Differential Equations: Review with YouTube Examples

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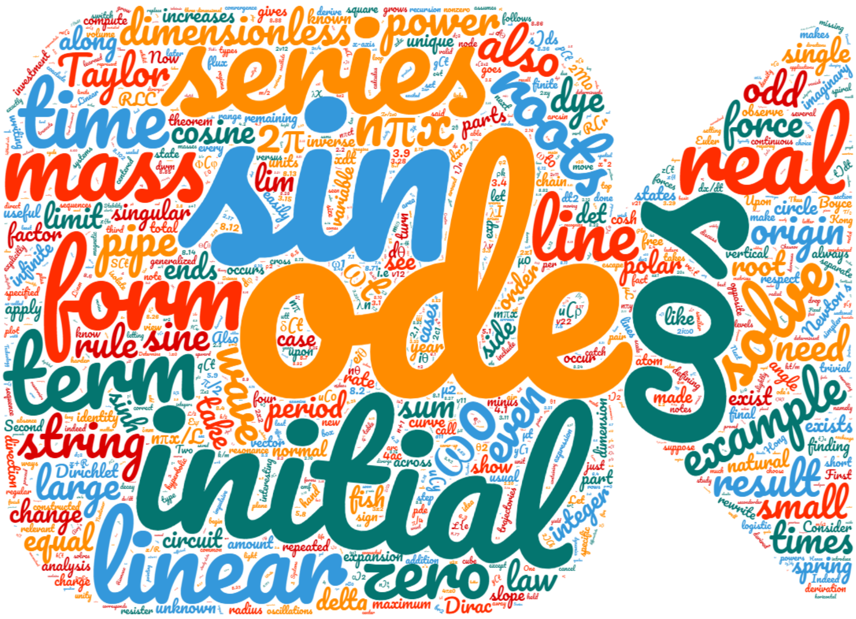

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Preface

This review book, used in conjuction with free online YouTube videos, is designed to help students prepare for exams, or for self-study. The topics covered here are most of the standard topics covered in a first course in differential equations.

The chapters and sections of this review book, organized by topics, can be read independently. Each chapter or section consists of three parts: (1) *Theory*; (2) *YouTube Example*; and (3) *Additional Practice*. In *Theory*, a summary of the topic and associated solution method is given. It is assumed that the student has seen the material before in lecture or in a standard textbook so that the presentation is concise. In *YouTube Example*, an online YouTube video illustrates how to solve an example problem given in the review book. Students are encouraged to view the video before proceeding to *Additional Practice*, which provides additional practice exercises similar to the YouTube example. The solutions to all of the practice exercises are given in this review book's Appendix.

For students who self-study, or desire additional explanatory materials, a complete set of free lecture notes by the author entitled *An Introduction to Differential Equations* can be downloaded by clicking **HERE**. This set of lecture notes also contains links to additional YouTube tutorials. The lecture notes and tutorials have been extensively used by the author over several years when teaching an introductory differential equations course at the Hong Kong University of Science and Technology.

Contents

Chapter 1

First-order Differential Equations

1.1 Separable Equations

1.1.1 Theory

A separable first-order ode for $y = y(x)$ can be written in the form

$$y' = \frac{f(x)}{g(y)},$$

with initial conditions $y(x_0) = y_0$. To solve a separable equation, treat $y' = dy/dx$ like a fraction and multiply by $g(y)dx$ to obtain

$$g(y)dy = f(x)dx.$$

Then integrate both sides to obtain

$$\int_{y_0}^{y} g(y)dy = \int_{x_0}^{x} f(x)dx.$$

Perform the integrations and solve for y when possible. If there are multiple solutions for y, choose the one that satisfies the initial condition.

1.1.2 YouTube Example

To review separable odes, click **HERE**, which solves

$$(1 + y)y' = x, \qquad y(0) = 0.$$

1.1.3 Additional Practice

1. Solve the following separable odes for $y = y(x)$.

a) $y' = \sqrt{xy}, \quad y(1) = 0$

b) $y^2 - xy' = 0, \quad y(1) = 1$

c) $e^{x-y}y' + e^{y-x} = 0, \quad y(0) = 0$

d) $y' + (\sin x)y = 0, \quad y(\pi/2) = 1$

e) $y' = y(1 - y), \quad y(0) = y_0 \quad (y_0 > 0)$

Solutions to the Additional Practice

1.2 Linear Equations

1.2.1 Theory

A linear first-order ode for $y = y(x)$ can be written in the form

$$y' + p(x)y = g(x),$$

with initial condition $y(x_0) = y_0$. To solve a linear equation, if necessary multiply by an integrating factor given by

$$\mu(x) = \exp\left(\int_{x_0}^{x} p(x)dx\right)$$

to obtain

$$\left(\mu(x)y(x)\right)' = \mu(x)g(x).$$

Then integrate to obtain

$$y(x) = \frac{1}{\mu(x)}\left(y_0 + \int_{x_0}^{x} \mu(x)g(x)dx\right).$$

1.2.2 YouTube Example

To review linear odes, click **HERE**, which solves

$$xy' + y = e^x, \qquad y(1) = 0.$$

1.2.3 Additional Practice

1. Solve the following linear odes for $y = y(x)$.

a) $x^2y' = 1 - 2xy, \quad y(1) = 2$

b) $x^4y' + 4x^3y = e^{-x}, \quad y(1) = -1/e$

c) $y' + 2xy = x, \quad y(0) = 1/2$

d) $(1 + x^2)y' + 2xy = 2x, \quad y(0) = 0$

e) $y' + \lambda y = a + be^{-\lambda x}, \quad y(0) = 0 \quad (\lambda > 0)$

Solutions to the Additional Practice

1.3 Exact Equations

1.3.1 Theory

An exact first-order ode for $y = y(x)$ can be written in the form

$$M(x,y) + N(x,y)y' = 0,$$

or equivalently,

$$M(x,y)dx + N(x,y)dy = 0,$$

where for some function $f = f(x,y)$, the functions $M(x,y)$ and $N(x,y)$ satisfy $M = \partial f/\partial x$ and $N = \partial f/\partial y$. An exact ode can therefore be rewritten in the form

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0,$$

or equivalently,

$$\frac{d}{dx}f(x,y(x)) = 0.$$

Integrate to obtain

$$f(x,y) = c,$$

where c is a constant.

To test whether a given differential equation is exact, compute $\partial^2 f/\partial x\partial y$ in two ways to obtain the necessary condition

$$\partial M/\partial y = \partial N/\partial x.$$

If this condition is satisfied, determine the function $f(x,y)$ by first integrating with respect to x the equation

$$\frac{\partial f}{\partial x} = M(x,y)$$

to obtain an equation in the form

$$f(x,y) = F(x,y) + h(y),$$

where $h(y)$ is the constant (independent of x) of integration. To determine $h(y)$, differentiate with respect to y to obtain

$$h'(y) = N(x,y) - \frac{\partial F}{\partial y}.$$

Then integrate to obtain $h(y)$. Absorb all the constants of integration into the final constant c.

1.3.2 YouTube Example

To review exact odes, click **HERE**, which solves

$$2xy + (x^2 - y^2)y' = 0.$$

1.3.3 Additional Practice

1. Show that the following odes are exact and find the general solutions.

a) $(2x - 3y) + (2y - 3x)y' = 0$

b) $(x^2 + 2xy - y^2) + (x^2 - 2xy - y^2)y' = 0$

c) $\dfrac{y}{x} + (\ln x)y' = 0$

d) $(ax + by)dx + (bx + cy)dy = 0$

e) $(\cos\theta + 2r\sin^2\theta)dr + r\sin\theta(2r\cos\theta - 1)d\theta = 0$

Solutions to the Additional Practice

1.4 Bernoulli Equations

1.4.1 Theory

A Bernoulli first-order ode for $y = y(x)$ can be written in the form

$$y' + p(x)y = q(x)y^n, \quad n \neq 0, 1.$$

To solve a Bernoulli equation, write as

$$y^{-n}y' + p(x)y^{1-n} = q(x),$$

and let

$$u = y^{1-n},$$

with derivative

$$u' = (1-n)y^{-n}y'.$$

Substitute into the Bernoulli equation to obtain the linear equation

$$u' + (1-n)p(x)u = (1-n)q(x).$$

Solve this equation using an integrating factor and then transform back to y.

1.4.2 YouTube Example

To review Bernoulli odes, click **HERE**, which solves

$$xy' + y = x^2y^2.$$

1.4.3 Additional Practice

1. Solve the following Bernoulli odes for $y = y(x)$.

a) $xy' + y = x^4y^3, \quad y(1) = 1$

b) $xy' + y = y^2$

c) $x^2y' - y^2 = 2xy$

d) $xy^2y' + y^3 = x, \quad y(1) = 1$

e) $y' = y(1-y), \quad y(0) = y_0$

Solutions to the Additional Practice

1.5 Homogeneous Equations

1.5.1 Theory

A first-order homogeneous ode for $y = y(x)$ can be written in the form

$$y' = F(y/x)$$

for some function F. To solve a homogeneous equation, let

$$u = y/x.$$

Then

$$y' = xu' + u.$$

Substitute into the homogeneous equation to obtain the separable equation

$$xu' + u = F(u).$$

Solve by separating variables and then transform back to y.

1.5.2 YouTube Example

To review homogeneous odes, click **HERE**, which solves

$$x^2y' = x^2 + xy + y^2.$$

1.5.3 Additional Practice

1. Solve the following homogeneous odes for $y = y(x)$.

a) $(x - y)y' = x + y$

b) $(x + y)y' = x - y$

c) $xy' = y + \sqrt{x^2 + y^2}$

d) $(x^2 + y^2)y' = xy$

e) $xyy' = x^2 + y^2$

Solutions to the Additional Practice

1.6 Riccati Equations

1.6.1 Theory

A Riccati first-order ode for $y = y(x)$ can be written in the form

$$y' = P(x)y^2 + Q(x)y + R(x).$$

If a particular solution $y = Y(x)$ of this equation is known, then to solve a Riccati equation, let

$$y(x) = Y(x) + \frac{1}{u(x)},$$

with derivative

$$y' = Y' - \frac{u'}{u^2}.$$

Substitute into the Riccati equation to obtain

$$Y' - \frac{u'}{u^2} = P(x)(Y + \frac{1}{u})^2 + Q(x)(Y + \frac{1}{u}) + R(x).$$

Expand the quadratic, and use the equation satisfied by the particular solution $y = Y(x)$,

$$Y' = P(x)Y^2 + Q(x)Y + R(x),$$

to transform the Riccati equation into the linear equation

$$u' + \big(Q(x) + 2P(x)Y(x)\big)u = -P(x).$$

Solve using an integrating factor and then transform back to y.

