Crossover from Two- to Three-Dimensional Turbulence

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(Received 22 March 1996)

Forced rotating turbulence is simulated within a periodic box of small aspect ratio. Critical parameter values are found for the stability of a 2D inverse cascade of energy in the presence of 3D motions at small scales. There is a critical rotation rate below which 2D forcing leads to an equilibrated 3D state, while for a slightly larger rotation rate, 3D forcing drives a 2D inverse cascade. It is shown that inverse and forward cascades of energy can coexist. This study is relevant to geophysical flows, and contains physics beyond the scope of quasigeostrophic models. [S0031-9007(96)01175-1]

PACS numbers: 47.27.Eq, 47.60.+i

Random forcing at wave number k_f of an incompressible fluid in three space dimensions leads to a forward cascade of energy from k_f to the dissipation wave number $k_d > k_f$ [1]. The energy dissipation rate ϵ_d is equal to the energy input rate ϵ_f and all scales are statistically steady. However, in two space dimensions, an inverse cascade of energy to $k < k_f$ develops [2]. In the 2D case, $\epsilon_d < \epsilon_f$ and only scales small compared to the largest populated scale $l_0 = 2\pi/k_0$ are statistically steady. Dimensional analysis leads to an energy spectrum $E(k) \propto k^{-5/3}$ for the range $k_0 \ll k \ll k_f$ in 2D, and for the range $k_f \ll k \ll k_d$ in 3D. Geophysical data in which the turbulence is both three dimensional and high Reynolds number support an approximate $k^{-5/3}$ scaling of the energy spectrum (e.g., [3]). Recent numerical data using hyperviscosities in 2D also exhibit $E(k) \propto k^{-5/3}$ for $k < k_f$ (see, e.g., [4]).

Here we consider forced turbulence in a thin layer of fluid rotating about the $\hat{\mathbf{z}}$ axis. The flow is simulated in a periodic box with $L_z \ll L_x = L_y$. This provides a single system in which one can study the transition and interaction between 2D and 3D behavior by varying the box height L_z or the rotation rate Ω.

In addition to its fundamental nature, this study has applications to flows in oceans and atmospheres where vertical length scales are much smaller than horizontal length scales and the large-scale eddy turnover time is much larger than the rotation time scale. Mathematical models of geophysical flows are usually quasi-2D in the sense that the motions are described only by the $\hat{\mathbf{z}}$ component of vorticity [5]. In such models, the effects of the small aspect ratio, stratification and, rotation are expressed through a single parameter, such as the Burger number characterizing the rotating shallow water equations [6]. Here we would like to separate these effects, and to establish when a quasi-2D model is a reasonable approximation to the large scales of small-aspect-ratio 3D (SAR3D) flows. We first seek to establish conditions for which 2D dynamics are stable to smaller-scale 3D fluctuations. Our ultimate goal is to determine conditions under which 3D forcing at small scales actually drives a 2D inverse cascade of energy to large scales.

It is often mistakenly argued on the basis of the Taylor-Proudman theorem that rotation two dimensionalizes the flow. The Taylor-Proudman theorem [7] states that $\partial_t(\nabla \times \mathbf{u}) = 2\Omega\,\partial_z \mathbf{u}$ in the limit of fast rotation and small viscosity. This implies only that motions slow with respect to the rotation rate are independent of z. However, if the initial conditions or the forcing contain a significant z variation, this variation remains and leads to rapid oscillations on the rotation time scale (inertial waves). Rotation does in fact lead to two dimensionalization, but this is a subtle *nonlinear* effect that has only begun to be understood [8,9].

We solve the 3D incompressible Navier-Stokes equations in a frame rotating at constant $\Omega = \Omega\hat{\mathbf{z}}$, given by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = \mathbf{f} - \nabla p + \nu \nabla^2 \mathbf{u}, \quad (1)$$

where $\mathbf{u}$ is the divergence-free velocity, the constant density has been absorbed into the pressure p, and ν is the kinematic viscosity. To obtain a sufficiently large range of inertial scales, we have used both an eddy-viscosity subgrid model [11] as well as a hyperviscosity operator $\nu_q(-1)^{q+1}\nabla^{2q}$ in place of $\nu\nabla^2$ in (1). The choice of a subgrid model affects the shape of the energy spectrum in the high wave-number range, but does not affect the results for the critical values of the parameters $S = L_f/L_z$ and $\mathrm{Ro} = (\epsilon_f k_f^2)^{1/3}/\Omega$ that define the crossover from 2D to 3D behavior, where $L_f = 2\pi/k_f$. Here we present mainly spectra as calculated using the eddy-viscosity model developed by Kraichnan and Chollet and Lesieur [11].

