HKUST

Final Examination

MATH 005 Algebra and Calculus I

20 December 2000 **12:30—15:30**

Answer ALL questions

Question 1 (a) Find the limits of the following expressions.

(i) $\lim_{n\to+\infty} \frac{5n^2-4n+1}{3n^3-n^2-1}$, (ii) $\lim_{n\to+\infty} \frac{n-3n^2-7n^3}{4n^3-5n+1}$. [6]

(b) Compute the following limit:

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h},$$

where $f(x) = ax^2 + b$, a and b are certain constants. [4]

(c) Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} 1-3x, & x < 4 \\ kx^2+2x-3, & x \geq 4 \end{cases}$$

where k is a constant.

(i) Find $\lim_{x\to 4-} f(x)$ and $\lim_{x\to 4+} f(x)$. [2]

(ii) Determine the value of k so that the function f has a limit at $x = 4$. [2]

(iii) Is the function f continuous at $x = 4$ for the value k found in (ii) ? [1]

Question 2 Differentiate the following functions:

(i) $f(x) = (x+2)^2 e^{x^2+2}$,

(ii) $f(x) = \frac{5x^2-3x+1}{4x-3}$, [5]

(iii) $f(x) = \sqrt{\ln(x^2+1)}$,

(iv) $f(x) = \frac{e^{x^2}\sqrt{4x-2}}{5x^2-4}$. [6]

Question 3 (a) Solve the following systems of equations.

(i)

$$\begin{aligned} x + 2y \quad &= 4, \\ -x + 3y + 3z &= -2, \\ y + \quad z &= 0. \end{aligned}$$

[5]

(ii)

$$\begin{aligned} 3x - 2y + z + 2w &= 0, \\ x + y - z - w &= 0, \\ 2x - 2y + 3z \quad &= 0. \end{aligned}$$

[5]

(b) Find the polynomial $y = f(x) = a_0 + a_1x + a_2x^2$ that passes through the three points $(1, 12), (2, 15)$ and $(3, 16)$. [6]

Question 4 **(a)** Find the present value of \$55,000 in 10 years' time if the interest rate is 6% compounded
(i) monthly,
(ii) continuously. [4]

(b) Suppose the payments of an annuity are paid at the end of each year for 8 years under an annual interest rate 6% such that the payments for the first three years is \$10,000 each year and \$12,000 each year for the remaining years. Find the present value of the annuity. [6]

Question 5 Let the cost function and demand function of a certain commodity of a company are given, respectively, by

$$c(q) = \frac{1}{8}q^2 + 4q + 200 \qquad \text{and} \qquad p(q) = 49 - q.$$

(a) Find the marginal cost and marginal revenue when $q = 5$ and interpret your answers, [4]

(b) Determine the value of q at which the company generates the maximum profit, [4]

(c) Determine the minimum level of the average revenue. What can you say about your answer in relation to (b) ? [2]

Question 6 Let $C(I)$ and $S(I)$ denote respectively the national consumption and national saving functions, where the variable I denotes the national income in billions. It is clear that $C(I) + S(I) = I$.

(a) Describe the relationship between $C(I)$ and $S(I)$ in terms of derivatives. [2]

(b) Find the expressions for the national consumption and national saving functions given that the marginal propensity to consume is

$$MPC = 0.5 + \frac{0.2}{\sqrt{I}}$$

and the consumption is 85 when the income is 100. [6]

(c) Interpret the meaning of MPC when $I = 49$, and use it to approximate the value of national consumption when $I = 48$. [3]

Question 7 **(a)** Integrate the following functions:
(i) $f(x) = (x + 1/x^2)^2$, (ii) $f(x) = \dfrac{x}{2x^2 + 1}$. [4]

(b) Evaluate $\displaystyle\int_0^2 \frac{1}{2} x e^{x^2+3}\, dx$. [3]

(c) Find the equation of tangent of $f(x) = \ln x + e^{\sqrt{\ln x}}$ at $x = e$. [4]

Question 8 Let y be a function of x given by

$$y = f(x) = -x^4 - 3x^2 + 1.$$

(i) Write down the y coordinate when $x = 0$. [1]

(ii) Evaluate the limits $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. [2]

(iii) Determine the regions on the x-axis that f is increasing/decreasing. [2]

(iv) Hence find the critical points of $f(x)$ and determine the nature of the critical points. [7]

(v) Sketch the graph of $f(x)$. [4]
