Turbulence spectrum of strongly conductive temperature field in a rapidly stirred fluid

J. Chasnov[a] V. M. Canuto, and R. S. Rogallo[b]
NASA Goddard Space Flight Center, Institute for Space Studies, 2880 Broadway, New York, New York 10025

(Received 10 May 1989; accepted 7 July 1989)

In an earlier paper [Phys. Fluids **31**, 2065 (1988)], a numerical simulation of a passive scalar field convected by a frozen velocity field, i.e., a velocity field with an infinite correlation time, was performed. In this paper, a simulation of a passive scalar field convected by a velocity field which is rapidly stirred at all scales of motion, i.e., a velocity field with near zero correlation time, is performed. For an energy spectrum of the form $E(k) \propto k^{-5/3}$, the temperature spectrum $G(k)$ is found to obey $G(k) \propto k^{-11/3}$ when conductive effects are dominant. A theoretical model is proposed which obtains the above result by representing the transfer of scalar variance by an eddy conductivity, whose correlation time is limited by the correlation time of the velocity field.

I. INTRODUCTION

In a recent paper[1] the turbulence spectrum of a passive scalar field (e.g., temperature fluctuations when the effects of buoyancy may be neglected) in a fluid of Prandtl number much less than unity was determined by means of a numerical simulation. Specifically, the universal region in wavenumber space, called the inertial-conductive subrange, where the molecular viscosity ν is much smaller than the turbulent viscosity while the molecular conductivity χ is much larger than the turbulent conductivity, was resolved in the numerical simulation. The Prandtl number of the fluid is defined as $\sigma = \nu/\chi$.

The passive temperature field $\theta(\mathbf{x})$ was taken to satisfy the usual equation

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \chi\nabla^2\theta\,, \tag{1}$$

where $\mathbf{u}$ is the velocity of the fluid. The temperature field is called "passive" since the equation determining the velocity field is assumed to be independent of θ. Chasnov *et al.*[1] performed simulations of Eq. (1) with *three different types* of velocity fields. First, and most simply, $\mathbf{u}(\mathbf{x})$ was chosen to be an isotropic, time-independent (frozen) velocity field, satisfying continuity, and whose Fourier components were assigned random phases (creating a so-called Gaussian velocity field). Furthermore, the spectrum of the velocity field $E(k)$, satisfying

$$\frac{1}{2}\langle u_i u_i\rangle = \int_0^\infty E(k)dk \tag{2}$$

was chosen to be proportional to $k^{-5/3}$, i.e., $E(k)$ was assumed to obey the Kolmogorov[2] inertial subrange spectrum,

$$E(k) = \mathrm{Ko}\,\epsilon^{2/3}k^{-5/3}\,, \tag{3}$$

where ϵ is the energy dissipation rate, defined as

$$\epsilon = 2\nu\int_0^\infty k^2E(k)dk\,, \tag{4}$$

and Ko is the Kolmogorov constant. Of course, by considering a frozen velocity field, ϵ loses all meaning as a dynamical variable so that one views Ko $\epsilon^{2/3}$ simply as a dimensional proportionality constant.

The approximation of a frozen, Gaussian velocity field was also made in the classical paper of Batchelor, Howells, and Townsend,[3] who predicted that the spectrum of the passive temperature field $G(k)$ would obey

$$G(k) = G_0\epsilon_\theta\epsilon^{2/3}\chi^{-3}k^{-17/3}\,, \tag{5}$$

where they determined the dimensionless constant G_0 to be

$$G_0 = \tfrac{1}{3}\,\mathrm{Ko}. \tag{6}$$

The spectrum $G(k)$ is defined so that

$$\langle\theta^2\rangle = \int_0^\infty G(k)dk\,, \tag{7}$$

while the cascade rate of the scalar variance ϵ_θ is defined as

$$\epsilon_\theta = 2\chi\int_0^\infty k^2G(k)dk\,. \tag{8}$$

Performing the numerical simulation of Eq. (1) using a frozen, Gaussian velocity field, Chasnov *et al.*[1] obtained *exactly* Eqs. (5) and (6) for the turbulence spectrum of the temperature field.