1.6.2 YouTube Example

To review Riccati odes, click **HERE**, which solves

$$y' + xy^2 = -\frac{1}{x^3}, \quad Y(x) = \frac{1}{x^2}.$$

1.6.3 Additional Practice

1. Solve the following Riccati odes for $y = y(x)$.

a) $y' = y^2 - y - 2$

i) $Y(x) = 2$

ii) $Y(x) = -1$

b) $y' = \frac{1}{x}\left(y^2 + y - 2\right)$

i) $Y(x) = 1$

ii) $Y(x) = -2$

c) $y' = -2y^2 + \dfrac{1}{x^2}, \quad Y(x) = \dfrac{1}{x}$

Solutions to the Additional Practice

Chapter 2

Second-order Differential Equations with Constant Coefficients

2.1 Homogeneous Equations

2.1.1 Theory

A homogeneous, second-order, constant-coefficient ode for $x = x(t)$ can be written in the form

$$a\ddot{x} + b\dot{x} + cx = 0,$$

where a, b, and c are constants. To solve, try

$$x(t) = e^{rt},$$

where r is a constant to be determined. Substitute into the ode and cancel the common exponential function to derive the characteristic equation

$$ar^2 + br + c = 0;$$

and factor or use the quadratic formula to obtain the two roots. Consider the following three cases.

1. Two real roots. Write the roots as $r = r_1, r_2$ and the general solution as

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

2. Complex conjugate roots. Write the roots as $r = \lambda + i\mu$ and its complex conjugate, and the general solution as

$$x(t) = e^{\lambda t}\left(A\cos\mu t + B\sin\mu t\right).$$

3. One real root. Write the root as r and the general solution as

$$x(t) = e^{rt}\left(c_1 + c_2 t\right).$$

2.1.2 YouTube Example

To review the case of two real roots, click **HERE**, which solves

$$\ddot{x} + 4\dot{x} + 3x = 0; \quad x(0) = 1,\ \dot{x}(0) = 1.$$

To review the case of complex conjugate roots, click **HERE**, which solves

$$\ddot{x} - 2\dot{x} + 5x = 0, \quad x(0) = 1, \dot{x}(0) = 1.$$

To review the case of one real root, click **HERE**, which solves

$$\ddot{x} + 4\dot{x} + 4x = 0; \quad x(0) = 1, \dot{x}(0) = 1.$$

2.1.3 Additional Practice

1. Solve the following homogeneous odes for $x = x(t)$.

a) Two real roots:

i) $\ddot{x} + 3\dot{x} + 2x = 0, \quad x(0) = 0,\ \dot{x}(0) = 1$

ii) $\ddot{x} - 3\dot{x} + 2x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0$

b) Complex conjugate roots:

i) $\ddot{x} - 2\dot{x} + 2x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0$

ii) $\ddot{x} + 2\dot{x} + 2x = 0, \quad x(0) = 0,\ \dot{x}(0) = 1$

c) One real root:

i) $\ddot{x} + 2\dot{x} + x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0$

ii) $\ddot{x} - 2\dot{x} + x = 0, \quad x(0) = 0,\ \dot{x}(0) = 1$

Solutions to the Additional Practice

2.2 Inhomogeneous Equations

2.2.1 Theory

An inhomogeneous, second-order, constant-coefficient ode for $x = x(t)$ can be written in the form

$$a\ddot{x} + b\dot{x} + cx = g(t),$$

where a, b, and c are constants, and $g(t)$ is nonzero. To solve, use a three-step method.

1. Find the general solution $x_h(t)$ of the homogeneous ode

   $$a\ddot{x} + b\dot{x} + cx = 0.$$

   Note that $x_h(t)$ must contain two free constants.

2. Find a particular solution $x_p(t)$ of the inhomogeneous ode. Use the method of undetermined coefficients described below.

3. Write the general solution of the inhomogeneous ode as the sum of the homogeneous and particular solutions,

   $$x(t) = x_h(t) + x_p(t),$$

   and use the initial conditions to determine the two free constants.

The general form of $g(t)$ commonly presented is

$$g(t) = Ct^n e^{\alpha t} \begin{cases} \cos \beta t \\ \sin \beta t \end{cases},$$

where n, α or β may be zero. Sometimes a sum of such functions is presented. Find particular solutions for each term in the sum separately and add them, or treat the sum as a whole.

To find a particular solution, try the trial function

$$x(t) = \left(a_0 t^n + a_1 t^{n-1} + \cdots + a_n\right) e^{\alpha t} \cos \beta t \\ + \left(b_0 t^n + b_1 t^{n-1} + \cdots + b_n\right) e^{\alpha t} \sin \beta t,$$

where the a's and b's are the undetermined coefficients. Substitution into the differential equation should result in a sufficient number of algebraic equations for the undetermined coefficients.

If any term in the trial function is a solution of the homogeneous equation, then multiply the trial function by an extra factor of t (or t^2 when the characteristic equation has repeated roots).

2.2.2 YouTube Example

To review how to find particular solutions for some common inhomogeneous terms, view the following. For an exponential function, click **HERE**, which solves

$$\ddot{x} + \dot{x} + x = 6e^{-2t};$$

for a trigonometric function, click **HERE**, which solves

$$\ddot{x} + \dot{x} + x = \sin 2t;$$

and for a polynomial, click **HERE**, which solves

$$\ddot{x} + \dot{x} + x = t^2.$$

To review how to find a particular solution when the inhomogeneous term is a solution of the homogeneous equation, click **HERE**, which solves

$$\ddot{x} + 3\dot{x} + 2x = 2e^{-t}.$$

To review the three-step solution of an inhomogeneous ode initial value problem, click **HERE**, which solves

$$\ddot{x} + 4\dot{x} + 3x = e^{-2t}, \quad x(0) = 1, \ \dot{x}(0) = 0.$$

2.2.3 Additional Practice

1. Find the particular solutions for the following inhomogeneous odes.

a) Exponential inhomogeneous term:

i) $\ddot{x} + 3\dot{x} + 2x = e^{2t}$

ii) $\ddot{x} + 3\dot{x} + 2x = e^{-2t}$

b) Sine or cosine inhomogeneous term:

i) $\ddot{x} + 3\dot{x} + 2x = \sin 2t$

ii) $\ddot{x} + 3\dot{x} + 2x = \cos 2t$

c) Polynomial inhomogeneous term:

i) $\ddot{x} + 3\dot{x} + 2x = 2t$

ii) $\ddot{x} + 3\dot{x} + 2x = t^2 + 2t$

2. Solve the inhomogeneous ode for $x = x(t)$.

a) $\ddot{x} + 3\dot{x} + 2x = e^{-2t}, \quad x(0) = 0, \ \dot{x}(0) = 0$

Solutions to the Additional Practice

Chapter 3

The Laplace Transform

3.1 Theory

Define the Laplace transform of a function $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, as

$$F(s) = \int_0^\infty e^{-st} f(t) dt.$$

Performing this integral for various functions $f(t)$ results in the table of Appendix A. In particular, the first and second derivatives of $x(t)$ are transformed as

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0), \quad \mathcal{L}\{\ddot{x}(t)\} = s^2 X(s) - sx(0) - \dot{x}(0).$$

To solve a constant-coefficient, inhomogenous differential equation for $x = x(t)$ of the form

$$a\ddot{x} + b\dot{x} + cx = g(t),$$

where $x(0)$ and $\dot{x}(0)$ are given, Laplace transform the differential equation for $x(t)$ into a solvable algebraic equation for $X(s)$, and then inverse Laplace transform $X(s)$ back into $x(t)$. To find $x(t)$ using the table, use the techniques of partial fraction decomposition and completing the square.

Typically, the Heaviside step function, $u_c(t)$, and the Dirac delta function, $\delta(t - c)$, are encountered when studying the Laplace transform technique. Both functions may appear in the inhomogeneous term and are used to model piecewise-continuous and impulsive forces.

3.2 YouTube Example

To review how to solve a standard inhomogeneous ode using the Laplace transform techique, click **HERE**, which solves

$$\ddot{x} + 2\dot{x} + 5x = e^{-t}, \quad x(0) = 0, \ \dot{x}(0) = 0.$$

To review how to solve an ode with a piecewise-continuous inhomogeneous term, click **HERE**, which solves

$$\ddot{x} + 3\dot{x} + 2x = \begin{cases} t & \text{if } 0 \leq t < 1, \\ 1 & \text{if } t \geq 1; \end{cases} \quad x(0) = 0, \ \dot{x}(0) = 0.$$

To review how to solve an ode with a Dirac delta-function inhomogeneous term, click **HERE**, which solves

$$\ddot{x} + 2\dot{x} + x = \delta(t - 1), \quad x(0) = 1, \ \dot{x}(0) = 0.$$

3.3 Additional Practice

1. Solve the following inhomogeneous odes using the Laplace transform technique.

a) $\ddot{x} + 2\dot{x} + 5x = e^{-2t}, \quad x(0) = 0, \ \dot{x}(0) = 0$

b) $\ddot{x} + 3\dot{x} + 2x = \begin{cases} 1 - t & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t \geq 1; \end{cases} \quad x(0) = 0, \ \dot{x}(0) = 0$

c) $\ddot{x} + 2\dot{x} + x = \delta(t - 1), \quad x(0) = 0, \ \dot{x}(0) = 1$

Solutions to the Additional Practice

Chapter 4

Series Solutions

4.1 Theory

A power series solution around $x = 0$ can be used to solve a linear, homogeneous equation for $y = y(x)$ of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials or convergent power series with no common polynomial factors, and $P(0) \neq 0$.

To solve, try

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Now write the sum $P(x)y'' + Q(x)y' + R(x)y$ as a single power series by shifting summation indices either up or down to match powers of x. Then set the coefficient of each power of x to zero. Determine a recursion relation for the unknown coefficients a_n. Solve this recursion relation to obtain two independent power series, each multiplied by a single free constant (usually a_0 and a_1). Write the general solution of the differential equation by summing these two power series. If initial conditions are specified, determine the values of the free constants.

