HKUST

Final Examination

MATH 005 Algebra and Calculus I

20 December 2000 **12:30—15:30**

Answer ALL questions

Question 1 (15 marks)

(a) Find the limits of the following expressions.

(i) $\lim_{n\to+\infty} \frac{5n^2-4n+1}{3n^3-n^2-1}$,

$$\lim_{n\to+\infty} \frac{5n^2-4n+1}{3n^2-n^2-1} = \frac{\lim(5/n-4/n^2+1/n^3)}{\lim(3-1/n-1/n^3)} = \frac{0}{3} = 0,$$

(ii) $\lim_{n\to+\infty} \frac{n-3n^2-7n^3}{4n^3-5n+1}$. [6]

$$\lim_{n\to+\infty} \frac{n-3n^2-7n^3}{4n^3-5n+1} = \frac{\lim(1/n^2-3/n-7)}{\lim(4-5/n^2+1/n^3)} = \frac{-7}{4}.$$

(b) Compute the following limit:

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h},$$

where $f(x) = ax^2+b$, a and b are certain constants. [4]

Let $f(x) = ax^2+b$,

$$\begin{aligned}\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0} \frac{a(2+h)^2+h-(a2^2+b)}{h} \\ &= \lim_{h\to 0} \frac{a(2^2+4h+h^2-2^2)}{h} \\ &= \lim_{h\to 0} a(4+h) \\ &= 4a.\end{aligned}$$

(c) Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} 1-3x, & x<4 \\ kx^2+2x-3, & x\ge 4 \end{cases}$$

where k is a constant.

(i) Find $\lim_{x\to 4-} f(x)$ and $\lim_{x\to 4+} f(x)$. [2]

$\lim_{x\to 4-} f(x) = \lim_{x\to 4-}(1-3x) = -11$, and $\lim_{x\to 4+} f(x) = \lim_{x\to 4+}(Kx^2+2x-3) = 16K+5$.

(ii) Determine the value of k so that the function f has a limit at $x=4$. [2]

f has a limit $\lim_{x\to 4+} f(x)$ if and only if $\lim_{x\to 4-} f(x) = \lim_{n\to 4} f(x)$, i.e.,

$$-11 = \lim_{x\to 4-} f(x) = \lim_{x\to 4+} f(x) = 16K+5,$$

i.e., $K=-1$.

(iii) Is the function f continuous at $x=4$ for the value k found in (ii) ? [1]

Since $f(4) = -1(4)^2+2(4)^2+2(4)-3 = -11$, and $\lim_{x\to 4} f(x) = -11$. Hence $\lim_{x\to 4} f(x) = f(4)$ and f is continous at 4 by definition.

Question 2 (11 marks)

(a) Differentiate the following functions:

(i) $f(x) = (x+2)^2 e^{x^2+2}$,

$$\begin{aligned}\frac{d}{dx}(x+2)^2 e^{x^2+2} &= e^{x^2+2}\frac{d}{dx}(x+2)^2 + (x+2)^2\frac{d}{dx}e^{x^2+2}\\ &= 2(x+2)e^{x^2+2} + (x+2)^2(2x)e^{x^2+2}\\ &= 2(x+2)\Big(1+x(x+2)\Big)e^{x^2+2}\\ &= 2(x+2)(x+1)^2 e^{x^2+2}.\end{aligned}$$

(ii) $f(x) = \dfrac{5x^2-3x+1}{4x-3}$, [5]

$$\begin{aligned}\frac{d}{dx}\Big(\frac{5x^2-3x+1}{4x-3}\Big) &= \frac{(4x-3)(5x^2-3x+1)' - (5x^2-3x+1)(4x-3)'}{(4x-3)^2}\\ &= \frac{(4x-3)(10x-3) - (5x^2-3x+1)(4)}{(4x-3)^2}\\ &= \frac{40x^2-42x+9-(20x^2-12x+4)}{(4x-3)^2}\\ &= \frac{20x^2-30x+5}{(4x-3)^2}\\ &= \frac{5(4x^2-6x+1)}{(4x-3)^2}\end{aligned}$$

(iii) $f(x) = \sqrt{\ln(x^2+1)}$,

$$\begin{aligned}\frac{d}{dx}\sqrt{\ln(x^2+1)} &\underline{\underline{u=\ln(x^2+1)}}\frac{d}{du}u^{1/2}\frac{du}{dx}\\ &= \frac{1}{2}u^{-1/2}\frac{d}{dx}\ln(n^2+1)\\ &= \frac{x}{(x^2+1)\sqrt{\ln(x^2+1)}}.\end{aligned}$$

(iv) $f(x) = \dfrac{e^{x^2}\sqrt{4x-2}}{5x^2-4}$. [6]

$$\ln f(x) = x^2\frac{1}{2}\ln(4x-2) - \ln(5x^2-4).$$

So

$$\frac{f'(x)}{f(x)} = 2x + \frac{2}{4x-2} - \frac{10x}{5x^2-4}.$$

Hence

$$f'(x) = \Big(2x + \frac{1}{2x-1} - \frac{10x}{5x^2-4}\frac{e^{x^2}\sqrt{4x-2}}{5x^2-4}\Big).$$

Question 3 (16 marks)

(a) Solve the following systems of equations.