Chasnov *et al.*[1] then proceeded to relax the assumption of a frozen, Gaussian velocity field. To determine separately the effect of these two approximations on the equilibrium spectrum of the temperature field, they first relaxed only the Gaussian field approximation. To eliminate this approximation, they time evolved an initially random velocity field by performing a numerical simulation of the Navier–Stokes equations. In order to reach very high Reynolds numbers, i.e., in order to resolve the inertial subrange and obtain an approximate $k^{-5/3}$ spectrum, a forcing scheme and subgrid scale model was employed.[4] After the Fourier components of the velocity field developed phase correlations, as deter-

[a] Also with the Department of Physics, Columbia University, New York, New York 10027.
[b] Permanent address: NASA Ames Research Center, Moffett Field, California 94035.

 0899-8213/89/101698-03$01.90

mined by the Navier–Stokes equations, the fully developed velocity field was then frozen in time and used to solve Eq. (1). It was shown that the temperature spectrum $G(k)$ again obeyed Eq. (5) but with a slightly higher value of G_0, i.e.,

$$G_0 = 0.39\ \mathrm{Ko}\,. \tag{9}$$

In the third and final simulation, the Navier–Stokes equations with a forcing scheme and subgrid model[4] was solved concurrently with Eq. (1). It was seen that this slightly affected the power-law dependence of $G(k)$, although as the value of the molecular conductivity χ was increased further, a $k^{-17/3}$ subrange was again obtained. This was to be expected, since the frozen field approximation should be accurate when the convective time scale, $(\epsilon k^2)^{-1/3}$, is much longer than the conductive time scale $(\chi k^2)^{-1}$, and the conductive time scale is seen to be shortened relative to the convective time scale with increasing χ.

II. NUMERICAL SIMULATION

In this paper, we perform a simulation of Eq. (1) using a velocity field not previously considered. In contrast to the case of a frozen velocity field, where the correlation time of the velocity field is infinite, we examine the case of a rapidly fluctuating velocity field, whose correlation time approaches zero. For example, suppose that some external stirring force applied to a fluid was able to make the velocity field fluctuate at all scales of motion with a correlation time much shorter than either the convective time scale or the conductive time scale. Such a physical experiment may be hard to perform in the laboratory, since when a fluid is "stirred," it is usually only the largest scales of the system that are directly affected, but is relatively easy to perform in the computer. (This same "thought experiment" was considered earlier by Kraichnan,[5] but in a different context.) In this paper, we will determine the turbulence spectrum of the passive temperature field in the conductive subrange by performing a numerical simulation using the code employed in Ref. 1.

To perform the above "experiment," we have solved Eq. (1) using an isotropic, Gaussian velocity field that satisfies continuity and whose spectrum is given by Eq. (3). The use of the Kolmogorov spectrum for the velocity field seems rather arbitrary, since one does not expect such a spectrum to develop when the velocity field is being *strongly* influenced at all scales of motion by an external stirring force. Nevertheless we choose such a spectrum for convenience and in order to compare the results to the spectrum of Batchelor *et al.*,[3] Eq. (5). Furthermore, in order to simulate a velocity field that is being stirred rapidly at all scales of motion, a different realization of Eq. (3) is chosen with each time step of the simulation. That is, the phases and velocity component distribution of the Fourier modes of the velocity field are rerandomized with each time step of the numerical simulation. If Δt is the time step of the simulation, then the correlation time of the velocity field, defined as the average time τ over which $\mathbf{u}(\mathbf{x},t)$ remains correlated with $\mathbf{u}(\mathbf{x},t+\tau)$, is given by $\tau = \Delta t/2$. The time scale τ is chosen to be much smaller than the convective and diffusive time scales of the turbulence.

In Fig. 1, we plot the three-dimensional temperature spectral function, $G(k)\times k^{11/3}$ vs k. Units are shown in the figure caption. Here k_{CO} is the Corrsin–Obukhov wavenumber. As clearly seen in Fig. 1, a new power-law subrange is obtained. The numerical simulation results distinctly show that $G(k) \propto k^{-11/3}$ when the effects of molecular conductivity strongly damp the temperature fluctuations.