4.2 YouTube Example

To review how to find a power series solution, click **HERE**, which solves the Airy differential equation given by

$$y'' - xy = 0.$$

To review how a power series substitution may yield a polynomial solution, click **HERE**, which solves the Hermite differential equation given by

$$y'' - 2xy' + 2\lambda y = 0,$$

where λ is a constant. Polynomial solutions occur when $\lambda = n$ is a nonnegative integer.

4.3 Additional Practice

1. Find two independent power series solutions to the following differential equations, where the highest power of x to be computed is specified.

a) $y'' + xy' + y = 0 \quad (x^6)$

b) $y'' + xy' - y = 0 \quad (x^6)$

c) $y'' + y' + xy = 0 \quad (x^5)$

2. The Chebychev equation is given by

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

a) Find the first three terms in each of two power series solutions.

b) If $\alpha = n$ is an integer, then one of the power series solutions becomes a polynomial. Find the polynomial solutions for $n = 0, 1, 2, 3$.

c) The Chebychev polynomials are the polynomial solutions $T_n(x)$ normalized so that $T_n(1) = 1$. Find $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$.

Solutions to the Additional Practice

Chapter 5

Cauchy-Euler Equations

5.1 Theory

The Cauchy-Euler equation for $y = y(x)$ can be written in the form

$$x^2y'' + \alpha xy' + \beta y = 0,$$

with α and β constants. Here, assume that $x > 0$. (To obtain solutions with $x < 0$, replace x by $-x$.) To solve, try

$$y(x) = x^r,$$

where r is a constant to be determined. Substitute into the ode and cancel the common power law factor to derive the characteristic or indicial equation

$$r^2 + (\alpha - 1)r + \beta = 0;$$

and factor or use the quadratic formula to obtain the two roots. Consider the following three cases.

1. Two real roots. Write the roots as $r = r_1, r_2$ and the general solution as

$$y(x) = c_1x^{r_1} + c_2x^{r_2}.$$

2. Complex conjugate roots. Write the roots as $r = \lambda + i\mu$ and its complex conjugate, and the general solution as

$$y(x) = x^\lambda\big(A\cos(\mu\ln x) + B\sin(\mu\ln x)\big).$$

3. One real root. Write the root as r and the general solution as

$$y(x) = x^r(c_1 + c_2\ln x).$$

5.2 YouTube Example

To review the case of two real roots, click **HERE**, which solves

$$x^2y'' + xy' - y = 0; \quad y(0) = 0,\ y(1) = 1.$$

To review the case of complex conjugate roots, click **HERE**, which solves

$$x^2y'' - xy' + (1 + \pi^2/4)y = 0; \quad y(1) = 1,\ y(e) = e.$$

To review the case of one real root, click **HERE**, which solves

$$x^2y'' + 3xy' + y = 0; \quad y(1) = 1,\ y(e) = 1.$$

5.3 Additional Practice

1. Solve the following Cauchy-Euler equations for $y = y(x)$, $x > 0$.

a) Two real roots:

i) $x^2y'' - 2xy' + 2y = 0; \quad y'(0) = 1,\ y(1) = 0$

ii) $2x^2y'' - xy' + y = 0; \quad y(1) = 0,\ y(4) = 1$

b) Complex conjugate roots:

i) $x^2y'' - xy' + (1+\pi^2)y = 0; \quad y(1) = 1,\ y(\sqrt{e}) = \sqrt{e}$

ii) $x^2y'' + 3xy' + (1+\pi^2)y = 0; \quad y(1) = 1,\ y(\sqrt{e}) = \sqrt{e}$

c) One real root:

i) $x^2y'' - xy' + y = 0; \quad y(1) = 1,\ y'(1) = 0$

ii) $4x^2y'' + y = 0; \quad y(1) = 1,\ y(e) = 0$

Solutions to the Additional Practice

Chapter 6

Systems of Linear Equations

6.1 Theory

A system of first-order, linear, homogeneous odes can be written in matrix form as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where $\mathbf{x} = \mathbf{x}(t)$ is an n-dimensional column vector and $\mathbf{A}$ is a constant n-by-n square matrix. To solve, try

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t},$$

where $\mathbf{v}$ is a constant n-dimensional column vector and λ is a constant scalar. Substitute into the ode and cancel the common exponential function to obtain the eigenvalue problem

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v},$$

with characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Find n linearly independent solutions and use the principle of superposition to find the general solution. Consider eigenvalues of three different types.

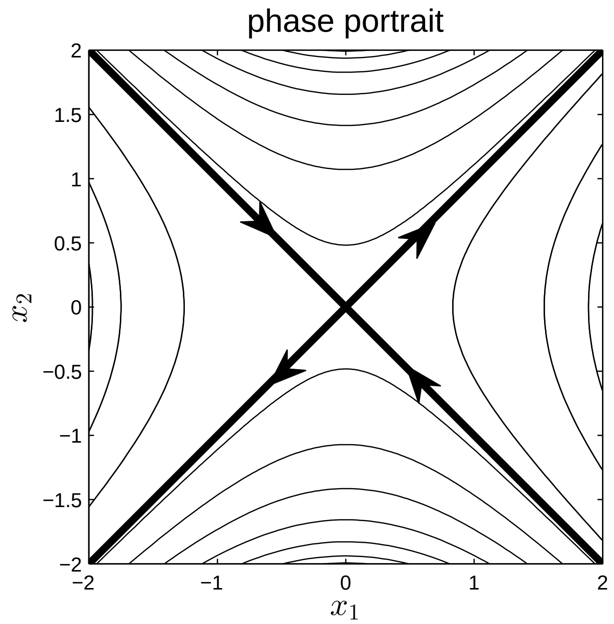
1. Real eigenvalue. With eigenvalue λ and eigenvector $\mathbf{v}$, write one solution as $\mathbf{x_1}(t) = \mathbf{v}e^{\lambda t}$.

2. Complex conjugate eigenvalues. With complex eigenvalues λ and $\bar{\lambda}$, and complex eigenvectors $\mathbf{v}$ and $\bar{\mathbf{v}}$, write two solutions as $\mathbf{x_1}(t) = \text{Re}\,(\mathbf{v}e^{\lambda t})$ and $\mathbf{x_2}(t) = \text{Im}\,(\mathbf{v}e^{\lambda t})$.

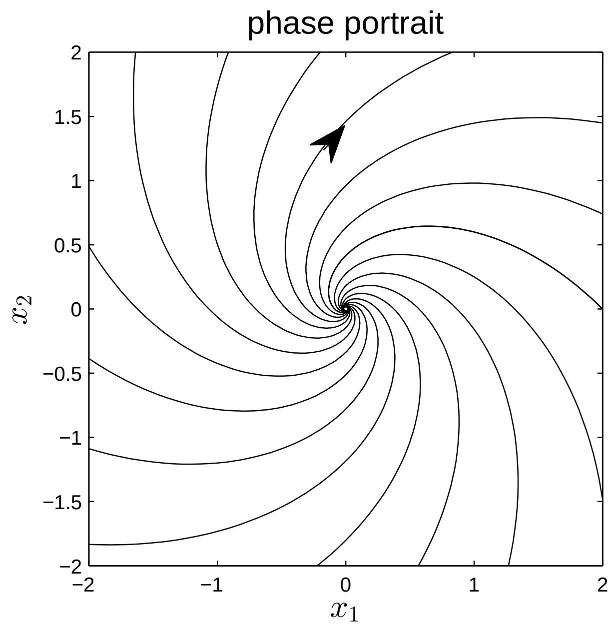
3. Repeated eigenvalue with fewer eigenvectors than eigenvalues. If the real eigenvalue λ has multiplicity 2, say, and there is only one linearly independent eigenvector $\mathbf{v}$, then write one solution as $\mathbf{x_1}(t) = \mathbf{v}e^{\lambda t}$ and seek a second solution by trying $\mathbf{x}(t) = (\mathbf{w} + t\mathbf{v})e^{\lambda t}$ with $\mathbf{w}$ an unknown constant vector. Solve the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$ for $\mathbf{w}$. Higher multiplicities can also be treated.

If $\mathbf{A}$ is a two-by-two matrix, then write the characteristic equation as $\lambda^2 - \text{Tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$. Represent the solutions in a phase portrait, which plots the trajectories of x_2 versus x_1 for various initial conditions. To sketch a phase portrait, consider three cases.

1. Two real eigenvalues. Draw two lines through the origin corresponding to trajectories following a single eigenvector. If the eigenvalue is negative, then draw arrows on the corresponding line pointing toward the origin; if the eigenvalue is positive, then draw arrows pointing away from the origin. Sketch trajectories corresponding to initial conditions with mixed eigenvectors. If both eigenvalues are negative, call the origin a sink or a stable node; if both eigenvalues are positive, call the origin a source or an unstable node; and if the eigenvalues have opposite sign, call the origin a saddle point. Below is a sample phase portrait for eigenvalues of opposite sign.

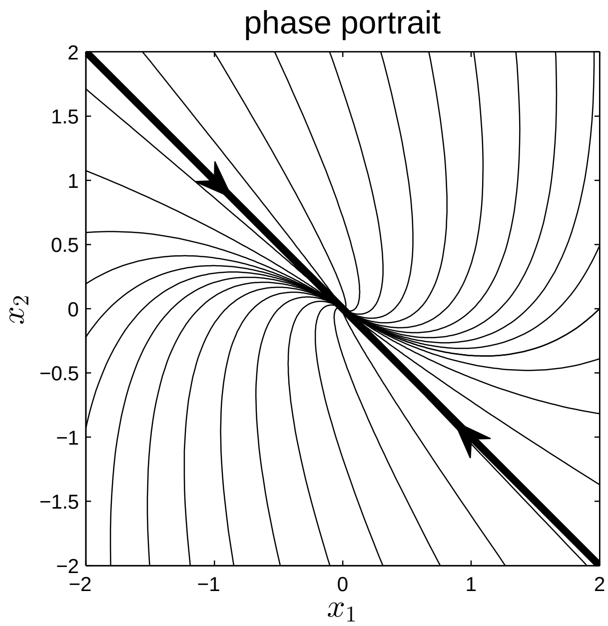


2. Complex conjugate eigenvalues. If the real part is negative, call the origin a stable spiral point and draw trajectories that spiral into the origin. If the real part is positive, call the origin an unstable spiral point and draw trajectories that spiral out of the origin. If the eigenvalues are pure imaginary, call the origin a center and draw trajectories that are closed ellipses. To determine the direction of rotation, compute $L = x_1\dot{x}_2 - x_2\dot{x}_1$, using the odes to eliminate $\dot{x}_1$ and $\dot{x}_2$ in favor of x_1 and x_2. If L is positive, then draw counterclockwise spirals and if L is negative, draw clockwise spirals. Below is a sample phase portrait for complex conjugate eigenvalues with a positive real part and with $L < 0$.