The force $\mathbf{f}$ is taken to be Gaussian and white in time with forcing spectrum $F(k_h)$ given by

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$$F(k_h) = \epsilon_f \frac{\exp\left(-0.5\{[k_h - (k_f)_h]/\sigma\}^2\right)}{(2\pi)^{1/2}\sigma}, \quad (2)$$

where ϵ_f is the energy input rate, $k_h = (k_x^2 + k_y^2)^{1/2}$ is the horizontal wave number, and $(k_f)_h = [(k_f)_x^2 + (k_f)_y^2]^{1/2}$ is the horizontal component of the peak wave number k_f of the force. The standard deviation of $F(k_h)$ has been chosen as $\sigma = 1 \gg 2\pi/L_x$. Here we present primarily the results for SAR3D turbulence driven by a two-dimensional, two-component (2D2C) force $\mathbf{f} = f_x\hat{\mathbf{x}} + f_y\hat{\mathbf{y}}$ with $(k_f)_h = k_f$. The case of 2D2C forcing investigates the stability of calculations such as those reported in [4]. We also briefly discuss 3D3C "conical" forcing for which the vertical component of the forcing wave number is $|(k_f)_z| = 2\pi/L_z$. The forcing is turned on at time zero, with initial 3D motions present only at the level of round-off error. Equations (1) and (2), modified to account for the subgrid model, are solved using a pseudospectral code.

The problem as posed is characterized by four nondimensional parameters: the aspect ratio $A \equiv L_z/L_x$, the Reynolds number $\mathrm{Re} = \epsilon_f^{1/3} k_f^{-4/3}/\nu$, the Rossby number $\mathrm{Ro} = (\epsilon_f k_f^2)^{1/3}/\Omega$, and the relative scale of the force $S = L_f/L_z$. Here we consider the limit of a very large Reynolds number by using subgrid models, and a small enough aspect ratio such that finite-size effects in the horizontal directions are not important for the times considered in our simulations. Therefore we are primarily concerned with the behavior of SAR3D flows as a function of Ro and S for a large Reynolds number and $A \to 0$.

The choice of forcing plays a role in the determination of the critical curve in (S, Ro) space defining the crossover from 2D to 3D behavior. We expect the shape of the force, its dimensionality, and its coherence properties to shift the critical curve, perhaps even significantly. Table I provides a few approximate critical values for the case of 2D2C forcing. Our purpose for this initial study is to illustrate a range of behaviors in (S, Ro) parameter space for the particular force (2) described above.

We have performed three sets of simulations with three different resolutions (see Table I): (I) $128 \times 128 \times 8$ ($A = 1/16$), (II) $512 \times 512 \times 8$ ($A = 1/64$), (III) $256 \times 256 \times 32$ ($A = 1/8$). In all cases, the grid is taken to be isotropic at small scales. The first and second data sets are used to study separation distances $S \leq 1$ (forcing in the 2D range of wave numbers). The third data set confirms that the nature of the results does not change when the resolution of the small scales is increased, and to study separation distances $S \geq 1$ (forcing in the 3D range of wave numbers).