III. THEORETICAL INTERPRETATION

A heuristic argument is presented below, which accounts for the above behavior of the temperature field. Figure 2 represents a typical interaction among the Fourier components of $\mathbf{u}$ and θ, which gives rise to the transfer of scalar variance in the inertial-conductive subrange, be it that resulting from a frozen velocity field or a rapidly varying velocity field (or by a more realistic freely decaying velocity field). In the graph, we have assumed that $E(k)$ decreases much slower than $G(k)$ in the conductive subrange. This is certainly true if $E(k) \propto k^{-5/3}$ and $G(k) \propto k^{-17/3}$ or $k^{-11/3}$. If this is the case, it is reasonable, then, to view the transfer of scalar variance, which is a product of two temperature Fourier components and one velocity Fourier component, to be a consequence of the following nonlocal interaction. One of the temperature Fourier components must be at wavenumber $\mathbf{k}$. This is the wavenumber to which the scalar variance is being transferred. Since most of the scalar variance resides at much lower wavenumbers (say $q \ll k$), at least relative to the velocity fluctuations, the other Fourier component of the temperature field must be approximately $\theta(\mathbf{q})$. Finally, since the wavenumbers $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$ must form a triangle, i.e., $\mathbf{p}+\mathbf{q}=\mathbf{k}$, the Fourier velocity component

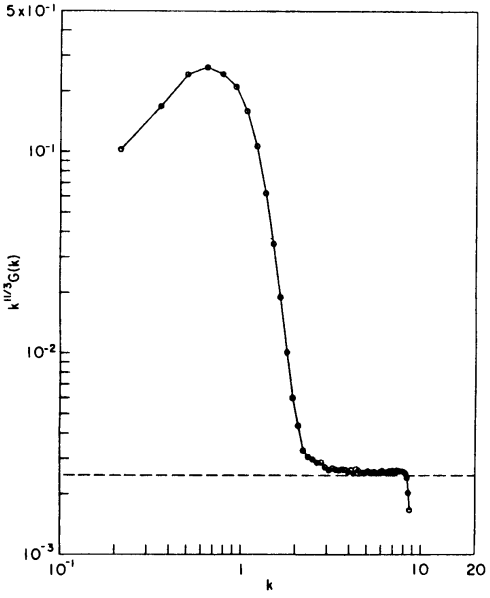

FIG. 1. Three-dimensional temperature spectral function, $G(k)\times k^{11/3}$, convected by a rapidly varying velocity field that satisfies Eq. (3). Here, k is in units of $\mathrm{Ko}^{1/8} k_{c\cdot\ 0}$, while G is in units of $\mathrm{Ko}^{-9/8}(\epsilon\chi^5)^{1/4}(\epsilon_0/\epsilon)$. The open circles represent the numerical simulation data while the dashed line represents the theoretical result.

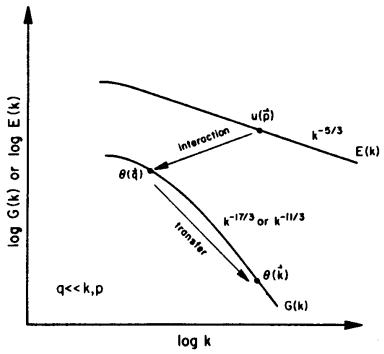

FIG. 2. Nonlocal transfer of scalar variance in the inertial-conductive subrange. This transfer may be successfully modeled by an eddy conductivity.

must be approximately $\mathbf{u}(\mathbf{p})$, where $|\mathbf{p}| \sim |\mathbf{k}|$. Hence the transfer of scalar variance is a result of a nonlocal interaction between $\mathbf{u}(\mathbf{p})$ and $\theta(\mathbf{q})$, causing a nonlocal transfer of scalar variance to $\theta(\mathbf{k})$. It is plausible to represent such a nonlocal interaction and transfer by an eddy conductivity (as made popular by Heisenberg[2]). Accordingly, we write the eddy conductivity acting on wavenumbers less than k by[6]

$$\chi_t(k) = \frac{2}{3}\int_k^\infty \frac{E(k)}{n_c(k)}dk\,, \tag{10}$$

where $\frac{2}{3}E(k)dk$ is the root-mean-square turbulent velocity of the fluid between wavenumber k and $k+dk$, and $n_c(k)^{-1}$ is the correlation time among the Fourier components associated with the nonlinear transfer. Then the equation for the transfer of scalar variance in the inertial-conductive subrange can be written as[6]

$$\epsilon_\theta = 2[\chi + \chi_t(k)]\int_0^k k^2 G(k)dk\,. \tag{11}$$

One can easily solve Eq. (11) for $G(k)$, i.e.,

$$G(k) = \tfrac{1}{3}\epsilon_\theta E(k)k^{-2}[\chi + \chi_t(k)]^{-2}n_c(k)^{-1}\,. \tag{12}$$

