for hyper-viscosity DNS examined by Carnavale *et al.* (1991) would be explained this way remains to be seen.

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