First we consider 2D2C forcing in the limit $\mathrm{Ro} \to \infty$ ($\Omega = 0$). We have determined a critical value $S_c \approx 0.5$ above which an overall inverse cascade was not found to exist and below which a 2D-like inverse cascade of energy to large scales was found to be stable to 3D motions at scales smaller than the forcing scale L_f [11]. For the case of $A = 1/64$ and $S = 0.75$, Fig. 1 (solid line) shows the time evolution of the total energy K and the ratio ϵ_d/ϵ_f. In this figure, time is nondimensionalized by $(\epsilon_f k_f^2)^{-1/3}$ and kinetic energy by $(\epsilon_f/k_f)^{2/3}$. After an initial period of development of the nonlinear interactions, the energy remains constant in time and $\epsilon_d/\epsilon_f \approx 1$ indicating that a 3D statistically steady state is established for all wave numbers. Most of the energy input by the random forcing is transferred directly to higher wave numbers and eventually dissipated by viscosity. Figure 2(a) shows energy spectra at two values of the nondimensional time $t \approx 42$ and $t \approx 52$. These spectra $E_h(k_h)$ are obtained by summing $u_i(\mathbf{k})u_i(-\mathbf{k})/2$ over k_z and binning into rings of radius k_h with $\Delta k_h = 2\pi/L_x$. One sees that the flow at these late times is statistically steady, with large fluctuations in the spectra at low wave numbers $k < k_f = 12$. The spectra at high wave numbers $k > k_f$ show an approximate scaling $E_h(k_h) \propto k^{-5/3}$. Notice that the small aspect ratio leads to jumps in the spectra at multiples of the smallest vertical wave number $k_z = 16$. As the parameter S is increased, less and less energy resides in the low wave numbers $k < k_f$.

The 3D behavior for $S = 0.75$ [Figs. 1 (solid line) and 2(a)] should be contrasted with the 2D behavior at large scales for $S = 0.375$ [Figs. 1 (dashed line) and 2(b), $A = 1/64$]. Notice that in both cases the forcing is in the 2D range of wave numbers. In the $S = 0.375$ case, the forcing is at wave number $k_f = 6$ and the smallest positive vertical wave number is $k_z = 16$ (and thus the smallest wave number of the 3D range is $k = 16$). The energy spectra $E_h(k_h)$ show two different cascade regions: for $k_h > k_f$ there exists a forward cascade of energy that quickly achieves a statistically steady state, while for $k_h < k_f$ there is an inverse cascade of energy leading to a time-dependent horizontal integral scale l_0 that is increasing in time, and only scales small compared to l_0 are statistically steady. The inverse-cascade and forward-cascade regions exist simultaneously, and both show rough power-law scaling $E(k_h) \propto k_h^{-5/3}$. Notice that the spectra decrease rapidly in the region of 2D scales $6 < k_h < 16$ greater than the forcing wave number before exhibiting the approximate $k_h^{-5/3}$ scaling in the 3D region $k_h > 16$. For this flow, only a fraction of the energy input by the forcing is dissipated by viscosity at small scales, the remainder of which is transferred to larger scales, and this is reflected in the fact that $\epsilon_d/\epsilon_f < 1$ [Fig. 1 (dashed line)]. Changing from the eddy viscosity subgrid model to the hyperviscosity subgrid model changes only the shape of the energy spectra for wave numbers in the 3D range [see Figs. 1 and 2(b)].

Next we consider rotating SAR3D turbulence driven by the 2D2C force with spectrum given by (2). We vary both the Rossby number Ro and the relative scale S of the force to determine the portion of (S, Ro) parameter space where an inverse cascade of energy to wave numbers $k_h < k_f$ will exist. Table I presents our results for selected values of (S, Ro). Recall that the critical value of the Rossby number $\mathrm{Ro}_c \to \infty$ as $S \to 0.5$. As can be extracted from Table I, $\mathrm{Ro}_c \approx 1.6$ for $S = 0.75$ ($A = 1/64$) and

TABLE I. The behavior (2D, 3D or ↕ for critical) of the turbulence for various (S, Ro) and 2D2C forcing.

(I) 128 × 128 × 8			(II) 512 × 512 × 8			(III) 256 × 256 × 32	
$(0.75, \infty)$	3D	$(12/16, \infty)$	3D	(0.75, 1.6)	↕	(2.0, 1.5)	3D
(0.75, 2.0)	3D	$(9/16, \infty)$	3D	(0.75, 1.3)	2D	(4.0, 1.8)	3D
(0.75, 1.4)	3D	$(8/16, \infty)$	↕	(0.75, 1.25)	2D	(4.0, 1.4)	↕
(0.75, 1.25)	↕	$(7/16, \infty)$	2D	(0.75, 0.5)	2D	(4.0, 1.25)	2D
(0.75, 1.2)	2D	$(6/16, \infty)$	2D	(1.0, 0.25)	2D	(4.0, 1.1)	2D
(0.75, 1.1)	2D	$(4/16, \infty)$	2D	(0.5, 2)	2D	(4.0, 1.0)	2D
(0.75, 0.75)	2D	(0.75, 2)	3D			(8.0, 0.7)	2D
(0.75, 0.5)	2D						

decreases slowly for $S > 0.75$. The data suggest that Ro_c may asymptote to a nonzero constant for large S.