3. Repeated eigenvalue with only one eigenvector. Draw a line through the origin corresponding to the trajectory following the single eigenvector. If the eigenvalue is negative, then draw arrows on the line pointing toward the origin; if the eigenvalue is positive, then draw arrows pointing away from the origin. Draw rotating trajectories that are blocked by the drawn line and call the origin

an improper node. To determine the direction of rotation, compute the sign of $L = x_1\dot{x}_2 - x_2\dot{x}_1$. Below is a sample phase portrait for a negative repeated eigenvalue with $L < 0$.



6.2 YouTube Example

All examples are for two-by-two matrices. To review the case of two real eigenvalues, click **HERE**, which solves

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1.$$

To review the case of complex conjugate eigenvalues, click **HERE**, which solves

$$\dot{x}_1 = x_1 + x_2, \quad \dot{x}_2 = -x_1 + x_2.$$

To review the case of repeated eigenvalues with only one linearly independent eigenvector, click **HERE**, which solves

$$\dot{x}_1 = -x_1 + x_2, \quad \dot{x}_2 = -x_1 - 3x_2.$$

6.3 Additional Practice

1. Find the general solution of the following system of odes. Sketch the phase portraits.

a) Two real eigenvalues:

i) $\dot{x}_1 = 7x_1 - 2x_2, \quad \dot{x}_2 = 2x_1 + 2x_2$

ii) $\dot{x}_1 = -x_2, \quad \dot{x}_2 = -2x_1 - x_2$

b) Complex conjugate eigenvalues:

i) $\dot{x}_1 = x_1 - 2x_2, \quad \dot{x}_2 = x_1 + x_2$

ii) $\dot{x}_1 = -x_1 - x_2, \quad \dot{x}_2 = x_1 - x_2$

c) Repeated eigenvalues:

i) $\dot{x}_1 = x_1 + x_2, \quad \dot{x}_2 = -4x_1 - 3x_2$

Solutions to the Additional Practice

Chapter 7

Non-linear Differential Equations

7.1 Fixed Points and Linear Stability Analysis

7.1.1 Theory

An autonomous, nonlinear ode for $x = x(t)$ can be written in the form

$$\dot{x} = f(x),$$

where $f(x)$ is a nonlinear function of x and independent of t. To determine the fixed points of the ode, solve the equation $f(x) = 0$ for $x = x_*$. To determine the linear stability of a fixed point, compute $f'(x)$. If $f'(x_*) < 0$, then the fixed point is stable, and if $f'(x_*) > 0$, then the fixed point is unstable.

A two-dimensional, autonomous, system of nonlinear odes can be written in the form

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y).$$

To determine the fixed points of this system, solve the simultaneous equations $f(x, y) = 0$ and $g(x, y) = 0$ for $(x, y) = (x_*, y_*)$. To determine the linear stability of a fixed point, compute the the Jacobian matrix given by

$$\mathbf{J} = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}.$$

Evaluate the Jacobian matrix at the fixed point and compute its eigenvalues. If both eigenvalues have negative real parts, then the fixed point is stable. If at least one of the eigenvalues has a positive real part, then the fixed point is unstable. Use of the Jacobian to determine stability can be generalized to higher-dimensional systems.

7.1.2 YouTube Example

To review how to find the fixed points of a nonlinear ode and classify their stability, click **HERE**, which considers

$$\dot{x} = x^2 - 1.$$

To review how to find the fixed points of a system of nonlinear odes and to determine their stability, click **HERE**, which considers

$$\dot{x} = x(1 - x - 2y), \quad \dot{y} = y(1 - 2x - y).$$

7.1.3 Additional Practice

1. Find all the fixed points of the following odes and classify their stability.

a) $\dot{x} = 4x^2 - 16$

b) $\dot{x} = x(1 - x^2)$

c) $\dot{x} = x(1 - 2x - y), \quad \dot{y} = y(1 - x - 2y)$

Solutions to the Additional Practice

7.2 Bifurcation Theory

7.2.1 Theory

A bifurcation occurs in a nonlinear differential equation when a small change in a parameter results in a qualitative change in the asymptotic solution. For example, bifurcations occur when fixed points are created or destroyed, or change their stability.

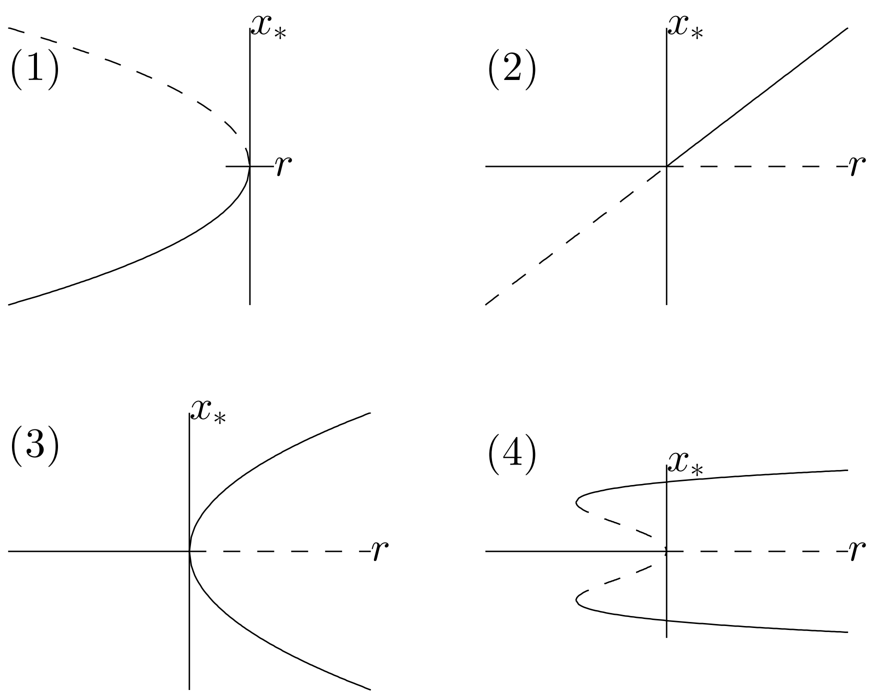
A nonlinear differential equation with a bifurcation parameter r can be written in the form

$$\dot{x} = f_r(x).$$

At a bifurcation point, multiple fixed points coalesce, resulting in four classic one-dimensional bifurcations.

1. Saddle-node bifurcation. Two fixed points—one stable and the other unstable—are created or destroyed.
2. Transcritical bifurcation. Two fixed points cross and exchange stability.
3. Supercritical pitchfork bifurcation. A stable fixed point becomes unstable and two symmetric stable fixed points are created.
4. Subcritical pitchfork bifurcation. A stable fixed point becomes unstable and two symmetric unstable fixed points are destroyed. There are no local stable fixed points above the bifurcation point, and the system usually jumps to a far away stable fixed point that may have been created in two symmetric saddle-node bifurcations below the bifurcation point.

Identify a bifurcation point by setting both $f_r(x)$ and $f_r'(x)$ equal to zero. The bifurcation diagrams representing the four classic one-dimensional bifurcations are shown below.



7.2.2 YouTube Examples

To review the normal form of a saddle-node bifurcation, click **HERE**, which considers

$$\dot{x} = r + x^2.$$

To review the normal form of a transcritical bifurcation, click **HERE**, which considers

$$\dot{x} = rx - x^2.$$

To review the normal form of a supercritical pitchfork bifurcation, click **HERE**, which considers

$$\dot{x} = rx - x^3.$$

To review the normal form of a subcritical pitchfork bifurcation, click **HERE**, which considers

$$\dot{x} = rx + x^3.$$

7.2.3 Additional Practice

1. Determine the value of r for which a bifurcation occurs, and identify the type of bifurcation. The functions $\cosh x$ and $\sinh x$ are the usual hyperbolic cosine and sine functions and are related to the exponential functions by $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.

a) $\dot{x} = r - \cosh x$

b) $\dot{x} = x(r - \sinh x)$

c) $\dot{x} = rx - \sinh x$

d) $\dot{x} = rx - \sin x$

Solutions to the Additional Practice

Chapter 8

Fourier Series

8.1 Theory

The Fourier series of a periodic function $f(x)$ with period $2L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

Determine the Fourier coefficients using the orthogonality relations for sine and cosine to obtain

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx.$$

If $f(-x) = f(x)$, then $f(x)$ is an even function and the Fourier series becomes a Fourier cosine series with all the b_n's equal to zero and

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx.$$

If $f(-x) = -f(x)$, then $f(x)$ is an odd function and the Fourier series becomes a Fourier sine series with all the a_n's equal to zero and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

8.2 YouTube Example

To review how to find a Fourier series, click **HERE**, which determines the Fourier series for the periodic extension of

$$f(x) = x, \quad -\pi < x \leq \pi.$$

8.3 Additional Practice

1. Find the Fourier series for the periodic extensions of

a) $f(x) = x^2, \quad -\pi < x \leq \pi$

b) $f(x) = \begin{cases} -1, & -\pi < x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}$

c) $f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}$

Solutions to the Additional Practice

Appendix

Appendix A

Table of Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. $e^{at}f(t)$	$F(s-a)$
2. 1	$\frac{1}{s}$
3. e^{at}	$\frac{1}{s-a}$
4. t^n	$\frac{n!}{s^{n+1}}$
5. $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
6. $\sin bt$	$\frac{b}{s^2+b^2}$
7. $\cos bt$	$\frac{s}{s^2+b^2}$
8. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
9. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
10. $t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
11. $t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
12. $u_c(t)$	$\frac{e^{-cs}}{s}$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $\delta(t-c)$	e^{-cs}
15. $\dot{x}(t)$	$sX(s)-x(0)$
16. $\ddot{x}(t)$	$s^2X(s)-sx(0)-\dot{x}(0)$

Appendix B

Problem solutions

Solutions to the Additional Practice of §1.1.3

1.

a) From $\int_0^y dy/y^{1/2} = \int_1^x x^{1/2}dx$, we obtain $y(x) = \frac{1}{9}\left(x^{3/2}-1\right)^2$.

b) From $\int_1^y dy/y^2 = \int_1^x dx/x$, we obtain $y(x) = \frac{1}{1-\ln x}$.

c) From $\int_0^y e^{-2y}dy = -\int_0^x e^{-2x}dx$, we obtain $y(x) = -\frac{1}{2}\ln\left(2-e^{-2x}\right)$.

d) From $\int_1^y dy/y = -\int_{\pi/2}^x \sin x dx$, we obtain $y(x) = e^{\cos x}$.

e) From $\int_{y_0}^y \frac{dy}{y(1-y)} = \int_0^x dx$ and the partial fraction decomposition $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$, we obtain $y(x) = \frac{y_0}{y_0+(1-y_0)e^{-x}}$.