(i)

$$\begin{aligned} x+2y \qquad &= \;\;4 \\ -x+3y+3z &= -2 \\ y+\;\; z &= \;\;0 \end{aligned}$$

[5]

$$\begin{pmatrix} 1 & 2 & 0 & \vdots & 4 \\ -1 & 3 & 3 & \vdots & -2 \\ 0 & 1 & 1 & \vdots & 0 \end{pmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{pmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & 5 & 3 & \vdots & 2 \\ 0 & 1 & 1 & \vdots & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 5 & 3 & \vdots & 2 \end{pmatrix} \xrightarrow{R_3 \to -5R_2 + R_3} \begin{pmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & -2 & \vdots & 2 \end{pmatrix}.$$

The system becomes, in echelon form,

$$\begin{aligned} x+2y \qquad &= 4 \\ y+\;\; z &= 0 \\ -2z &= 2. \end{aligned}$$

We deduce $z=-1$, and $y=-1$ and $x=4-2y=2$. So the original system has a unique solution $(2,1,-1)$.

(ii)

$$\begin{aligned} 3x-2y+\;z+2w &= 0 \\ x+\;y-\;\;z-\;w &= 0 \\ 2x-2y+3z \qquad &= 0 \end{aligned}$$

[5]

Since the given system has 4 unknowns and 3 equations, so it must admit an infinite number of solutions if the equations are consistant.

$$\begin{pmatrix} 3 & -2 & 1 & 2 & \vdots & 0 \\ 1 & 1 & -1 & -1 & \vdots & 0 \\ 2 & -1 & 3 & 0 & \vdots & 0 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 1 & 1 & -1 & -1 & \vdots & 0 \\ 3 & -2 & 1 & 2 & \vdots & 0 \\ 2 & -1 & 3 & 0 & \vdots & 0 \end{pmatrix}.$$

$$\xrightarrow[R_2 \to -3R_1 + R_2]{R_3 \to -2R_1 + R_3} \begin{pmatrix} 1 & 1 & -1 & -1 & \vdots & 0 \\ 0 & -5 & 4 & 5 & \vdots & 0 \\ 0 & -3 & 5 & 2 & \vdots & 0 \end{pmatrix} \xrightarrow{R_3 \to -\frac{3}{5}R_2 + R_3} \begin{pmatrix} 1 & 1 & -1 & -1 & \vdots & 0 \\ 0 & -5 & 4 & 5 & \vdots & 0 \\ 0 & 0 & 13/5 & -1 & \vdots & 0 \end{pmatrix}.$$

The system becomes

$$\begin{aligned} x+\;y-\;\;z-\;w &= 0 \\ -5y+4z+5w &= 0 \\ 13/5z-\;\;w &= 0. \end{aligned}$$

writing $w=t$, then $z=\dfrac{5}{13w}=\dfrac{5}{13}t$ and thus

$$y=\frac{1}{5}(4z+5w)=\frac{4}{5}\Big(\frac{5}{13}t=\frac{17}{13}\Big)t.$$

Finally $x=-y+w+z=\dfrac{17}{13}t+\dfrac{5}{13}t+t=\Big(1-\dfrac{17}{13}+\dfrac{5}{13}\Big)t=\dfrac{1}{13}t$. Hence the solution is given by $x=t/13, y=17t/13$ and $z=5t/13, w=t.\Big($or $(x,y,z,w)=(1/13,17/13,5/13,1)t.\Big)$

Since the quadratic $f(x)=a_0+a_1x+a_2x^2$ passes through (1,12), and (3, 16). So

$$a_0+\;a_1+\;a_2=12,$$

$$a_0+2a_1+4a_2=15,$$

$$a_0 + 3a_1 + 9a_2 = 16.$$

$$\begin{pmatrix} 1 & 1 & 1 & \vdots & 12 \\ 1 & 2 & 4 & \vdots & 15 \\ 1 & 3 & 9 & \vdots & 16 \end{pmatrix} \xrightarrow[R_2 \to -R_1 + R_2]{R_3 \to -R_1 + R_3} \begin{pmatrix} 1 & 1 & -1 & \vdots & 12 \\ 0 & 1 & 3 & \vdots & 3 \\ 0 & 2 & 8 & \vdots & 4 \end{pmatrix}.$$

$$\xrightarrow{R_3 \to -2R_2 + R_3} \begin{pmatrix} 1 & 1 & 1 & \vdots & 12 \\ 0 & 1 & 3 & \vdots & 3 \\ 0 & 0 & 2 & \vdots & -2 \end{pmatrix}.$$