Numerical data sets I and II were used to explore $S \leq 1$. For aspect ratios $A = 1/16$ and $1/64$, the range of S is limited by the dimensions of the box to approximately $0.1 \leq S \leq 1$. For $S = 0.75$, we found the near-critical values $\mathrm{Ro} = 1.25$ for $A = 1/16$ and $\mathrm{Ro} = 1.6$ for $A = 1/64$ (Table I). The critical Ro for fixed S changes by (20–25)% when the aspect ratio is lowered from $A = 1/16$ to $1/64$, suggesting that finite-size effects in the horizontal directions may be influencing the results for the case $A = 1/16$. For fixed S and decreasing Ro, the value of ϵ_d/ϵ_f decreases approximately linearly for Ro less than the critical value, indicating that a larger fraction of energy is cascaded to large scales as the Rossby number is decreased below critical. Figure 1 compares $\mathrm{Ro} = \infty$ (solid) and $\mathrm{Ro} = 0.5$ (dot-dashed line) for $A = 1/64$ and $S = 0.75$.

The scaling $\epsilon_d = O(\mathrm{Ro})$ indicated by our results for 2D2C forcing can be understood in light of the closures [9,11] which show that, in a statistically steady state, the energy flux to $k > k_f$ is proportional to a decorrelation time scale (θ_{kpq}). In the absence of rotation, this time scale is determined by the nonlinear interactions and is $O(1)$ with respect to the variables nondimensionalized by ϵ_f and

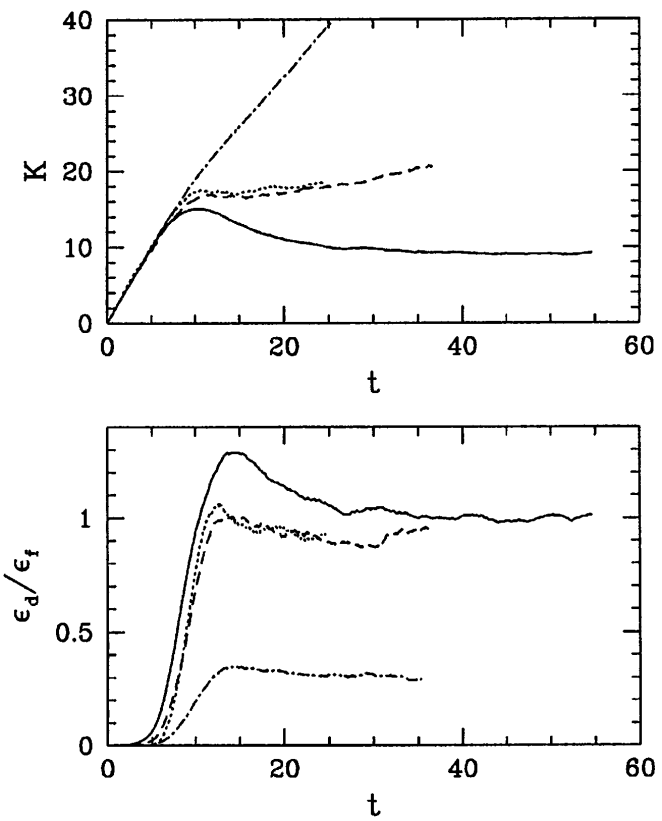

FIG. 1. $A = 1/64$, $\mathrm{Ro} = \infty$, $S = 0.75$ (solid line); $A = 1/64$, $\mathrm{Ro} = \infty$, $S = 0.375$, eddy viscosity model (dashed line); $A = 1/64$, $\mathrm{Ro} = \infty$, $S = 0.375$, hyperviscosity operator ∇^4 (dotted line); $A = 1/64$, $\mathrm{Ro} = 0.5$, $S = 0.75$ (dot-dashed line).