The Batchelor *et al.*[3] result can be obtained (as pointed out in their paper) by assuming $\chi \gg \chi_t(k)$ and

$$n_c(k) = \chi k^2\,. \tag{13}$$

Using Eqs. (12) and (13), we find

$$G(k) = \tfrac{1}{3}\epsilon_\theta \chi^{-3}k^{-4}E(k)\,, \tag{14}$$

which is equivalent to Eq. (5) when one uses Eq. (3) for $E(k)$. Presumably, Eq. (14) is of more general validity, as long as $E(k)$ decreases much less rapidly then $G(k)$ in the conductive subrange. It should be noted that the choice of the conductive time scale for $n_c(k)^{-1}$, Eq. (13) implies that molecular conduction interferes destructively with the effectiveness of the nonlinear cascade of scalar variance. If this was not the case, then a more reasonable choice for $n_c(k)$ would be $n_c(k) = \chi_t(k)k^2$, which would result in $G(k) \propto k^{-13/3}$, as originally predicted by Ogura.[7] However, recent numerical simulation results[1] are seen to be in agreement with the $k^{-17/3}$ power law, implying the essential correctness of Eq. (13).

The power-law subrange found in the present simulation may be derived in an analogous heuristic manner. The concept of an eddy conductivity is also assumed to be valid in this case. Now, however, the correlation time of the nonlinear transfer is given by τ, i.e., in place of Eq. (13), we have

$$n_c = \tau^{-1}\,, \tag{15}$$

independent of k. Then, instead of Eq. (14), we may derive, assuming $\chi \gg \chi_t(k)$,

$$G(k) = \tfrac{1}{3}\tau\epsilon_\theta \chi^{-2}k^{-2}E(k)\,, \tag{16}$$

which yields a $k^{-11/3}$ spectrum when use is made of Eq. (3). We have plotted the above theoretical result for $G(k)$ as the dashed line in Fig. 1. As can be seen, the agreement of Eq. (16) with the simulation data is quite good.

Although the above theoretical argument has been presented in a heuristic manner, Eq. (16) may in fact become exact as $\tau \to 0$. An eddy conductivity equivalent to Eqs. (10) and (15) has been previously derived by Kraichnan[8] in the context of second-order perturbation theory, which the author claims is exact in the limit $\tau \to 0$. More phenomenological closure theories, such as the eddy damped quasinormal Markovian (EDQNM) closure[9] can also be made to yield the same result as Eq. (16), as long as the choice of eddy damping rate in the scalar equation is made in analogy to Eq. (15).

ACKNOWLEDGMENTS

One of the authors (J. C.) wishes to thank Professor G. K. Batchelor for his encouragment.

One of the authors (J. C.) also gratefully acknowledges financial support from the NASA Ames Research Center.

[1] J. Chasnov, V. M. Canuto, and R. S. Rogallo, Phys. Fluids **31**, 2065 (1988).
[2] G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge U. P., Cambridge, 1953).
[3] G. K. Batchelor, I. D. Howells, and A. A. Townsend, J. Fluid Mech **5**, 134 (1959).
[4] E. D. Siggia and G. S. Patterson, J. Fluid Mech. **86**, 567 (1978).
[5] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[6] V. M. Canuto, I. Goldman, and J. Chasnov, Phys. Fluids **30**, 3391 (1987).
[7] Y. Ogura, J. Meteorol. **15**, 539 (1958).
[8] R. H. Kraichnan, Phys. Rev. A **25**, 3281 (1982).
[9] M. Lesieur, *Turbulence in Fluids* (Nijhoff, Dordrecht, 1987).