Hence $2a_2 = -2, a + 2 = -1$. So $a_1 + 3a_2 = 3$ or $a_1 = 3 - 3a_2 = 3 - 3(-1) = 6$. Finally, $a_0 = 12 - a_1 - a_2 = 12 - 6 - (-1) = 7$. So the polynomial is $f(x) = 7 + 6x - x^2$.

(b) Find the polynomial $y = f(x) = a_0 + a_1x + a_2x^2$ that passes through the three points $(1, 12), (2, 15)$ and $(3, 16)$. [6]

Question 4 (10 marks)

(a) Find the present value of $55,000 in 10 years' time if the interest rate is 6% compounded

(i) monthly,

$$PV = 55,000\left(1 + \frac{6/12}{100}\right)^{-120}$$
$$\approx \$30,230.$$

(ii) continuously. [4]

$$PV = 55,000e^{-0.06(10)}$$
$$= 30,184.64.$$

(b) Suppose the payments of an annuity are paid at the end of each year for 8 years under an annual interest rate 6% such that the payments for the first three years is $10,000 each year and $12,000 each year for the remaining years. Find the present value of the annuity. [6]

The PV of the first three years is $\frac{10,000}{4}\left(1 - \frac{1}{(1+r)^3}\right)$, and the PV of the remaining 8 years is

$$\frac{12,000}{r}\left(1 - \frac{1}{(1+r)^8}\right) - \frac{12,000}{r}\left(1 - \frac{1}{(1+r)^3}\right).$$

Thus the PV of the annuity is

$$\begin{aligned}
&\frac{12,000}{r}\left(1 - \frac{1}{(1+r)^8}\right) + \left(\frac{10,000}{r} - \frac{12,000}{r}\right)\left(1 - \frac{1}{(1+r)^3}\right)\\
=&\frac{12,000}{0.06}\left(1 - \frac{1}{(1.06)^8}\right) - \frac{2,000}{0.06}\left(1 - \frac{1}{(1.06)^3}\right)\\
\approx&108,053.78 - 27,987.31\\
=&\underline{\underline{80,066.47}}
\end{aligned}$$

Question 5 (10 marks)

Let the cost function and demand function of a certain commodity of a company are given, respectively, by

$$c(q) = \frac{1}{8}q^2 + 4q + 200 \qquad \text{and} \qquad p(q) = 49 - q.$$

(a) Find the marginal cost and marginal revenue when $q = 5$ and interpret your answers, [4]

Marginal cost$=\frac{dc}{dq} = \frac{1}{4}q + 4$. Thus

$$\left.\frac{dc}{dq}\right|_{q=5} = \frac{5}{4} + 4 = \frac{21}{4},$$

and

Marginal revenue $\frac{d}{dq}\Big(qp(q)\Big) = 49 - 2q$. Thus

$$\left.\frac{dr}{dq}\right|_{q=5} = 49 - 10 = 39.$$

$c'(5) = 21/4$ means that the cost of producting the 6th unit of the commodity is approximately $21/4$. Similar interpretation for $r'(5)$.

(b) Determine the value of q at which the company generates the maximum profit, [4]

The profit function $p(q)$ is given by

$$\begin{aligned} P(q) &= \text{Total revenue} - \text{Total Cost} \\ &= qp(q) - c(q) \\ &= 49q - q^2 - \Big(\frac{1}{8}q^2 + 4q + 200\Big) \\ &= -\frac{9}{8}q^2 + 45q - 200. \end{aligned}$$

At the critical point

$$0 = P'(q) = -\frac{9}{4}q + 45.$$

Hence it is $q = 20$. Thus $P(20) = 250$ is the maximum since P is a quadratic with a negative leading coefficient.

(c) Determine the minimum level of the average revenue. What can you say about your answer in relation to (b) ? [2]

The average revenue function is given by

$$A(q) = \frac{c(q)}{q} = \frac{1}{8}q + 4 + 200/q.$$

The minimum level of $A(q)$ must occur at the critical point of $A(q)$, i.e.,

$$\begin{aligned} 0 &= A'(q) \\ &= \frac{1}{8} - \frac{200}{8^2}. \end{aligned}$$

Hence

$q^2 = 1600$ or $q = 40$.