Solutions to the Additional Practice of §1.2.3

1.

a) Since $x^2y' + 2xy = (x^2y)'$, we obtain from direct integration $y(x) = \dfrac{1+x}{x^2}$.

b) Since $x^4y' + 4x^3y = (x^4y)'$, we obtain from direct integration $y(x) = -\dfrac{e^{-x}}{x^4}$.

c) Using the integrating factor $\mu(x) = \exp\left(\int_0^x 2x dx\right) = e^{x^2}$, we obtain

$$y(x) = e^{-x^2}\left(\frac{1}{2} + \int_0^x xe^{x^2}dx\right) = \frac{1}{2}.$$

d) Since $(1+x^2)y' + 2xy = \Big((1+x^2)y\Big)'$, we obtain from direct integration $y(x) = \dfrac{x^2}{1+x^2}$.

e) Using the integrating factor $\mu(x) = \exp\left(\int_0^x \lambda dx\right) = e^{\lambda x}$, we obtain

$$y(x) = e^{-\lambda x}\int_0^x \left(a + be^{-\lambda x}\right)e^{\lambda x}dx = \frac{a}{\lambda}\left(1 - e^{-\lambda x}\right) + bxe^{-\lambda x}.$$

Solutions to the Additional Practice of §1.3.3

1.

a) Let $M = 2x - 3y$ and $N = 2y - 3x$. Then $M_y = N_x = -3$ so the equation is exact. From $f_x = M$ we obtain $f = x^2 - 3xy + h(y)$, and from $f_y = N$ we obtain $h'(y) = 2y$ or $h(y) = y^2$. The general solution is $x^2 - 3xy + y^2 = c$.

b) Let $M = x^2 + 2xy - y^2$ and $N = x^2 - 2xy - y^2$. Then $M_y = N_x = 2x - 2y$ so the equation is exact. From $f_x = M$ we obtain $f = x^3/3 + x^2y - xy^2 + h(y)$, and from $f_y = N$ we obtain $h'(y) = -y^2$ or $h(y) = -y^3/3$. The general solution is $x^3/3 + x^2y - xy^2 - y^3/3 = c$.

c) Since $y/x + (\ln x)y' = (y \ln x)'$, the general solution is $y \ln x = c$.

d) Let $M = ax + by$ and $N = bx + cy$. Then $M_y = N_x = b$ so the equation is exact. From $f_x = M$ we obtain $ax^2/2 + bxy + h(y)$, and from $f_y = N$ we obtain $h'(y) = cy$ or $h(y) = cy^2/2$. The general solution is $x^2/2 + bxy + y^2/2 = k$.

e) Let $M(r, \theta) = \cos\theta + 2r\sin^2\theta$ and $N(r, \theta) = r\sin\theta(2r\cos\theta - 1)$. Then $M_\theta = N_r = 4r\sin\theta\cos\theta - \sin\theta$ so the equation is exact. From $f_r = M$ we obtain $f = r\cos\theta + r^2\sin^2\theta + h(\theta)$, and from $f_\theta = N$ we obtain $h'(\theta) = 0$. The general solution is $r\cos\theta + r^2\sin^2\theta = c$.

Solutions to the Additional Practice of §1.4.3

1.

a) Write as $xy^{-3}y' + y^{-2} = x^4$ and use the substitution $u = y^{-2}$ to obtain $u' - 2u/x = -2x^3$. With $u(1) = 1$, we have the integrating factor $\mu = 1/x^2$ and

$$u = x^2\left(1 - \int_1^x 2x dx\right) = x^2(2 - x^2).$$

With $y(1) = 1$ the solution is $y = 1/\sqrt{u}$, or $y(x) = \dfrac{1}{x\sqrt{2 - x^2}}$.

b) Write as $xy^{-2}y' + y^{-1} = 1$ and use the substitution $u = 1/y$ to obtain $u' - u/x = -1/x$. We have the integrating factor $\mu = 1/x$ and $u = x(c - \int x^{-2}dx) = 1 + cx$. The general solution is $y(x) = 1/(1 + cx)$.

c) Write as $x^2y^{-2}y' - 2xy^{-1} = 1$ and use the substitution $u = 1/y$ to obtain $x^2u' + 2xu = -1$. Since $x^2u' + 2xu = (x^2u)'$, we obtain from direct integration $u = (c - x)/x^2$, so that $y(x) = x^2/(c - x)$.

d) Write as $3y^2y' + 3y^3/x = 3$ and use the substitution $u = y^3$ to obtain $u' + 3u/x = 3$. With $u(1) = 1$ we have the integrating factor $\mu = x^3$ and $u = x^{-3}\left(1 + \int_1^x 3x^3 dx\right) = \dfrac{1}{x^3}\left(\dfrac{1}{4} + \dfrac{3}{4}x^4\right)$. With $y(1) = 1$ the solution is $y(x) = \dfrac{1}{x}\left(\dfrac{1}{4} + \dfrac{3}{4}x^4\right)^{1/3}$.

e) Write as $y^{-2}y' - y^{-1} = -1$ and use the substitution $u = 1/y$ to obtain $u' + u = 1$. With $u(0) = 1/y_0$, we have the integrating factor $\mu = e^x$ and

$$u = e^{-x}\left(\frac{1}{y_0} + \int_0^x e^x dx\right) = 1 + \left(\frac{1}{y_0} - 1\right)e^{-x}.$$

The solution is $y(x) = \dfrac{y_0}{y_0 + (1 - y_0)e^{-x}}$.

Solutions to the Additional Practice of §1.5.3

1.

a) Write as $(1 - y/x)y' = 1 + y/x$ and let $u = y/x$ to obtain

$$xu' = \frac{1+u^2}{1-u},$$

$$\int \frac{1-u}{1+u^2}du = \int \frac{dx}{x},$$

$$\int \frac{1}{1+u^2} - \frac{1}{2}\int \frac{2u}{1+u^2}du - \int \frac{dx}{x} = 0,$$

$$\tan^{-1} u - \frac{1}{2}\ln(1+u^2) - \ln x = c,$$

$$\tan^{-1}(y/x) - \ln\sqrt{x^2+y^2} = c.$$

b) Write as $(1 + y/x)y' = 1 - y/x$ and let $u = y/x$ to obtain

$$xu' = \frac{1-2u-u^2}{1+u},$$

$$\int \frac{1+u}{1-2u-u^2}du = \int \frac{dx}{x},$$

$$-\frac{1}{2}\int \frac{-2-2u}{1-2u-u^2}du - \int \frac{dx}{x} = 0,$$

$$-\frac{1}{2}\ln|1-2u-u^2| - \ln|x| = c_1,$$

$$\ln|1-2u-u^2| + \ln x^2 = c_2,$$

$$x^2 - 2xy - y^2 = c.$$

c) Write as $y' = y/x + \sqrt{1+(y/x)^2}$ and let $u = y/x$ to obtain

$$xu' = \sqrt{1+u^2},$$

$$\int \frac{du}{\sqrt{1+u^2}} - \int \frac{dx}{x} = 0,$$

$$\ln|u + \sqrt{1+u^2}| - \ln|x| = c_1,$$

$$y + \sqrt{x^2+y^2} = cx^2.$$

d) Write as $(1 + (y/x)^2)y' = y/x$ and let $u = y/x$ to obtain

$$(1+u^2)(xu'+u) = u,$$

$$\int \frac{1+u^2}{u^3}du + \int \frac{dx}{x} = 0.$$

$$-\frac{1}{2u^2} + \ln|u| + \ln|x| = c_1,$$

$$-\frac{x^2}{2y^2} + \ln|y| = c_1,$$

$$y = c \exp\left(x^2/2y^2\right).$$

e) Write as $(y/x)y' = 1 + (y/x)^2$, and let $u = y/x$ to obtain

$$u(xu' + u) = 1 + u^2,$$

$$\int \frac{dx}{x} = \int u du,$$

$$\ln|x| = \frac{1}{2}u^2 + c_1,$$

$$x = c \exp(y^2/2x^2).$$

Note that this solution could have been obtained from Question 4 by the substitutions $y \to x$ and $x \to y$.

Solutions to the Additional Practice of §1.6.3

1.

a) *i)* Let $y = 2 + 1/u$ to obtain $u' + 3u = -1$. With the integrating factor $\mu = e^{3x}$, the solution is $u = -1/3 + c_1 e^{-3x}$. We find $y(x) = \dfrac{2ce^{-3x} + 1}{ce^{-3x} - 1}$.

ii) Let $y = -1 + 1/u$ to obtain $u' - 3u = -1$. With the integrating factor $\mu = e^{-3x}$, the solution is $u = 1/3 + c_2 e^{3x}$. By redefining constants, the same result can be found as (a).

b) *i)* Let $y = 1 + 1/u$ to obtain $u' + 3u/x = -1/x$. With the integrating factor $\mu = x^3$, the solution is $u = c_1/x^3 - 1/3$. We find $y(x) = \dfrac{c + 2x^3}{c - x^3}$.

ii) Let $y = -2 + 1/u$ to obtain $u' - 3u/x = -1/x$. With the integrating factor $\mu = 1/x^3$, the solution is $u = 1/3 + c_2 x^3$. By redefining constants, the same result can be found as (a).

c) Let $y = 1/x + 1/u$ to obtain $u' - 4u/x = 2$. With the integrating factor $\mu = 1/x^4$, the solution is $u = c_1 x^4 - 2x/3$. We find $y(x) = \dfrac{cx^3 + 1}{cx^4 - 2x}$.