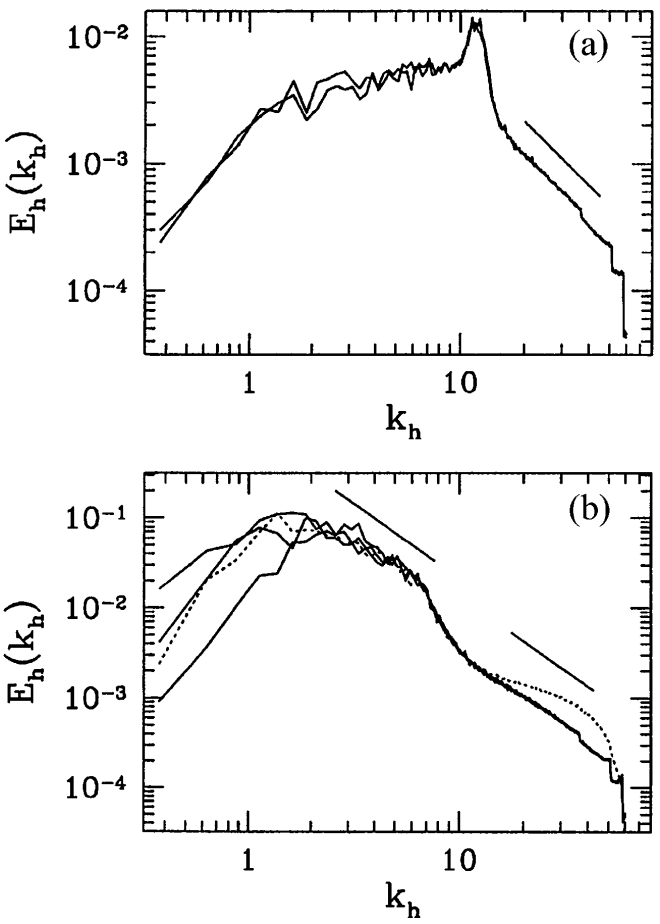

FIG. 2. (upper) $A = 1/64$, $\mathrm{Ro} = \infty$, $S = 0.75$ (statistically steady); (lower) $A = 1/64$, $\mathrm{Ro} = \infty$, $S = 0.375$: eddy viscosity (solid line) with time increasing upwards; hyperviscosity (dotted line). The lines are $E_h \propto k_h^{-5/3}$.

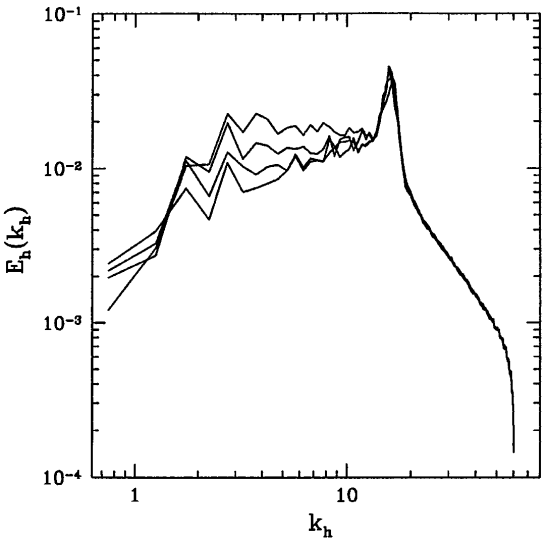

FIG. 3. $A = 1/8$, $S = 4.0$, Ro $= 1.8$ (approaching a statistically steady state with time increasing downwards).

k_f. Rotation introduces an additional linear decorrelation mechanism on a time scale $O(\mathrm{Ro})$. Hence the latter is the dominant decorrelation time scale when $\mathrm{Ro} < 1$. A subset of triad interactions is not decorrelated by rotation: the resonant triads for which $k_z/|\mathbf{k}| \pm p_z/|\mathbf{p}| \pm q_z/|\mathbf{q}| = O(\mathrm{Ro})$. However, there is no energy transfer between 2D and 3D modes resulting from the resonant triads [8], and thus we expect the scaling $\epsilon_d = O(\mathrm{Ro})$ for 2D2C forcing and Ro smaller than the $O(1)$ critical value. The scaling $\epsilon_d = O(\mathrm{Ro})$ may hold in general (i.e., for 3D3C forcing) since the number of resonant triads is $O(\mathrm{Ro})$ as $\mathrm{Ro} \to 0$.