Solutions to the Additional Practice of §2.1.3

1.

a) *i)* Try $x = e^{rt}$. We obtain $r^2 + 3r + 2 = (r+1)(r+2) = 0$ so that $x(t) = c_1 e^{-t} + c_2 e^{-2t}$. Initial conditions result in $c_1 + c_2 = 0$ and $-c_1 - 2c_2 = 1$, yielding $c_1 = 1$ and $c_2 = -1$. The solution is $x(t) = e^{-t} - e^{-2t} = e^{-t}(1 - e^{-t})$.

ii) Try $x = e^{rt}$. We obtain $r^2 - 3r + 2 = (r-1)(r-2) = 0$ so that $x(t) = c_1 e^{t} + c_2 e^{2t}$. Initial conditions result in $c_1 + c_2 = 1$ and $c_1 + 2c_2 = 0$, yielding $c_1 = 2$ and $c_2 = -1$. The solution is $x(t) = 2e^{t} - e^{2t} = -e^{2t}(1 - 2e^{-t})$.

b) *i)* Try $x = e^{rt}$. We obtain $r^2 - 2r + 2 = 0$ with roots $r_\pm = 1 \pm i$ so that $x(t) = e^{t}(A\cos t + B\sin t)$. Initial conditions result in $A = 1$ and $A + B = 0$, yielding $B = -1$. The solution is $x(t) = e^{t}(\cos t - \sin t)$.

ii) Try $x = e^{rt}$. We obtain $r^2 + 2r + 2 = 0$ with roots $r_\pm = -1 \pm i$ so that $x(t) = e^{-t}(A\cos t + B\sin t)$. Initial conditions result in $A = 0$ and $-A + B = 1$, yielding $B = 1$. The solution is $x(t) = e^{-t}\sin t$.

c) *i)* Try $x = e^{rt}$. We obtain $r^2 + 2r + 1 = (r+1)^2 = 0$ so that $x(t) = e^{-t}(c_1 + c_2 t)$. Initial conditions result in $c_1 = 1$ and $c_2 - c_1 = 0$, yielding $c_2 = 1$. The solution is $x(t) = e^{-t}(1 + t)$.

ii) Try $x = e^{rt}$. We obtain $r^2 - 2r + 1 = (r-1)^2 = 0$ so that $x(t) = e^{t}(c_1 + c_2 t)$. Initial conditions result in $c_1 = 0$ and $c_1 + c_2 = 1$, yielding $c_2 = 1$. The solution is $x(t) = te^{t}$.

Solutions to the Additional Practice of §2.2.3

1.

a) *i)* Try $x = Ae^{2t}$. We obtain $4A + 6A + 2A = 1$, or $A = 1/12$. The particular solution is $x_p = \frac{1}{12}e^{2t}$.

ii) The inhomogeneous term is a solution of the homogeneous equation. Try $x = ty$, with $y = Ae^{-2t}$ a solution of the homogeneous equation. We have $\dot{x} = y + t\dot{y}$, $\ddot{x} = 2\dot{y} + t\ddot{y}$. Since $\ddot{y} + 3\dot{y} + 2y = 0$, we have $2\dot{y} + 3y = e^{-2t}$ or $-4A + 3A = 1$, or $A = -1$. The particular solution is $x_p = -te^{-2t}$.

b) *i)* Try $x = A\sin 2t + B\cos 2t$ so that $\dot{x} = 2A\cos 2t - 2B\sin 2t$ and $\ddot{x} = -4A\sin 2t - 4B\cos 2t$. Then $\ddot{x} + 3\dot{x} + 2x = (-4A - 6B + 2A)\sin 2t + (-4B + 6A + 2B)\cos 2t = \sin 2t$. Matching coefficients, we have $-2A - 6B = 1$ and $6A - 2B = 0$, with solution $A = -1/20$, $B = -3/20$. The particular solution is $x_p = -\frac{1}{20}(\sin 2t + 3\cos 2t)$.

ii) Same as (a) except we have $-2A - 6B = 0$ and $6A - 2B = 1$, with solution $A = 3/20$, $B = -1/20$. The particular solution is $x_p = \frac{1}{20}(3\sin 2t - \cos 2t)$.

c) *i)* Try $x = At + B$, so that $\dot{x} = A$ and $\ddot{x} = 0$. Then $\ddot{x} + 3\dot{x} + 2x = 2At + (3A + 2B) = 2t$. Matching coefficients, we have $2A = 2$ and $3A + 2B = 0$, or $A = 1$ and $B = -3/2$. The particular solution is $x_p = t - 3/2$.

ii) Try $x = At^2 + Bt + C$, so that $\dot{x} = 2At + B$ and $\ddot{x} = 2A$. Then $\ddot{x} + 3\dot{x} + 2x = 2At^2 + (6A + 2B)t + (2A + 3B + 2C) = t^2 + 2t$. Matching coefficients, we have $2A = 1$, $6A + 2B = 2$, and $2A + 3B + 2C = 0$, or $A = 1/2$, $B = -1/2$, and $C = 1/4$. The particular solution is $x_p = \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4}$.

2.

a) The homogeneous solution is $x_h = c_1e^{-t} + c_2e^{-2t}$ and the particular solution is $x_p = -te^{-2t}$. The general solution is $x = c_1e^{-t} + c_2e^{-2t} - te^{-2t}$. Using $\dot{x} = -c_1e^{-t} - 2c_2e^{-2t} - e^{-2t} + 2te^{-2t}$, initial conditions result in $c_1 + c_2 = 0$ and $c_1 + 2c_2 = -1$, or $c_1 = 1$ and $c_2 = -1$. The solution is $x(t) = e^{-t}\big(1 - (1 + t)e^{-t}\big)$.

Solutions to the Additional Practice of §3.3

1.

a) The Laplace transform of the equation results in $X(s) = \dfrac{1}{(s+2)(s^2+2s+5)}$. Partial fraction decomposition results in $X(s) = \dfrac{1}{5}\left(\dfrac{1}{s+2}\right) - \dfrac{1}{5}\left(\dfrac{s}{s^2+2s+5}\right)$. Completing the square results in $s^2+2s+5 = (s+1)^2+2^2$ and
$X(s) = \dfrac{1}{5}\left(\dfrac{1}{s+2}\right) - \dfrac{1}{5}\left(\dfrac{s+1}{(s+1)^2+2^2} - \dfrac{1}{2}\left(\dfrac{2}{(s+1)^2+2^2}\right)\right)$. From the Laplace transform table, we have $x(t) = \dfrac{1}{5}e^{-2t} - \dfrac{1}{5}\left(e^{-t}\cos 2t - \dfrac{1}{2}e^{-t}\sin 2t\right)$, which we write as $x(t) = \dfrac{1}{5}e^{-t}\left(e^{-t} - \cos 2t + \dfrac{1}{2}\sin 2t\right)$.

b) We write the inhomogeneous term as $h(t) = (1-t) + u_1(t)(t-1)$. The Laplace transform of the equation results in
$X(s) = \left(\dfrac{s-1}{s^2(s+1)(s+2)}\right) + e^{-s}\left(\dfrac{1}{s^2(s+1)(s+2)}\right) = F(s) + e^{-s}G(s)$. The inverse Laplace transform yields $x(t) = f(t) + u_1(t)g(t-1)$. To determine $f(t)$ and $g(t)$ we need to take inverse Laplace transforms. Partial fraction decomposition results in
$F(s) = \dfrac{5}{4}\left(\dfrac{1}{s}\right) - \dfrac{1}{2}\left(\dfrac{1}{s^2}\right) - 2\left(\dfrac{1}{s+1}\right) + \dfrac{3}{4}\left(\dfrac{1}{s+2}\right)$ and
$G(s) = -\dfrac{3}{4}\left(\dfrac{1}{s}\right) + \dfrac{1}{2}\left(\dfrac{1}{s^2}\right) + \left(\dfrac{1}{s+1}\right) - \dfrac{1}{4}\left(\dfrac{1}{s+2}\right)$. From the Laplace transform table, we have
$f(t) = \dfrac{5}{4} - \dfrac{1}{2}t - 2e^{-t} + \dfrac{3}{4}e^{-2t}$ and
$g(t) = -\dfrac{3}{4} + \dfrac{1}{2}t + e^{-t} - \dfrac{1}{4}e^{-2t}$.

c) The Laplace transform of the equation results in $X(s) = \dfrac{1+e^{-s}}{(s+1)^2}$. From the Laplace transform table, we have $x(t) = te^{-t} + u_1(t)(t-1)e^{-(t-1)}$.

Solutions to the Additional Practice of §4.3

1.

a) Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation, we have

$$\begin{aligned} y'' + xy' + y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \big((n+2)(n+1)a_{n+2} + (n+1)a_n\big) x^n = 0. \end{aligned}$$

We obtain the recursion relation

$$a_{n+2} = -\frac{a_n}{(n+2)}.$$

Starting with a_0, we find $a_2 = -a_0/2$, $a_4 = a_0/8$, and $a_6 = -a_0/48$. Starting with a_1, we find $a_3 = -a_1/3$, $a_5 = a_1/15$, and $a_7 = -a_1/105$. The general solution to order x^7 is given by

$$y(x) = a_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \dots\right) + a_1 \left(x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \dots\right).$$

b) Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation, we have

$$\begin{aligned} y'' + xy' - y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \big((n+2)(n+1)a_{n+2} + (n-1)a_n\big) x^n = 0. \end{aligned}$$

We obtain the recursion relation

$$a_{n+2} = -\frac{(n-1)}{(n+2)(n+1)} a_n.$$

Starting with a_0, we find $a_2 = a_0/2$, $a_4 = -a_0/24$, and $a_6 = a_0/240$. Starting with a_1, we find $a_3 = 0$, $a_5 = 0$, etc. The general solution to order x^6 is given by

$$y(x) = a_0 \left(1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{240} - \dots\right) + a_1 x.$$

c) Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation, we have

$$\begin{aligned}
&y'' + y' + xy \\
&\quad = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\
&\quad = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n \\
&\quad = 2a_2 + a_1 + \sum_{n=1}^{\infty} \big((n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_{n-1}\big) x^n \\
&\quad = 0.
\end{aligned}$$

We obtain $a_1 + 2a_2 = 0$, and for $n \geq 1$,

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_{n-1} = 0.$$

The relevant system of equations for the first five coefficients are given by

$$\begin{aligned}
a_1 + 2a_2 &= 0, & 6a_3 + 2a_2 + a_0 &= 0, \\
12a_4 + 3a_3 + a_1 &= 0, & 20a_5 + 4a_4 + a_2 &= 0.
\end{aligned}$$

With a_0 and a_1 free, the solutions for the remaining coefficients are

$$\begin{aligned}
a_2 &= -\frac{1}{2}a_1, & a_3 &= -\frac{1}{6}a_0 + \frac{1}{6}a_1, \\
a_4 &= \frac{1}{24}a_0 - \frac{1}{8}a_1, & a_5 &= \frac{1}{120}a_0 - \frac{1}{20}a_1.
\end{aligned}$$

The solution to order x^5 is

$$\begin{aligned}
y(x) = a_0 &\left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots\right) \\
&+ a_1 \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{1}{20}x^5 + \dots\right).
\end{aligned}$$

2.

a) Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation, we have

$$\begin{aligned}
&(1-x^2)y'' - xy' + \alpha^2 y \\
&\quad = (1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \alpha^2 \sum_{n=0}^{\infty} a_n x^n \\
&\quad = \sum_{n=0}^{\infty} \Big((n+2)(n+1)a_{n+2} - n(n-1)a_n - n a_n + \alpha^2 a_n\Big) x^n \\
&\quad = \sum_{n=0}^{\infty} \Big((n+2)(n+1)a_{n+2} + (\alpha^2 - n^2)a_n\Big) x^n = 0.
\end{aligned}$$

We obtain the recursion relation

$$a_{n+2} = -\frac{(\alpha^2 - n^2)}{(n+2)(n+1)} a_n.$$

Starting with a_0, we find $a_2 = -\frac{\alpha^2}{2}a_0$, $a_4 = \frac{\alpha^2(\alpha^2-4)}{24}a_0$. Starting with a_1, we find $a_3 = -\frac{(\alpha^2-1)}{6}a_1$, $a_5 = \frac{(\alpha^2-9)(\alpha^2-1)}{120}a_1$. The first three terms in each of two power series solutions are given by

$$y(x) = a_0\left(1 - \frac{\alpha^2}{2}x^2 + \frac{\alpha^2(\alpha^2-4)}{24}x^4 + \dots\right) + a_1\left(x - \frac{(\alpha^2-1)}{6}x^3 + \frac{(\alpha^2-9)(\alpha^2-1)}{120}x^5 + \dots\right).$$

b) $y(x) = a_0$, $y(x) = a_1 x$, $y(x) = a_0(1-2x^2)$, $y(x) = a_1(x - \frac{4}{3}x^3)$.

c) $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$.

Solutions to the Additional Practice of §5.3

1.

a) *i)* Try $y = x^r$. We obtain $r(r-1) - 2r + 2 = 0$, or $(r-1)(r-2) = 0$, with roots $r = 1, 2$. The general solution is $y(x) = c_1 x + c_2 x^2$, with $y'(x) = c_1 + 2c_2 x$. Boundary conditions result in $c_1 = 1$ and $c_1 + c_2 = 0$, yielding $c_2 = -1$. The solution is $y(x) = x - x^2$.

ii) Try $y = x^r$. We obtain $2r(r-1) - r + 1 = 0$, or $(2r-1)(r-1) = 0$, with roots $r = 1/2, 1$. The general solution is $y(x) = c_1\sqrt{x} + c_2 x$. Boundary conditions result in $c_1 + c_2 = 0$ and $2c_1 + 4c_2 = 1$, yielding $c_1 = -1/2$ and $c_2 = 1/2$. The solution is $y(x) = \frac{1}{2}(x - \sqrt{x})$.

b) *i)* Try $y = x^r$. We obtain $r(r-1) - r + 1 + \pi^2 = 0$ with roots $r_\pm = 1 \pm i\pi$. The general solution is $y(x) = x\big(A\cos(\pi \ln x) + B\sin(\pi \ln x)\big)$. Boundary conditions result in $A = 1$ and $B = 1$. The solution is $y(x) = x\big(\cos(\pi \ln x) + \sin(\pi \ln x)\big)$.

ii) Try $y = x^r$. We obtain $r(r-1) + 3r + 1 + \pi^2 = 0$ with roots $r_\pm = -1 \pm i\pi$. The general solution is $y(x) = \frac{1}{x}\big(A\cos(\pi \ln x) + B\sin(\pi \ln x)\big)$. Boundary conditions result in $A = 1$ and $B = e$. The solution is $y(x) = \frac{1}{x}\big(\cos(\pi \ln x) + e\sin(\pi \ln x)\big)$.

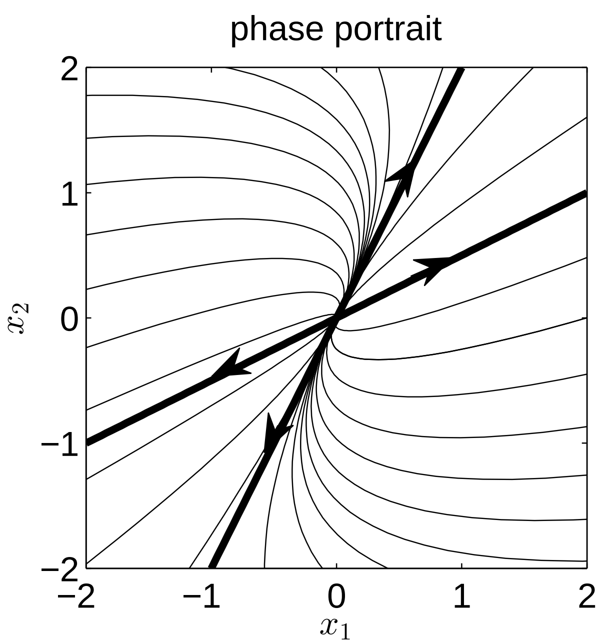
c) *i)* Try $y = x^r$. We obtain $r(r-1) - r + 1 = 0$, or $(r-1)^2 = 0$, with repeated root $r = 1$. The general solution is $y(x) = x(c_1 + c_2 \ln x)$, with $y'(x) = c_1 + c_2(1 + \ln x)$. Boundary conditions result in $c_1 = 1$ and $c_1 + c_2 = 0$, yielding $c_2 = -1$. The solution is $y(x) = x(1 - \ln x)$.

ii) Try $y = x^r$. We obtain $4r(r-1) + 1 = 0$, or $(2r-1)^2 = 0$, with repeated root $r = 1/2$. The general solution is $y(x) = \sqrt{x}(c_1 + c_2 \ln x)$. Boundary conditions result in $c_1 = 1$ and $\sqrt{e}(c_1 + c_2) = 0$, yielding $c_2 = -1$. The solution is $y(x) = \sqrt{x}(1 - \ln x)$.

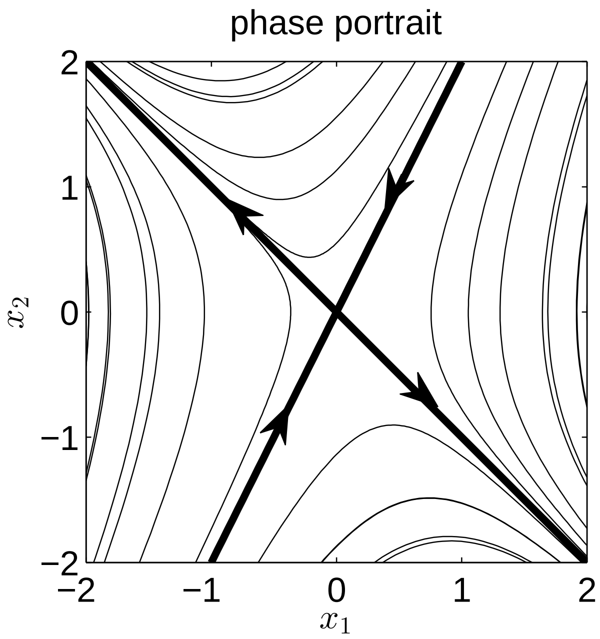
Solutions to the Additional Practice of §6.3

1.

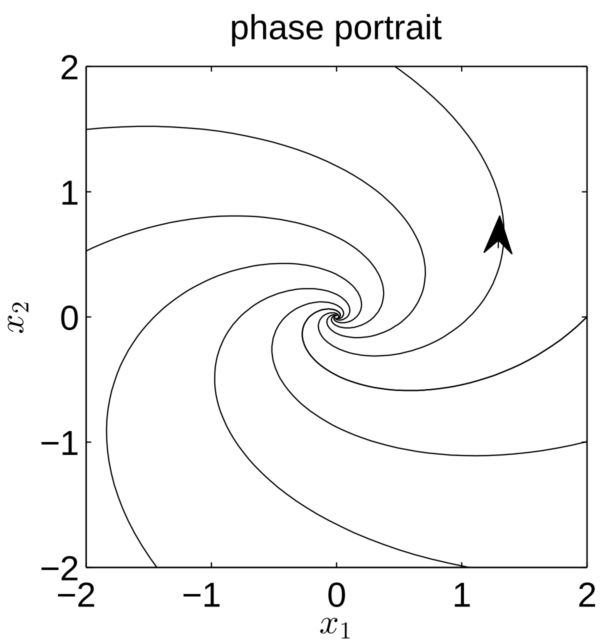
a) *i)* With $\mathbf{A} = \begin{pmatrix} 7 & -2 \\ 2 & 2 \end{pmatrix}$, the characteristic equation is $\lambda^2 - 9\lambda + 18 = (\lambda - 6)(\lambda - 3) = 0$, with roots $\lambda_1 = 6$ and $\lambda_2 = 3$. The corresponding eigenvectors are $\mathbf{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The general solution is $\mathbf{x} = c_1 e^{6t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The phase portrait is shown below.



ii) With $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}$, the characteristic equation is $\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$, with roots $\lambda_1 = 1$ and $\lambda_2 = -2$. The corresponding eigenvectors are $\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The general solution is $\mathbf{x} = c_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The phase portrait is shown below.