Figures 3 and 4 present results from data set III for 2D2C forcing in the 3D range of wave numbers. Figure 3 shows energy spectra $E_h(k_h)$ (time increasing downward) for the point in (S, Ro) space $S = 4.0$, Ro $= 1.8$ which is on the 3D side of the critical curve, meaning that there is

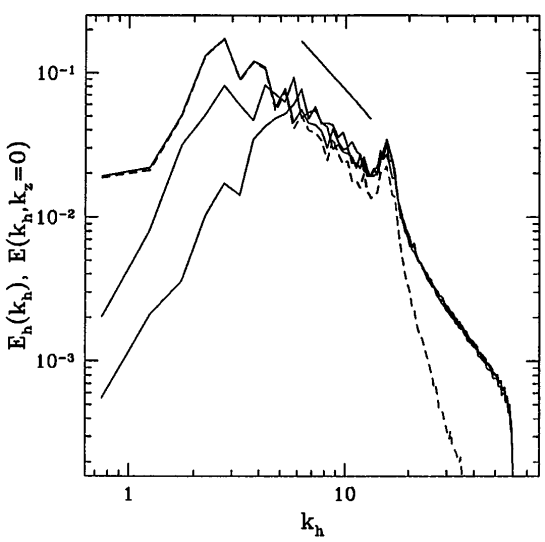

FIG. 4. $A = 1/8$, $S = 4.0$, Ro $= 1.0$: (solid line) $E_h(k_h)$ at three times (time increasing upwards); (dashed line) $E(k_h, k_z = 0)$ at the latest time. The line is $E_h \propto k_h^{-5/3}$.

no inverse cascade of energy to $k < k_f = 16$. Figure 4 shows spectra for $S = 4.0$, Ro $= 1.0$ which is on the 2D side of the critical curve and shows that a fraction of the energy input by the force is transferred to wave numbers smaller than the forcing wave number $k_f = 16$ by an inverse cascade of energy. The dashed line in Fig. 4 is the pure 2D spectrum $E(k_h, k_z = 0)$ to indicate the wave numbers containing significant 3D energy.

The critical curve in (S, Ro) space defining a crossover from 2D to 3D behavior will be shifted when the force is modified. The existence of an inverse cascade in SAR3D flows subjected to 3D3C forcing is probably most relevant to the geophysical applications, and is the subject of ongoing study. With 3D3C conical forcing (2), the critical curve shifts only slightly: We found a critical value $\mathrm{Ro}_c \approx 1.25$ for $A = 1/64$ and $S = 1.0$, while this value Ro $= 1.25$ is just below critical for 2D2C forcing, $A = 1/64$ and $S = 1.0$. We also found the persistence of a strong inverse cascade for the cases: $A = 1/64$, Ro $= 0.5$, and $S = 1.0$ leading to $\epsilon_d/\epsilon_f \approx 0.8$; and $A = 1/8$, Ro $= 0.1$, and $S = 2.0$ leading to $\epsilon_d/\epsilon_f \approx 0.6$ and representing 3D3C forcing deeper in the 3D range of scales. These results demonstrate that it is possible to drive a 2D inverse cascade from purely 3D forcing at the small scales in a rotating flow.

L. M. S. gratefully acknowledges the support of NSF under Grant No. CTS-9528491 and computing resources provided by the Cornell Theory Center (CTC) and the Maui High Performance Computing Center (MHPCC). The authors are indebted to R. Rogallo and A. Wray for the use of their software. The computations were begun at the Hong Kong University of Science and Technology (HKUST). L. M. S. and F. W. thank the Mathematics Department at HKUST and The Hong Kong Research Grant Council for making possible their visit to HKUST.

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