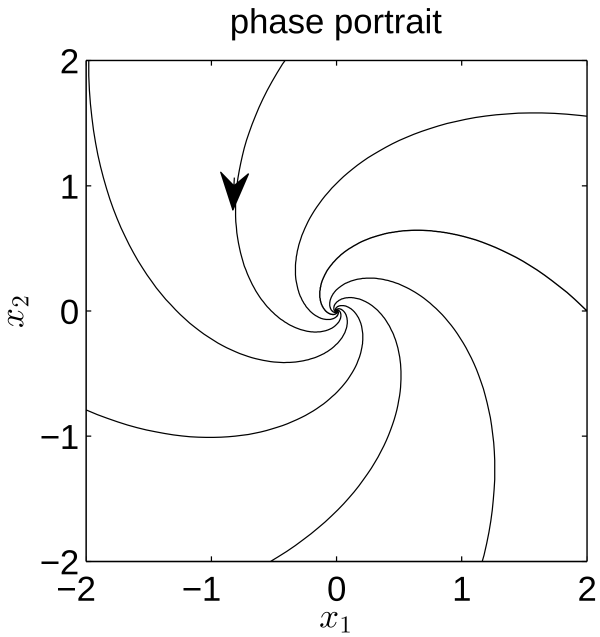


b) *i)* With $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, the characteristic equation is $\lambda^2 - 2\lambda + 3 = 0$, with root $\lambda = 1 + i\sqrt{2}$ and its complex conjugate. The corresponding eigenvector is $\mathbf{v} = \begin{pmatrix} 1 \\ -i\sqrt{2}/2 \end{pmatrix}$. The general solution is constructed from the linearly independent solutions $\mathbf{x_1} = \text{Re}\left(\mathbf{v}e^{\lambda t}\right)$ and $\mathbf{x_2} = \text{Im}\left(\mathbf{v}e^{\lambda t}\right)$, and is $\mathbf{x} = e^t \begin{pmatrix} c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) \\ \frac{\sqrt{2}}{2}\left(c_1 \sin(\sqrt{2}t) - c_2 \cos(\sqrt{2}t)\right) \end{pmatrix}$. To determine the direction of rotation, compute $L = x_1\dot{x}_2 - x_2\dot{x}_1 = x_1^2 + 2x_2^2 > 0$, and find that the rotation is counterclockwise. The phase portrait is shown below.

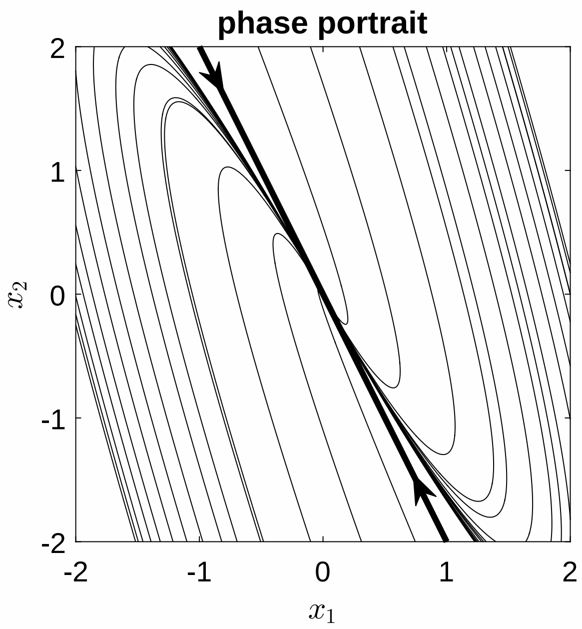


ii) With $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, the characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$, with root $\lambda =$

$-1+i$ and its complex conjugate. The corresponding eigenvector is $\mathbf{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. The general solution is constructed from the linearly independent solutions $\mathbf{x_1} = \mathrm{Re}\left(\mathbf{v}e^{\lambda t}\right)$ and $\mathbf{x_2} = \mathrm{Im}\left(\mathbf{v}e^{\lambda t}\right)$, and is $\mathbf{x} = e^{-t}\begin{pmatrix} c_1 \cos t + c_2 \sin t \\ c_1 \sin t - c_2 \cos t \end{pmatrix}$. To determine the direction of rotation, compute $L = x_1\dot{x}_2 - x_2\dot{x}_1 = x_1^2 + x_2^2 > 0$, and find that the rotation is counterclockwise. The phase portrait is shown below.



c) *i)* With $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}$, the characteristic equation is $\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$, with repeated root $\lambda = -1$. The corresponding eigenvector is $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. To find a second solution, try $\mathbf{x} = (\mathbf{w} + t\mathbf{v})e^{\lambda t}$, to obtain $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$. Write $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and find $w_2 = 1 - 2w_1$. Then $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + w_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, where we may choose $w_1 = 0$. The general solution is $\mathbf{x} = e^{-t}\left(c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)\right)$. The phase portrait is shown below.


phase portrait
2
1
0
-1
-2
x_2
-2
-1
0
1
2
x_1

Solutions to the Additional Practice of §7.1.3

1.

a) Fixed points are determined from $f(x) = 4x^2 - 16 = 0$, with solutions $x_* = \pm 2$. Linear stability is determined from $f'(x) = 8x$, with $f'(-2) = -16 < 0$ and $f'(2) = 16 > 0$. Therefore $x_* = -2$ is stable and $x_* = 2$ is unstable.

b) Fixed points are determined from $f(x) = x(1 - x^2) = 0$, with solutions $x_* = 0, \pm 1$. Linear stability is determined from $f'(x) = 1 - 3x^2$, with $f'(0) = 1 > 0$ and $f'(\pm 1) = -2 < 0$. Therefore $x_* = 0$ is unstable and $x_* = \pm 1$ is stable.

c) Fixed points are determined from $f(x, y) = x(1 - 2x - y) = 0$ and $g(x, y) = y(1 - x - 2y) = 0$. Three obvious fixed points are $(x_*, y_*) = (0, 0), (0, 1/2), (1/2, 0)$. The fourth fixed point satisfies $2x + y = 1$ and $x + 2y = 1$, or $(x_*, y_*) = (1/3, 1/3)$. The stability of these fixed points is determined from the eigenvalues of the Jacobian matrix J at the fixed points, where

$$\mathbf{J} = \begin{pmatrix} 1 - 4x - y & -x \\ -y & 1 - x - 4y \end{pmatrix}.$$

We have

$$\mathbf{J}|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{J}|_{(0,1/2)} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & -1 \end{pmatrix},$$

$$\mathbf{J}|_{(1/2,0)} = \begin{pmatrix} -1 & -1/2 \\ 0 & 1/2 \end{pmatrix}, \qquad \mathbf{J}|_{(1/3,1/3)} = \begin{pmatrix} -2/3 & -1/3 \\ -1/3 & -2/3 \end{pmatrix}.$$

The fixed points $(0, 0), (0, 1/2), (1/2, 0)$ have at least one positive eigenvalue and are unstable. The eigenvalues of the fourth fixed point are the two solutions of the quadratic equation $\lambda^2 + \frac{4}{3}\lambda + \frac{1}{3} = 0$, both of which are negative. Therefore, the fixed point $(1/3, 1/3)$ is stable.

Solutions to the Additional Practice of §7.2.3

1.

a) $f(x) = r - \cosh x$. At the bifurcation point, both $f(x) = 0$ ($\cosh x = r$) and $f'(x) = 0$ ($\sinh x = 0$). The bifurcation occurs when $r = 1$ and $x = 0$. Taylor series expanding $\cosh x$ to second order yields $f(x) = (r-1) - x^2/2$. This is a saddle-node bifurcation with fixed points $x = \pm\sqrt{2(r-1)}$, which only exist when $r \geq 1$. Stability of these two branches is computed using $f'(x) = -x$, so that the upper branch is stable and the lower branch is unstable.

b) $f(x) = x(r - \sinh x)$. At the bifurcation point, both $f(x) = 0$ ($x = 0$ or $\sinh x = r$) and $f'(x) = 0$ ($\sinh x + x\cosh x = r$). The bifurcation occurs when $r = 0$ and $x = 0$. Taylor series expanding $\sinh x$ to second order yields $f(x) = rx - x^2$, which is the normal form for a transcritical bifurcation.

c) $f(x) = rx - \sinh x$. At the bifurcation point, both $f(x) = 0$ ($\sinh x = rx$) and $f'(x) = 0$ ($\cosh x = r$). The bifurcation occurs when $r = 1$ and $x = 0$. Taylor series expanding $\sinh x$ to third order yields $f(x) = (r-1)x - x^3/6$. This is a supercritical pitchfork bifurcation with fixed points $x = 0$ and $x = \pm\sqrt{6(r-1)}$, the latter which only exist when $r \geq 1$.

d) $f(x) = rx - \sin x$. At the bifurcation point, both $f(x) = 0$ ($\sin x = rx$) and $f'(x) = 0$ ($\cos x = r$). We consider the bifurcation that occurs when $r = 1$ and $x = 0$. Taylor series expanding $\sin x$ to third order yields $f(x) = (r-1)x + \frac{1}{6}x^3$. This is a subcritical pitchfork bifurcation. Fixed points near the bifurcation point $r = 1$ are given by $x = 0$ and $x = \pm\sqrt{6(1-r)}$, the latter which only exist when $r \leq 1$.

Solutions to the Additional Practice of §8.3

1.

a) $f(x)$ is an even function and $L = \pi$. The coefficients of the Fourier cosine series are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx.$$

For $n = 0$, we have $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$. For $n > 0$, we have

$$\begin{aligned} a_n &= \frac{2}{n\pi}\left(-x^2 \sin nx\Big|_0^{\pi} - 2\int_0^{\pi} x \sin nx \, dx\right) \\ &= -\frac{4}{n^2\pi}\left(-x\cos nx|_0^{\pi} + \int_0^{\pi} \cos nx \, dx\right) \\ &= \frac{4\cos n\pi}{n^2} \\ &= \frac{4}{n^2} \times \begin{cases} -1, & \text{if } n \text{ odd;} \\ 1, & \text{if } n \text{ even.} \end{cases} \end{aligned}$$

The Fourier cosine series is

$$f(x) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right).$$

b) $f(x)$ is an odd function and $L = \pi$. The coefficients of the Fourier sine series are given by

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx \\ &= -\frac{2}{n\pi}\cos nx|_0^{\pi} \\ &= \frac{2}{n\pi}(1 - \cos n\pi) \\ &= \frac{4}{n\pi} \times \begin{cases} 1, & \text{if } n \text{ odd;} \\ 0, & \text{if } n \text{ even.} \end{cases} \end{aligned}$$

The Fourier sine series is

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right).$$

c) We have $f(x) = \frac{1}{2} + \frac{1}{2}g(x)$, where the Fourier sine series for $g(x)$ was given in Problem 2. Therefore,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right).$$