Question 6 (11 marks)

Let $C(I)$ and $S(I)$ denote respectively the national consumption and national saving functions, where the variable I denotes the national income in billions. It is clear that $C(I) + S(I) = I$.

(a) Describe the relationship between $C(I)$ and $S(I)$ in terms of derivatives. [2]

Since

$$C(I) + S(I) = I.$$

So

$$\frac{dC}{dI} + \frac{dS}{dI} = 1.$$

(b) Find the expressions for the national consumption and national saving functions given that the marginal propensity to consume is

$$MPC = 0.5 + \frac{0.2}{\sqrt{I}}$$

and the consumption is 85 when the income is 100. [6]

We have

$$\begin{aligned} C(I) &= \int 0.5 + \frac{0.2}{\sqrt{I}}\ dI \\ &= 0.5I + 0.4\sqrt{I} + c. \end{aligned}$$

But

$$\begin{aligned} 85 = C(100) &= 0.5(100) + 0.4\sqrt{100} + c \\ &= 50 + 4 + c. \end{aligned}$$

Thus $c = 31$. Hence $C(I) = 0.5I + 0.4\sqrt{I} + 31$.

$$\begin{aligned} S(I) &= \int \frac{dS}{dI}\ dI = \int 1 - \frac{dC}{dI}\ dI \\ &= \int 1 - \Big(\frac{0.5 + 0.2}{\sqrt{I}}\Big)\ dI = \int 0.5 - \frac{0.2}{\sqrt{I}} \\ &= 0.5I - 0.4\sqrt{I} + c. \end{aligned}$$

But $S(100) = 100 - 85 = 15$. Hence $15 = 0.5 - 0.4\sqrt{100} + c$. We deduce that $c = -31$. Thus

$$S(I) = 0.5I - 0.4\sqrt{I} - 31$$

(c) Interpret the meaning of MPC when $I = 49$, and use it to approximate the value of national consumption when $I = 48$. [3]

$$MPC(49) = 0.5 + \frac{0.2}{\sqrt{49}} \approx 0.53.$$

This means that when the national income is 49 billion the nation will save 0.53 billion when the income increases by 1 more billion.

By the first order approximation, we have

$$\begin{aligned} C(48) - C(49) &= C(49 - 1) - C(49) \\ &\approx C'(49)(48 - 49) \\ &= C'(49)(-1). \end{aligned}$$

Thus

$$\begin{aligned} C(48) &\approx C(49) + C'49(-1) \\ &\approx 58.3 - 0.53 \\ &\approx 57.8. \end{aligned}$$

Also

$$C(48) = 0.5(48) + 0.4\sqrt{48} + 31 \approx 57.8.$$

Question 7 (11 marks)

(a) Integrate the following functions:
(i) $f(x) = (x + 1/x^2)^2$, (ii) $f(x) = \dfrac{x}{2x^2+1}$. [4]

(i)

$$\int \left(x + \frac{1}{x^2}\right)^2 dx = \int x^2 + 2 + \frac{1}{5x^5} + c.$$

(ii)

$$\begin{aligned}\int \frac{x}{2x^2+1}\, dx \;\underline{\underline{u = 2x^2+1}} \int \frac{x}{u}\frac{1}{4x}\, du \\ &= \frac{1}{4}\int \frac{1}{u}\, du \\ &= \frac{1}{4}\ln u + c \\ &= \frac{1}{4}\ln(2x^2+1) + c.\end{aligned}$$

(b) Evaluate $\displaystyle\int_0^2 \frac{1}{2} x e^{x^2+3}\, dx$. [3]

$$\begin{aligned}\int \frac{1}{2} x e^{x^2+3}\, dx \int \frac{1}{2} x e^{u} \frac{1}{2x}\, du \\ &= \frac{1}{4}\int e^u\, du \\ &= \frac{1}{4}e^u + c \\ &= \frac{1}{4}e^{x^2+3} + c.\end{aligned}$$

when $x = 0, u = 0+3 = 3$ when $x = 2, u = 4+3 = 7$. Hence
$\displaystyle\int_0^2 \frac{1}{2} x e^{x^2+3}\, dx = \int_3^7 e^u\, du = e^7 - e^3.$

(c) Find the equation of tangent of $f(x) = \ln x + e^{\sqrt{\ln x}}$ at $x = e$. [4]

Given that $f(x) = \ln x + e^{\sqrt{\ln x}}$. When $x = e, f(e) = \ln e + e^{\sqrt{\ln e}} = 1 + e$. But

$$f'(x) = \frac{1}{x} + \frac{1}{2x\sqrt{\ln x}} e^{\sqrt{\ln x}},$$

and

$$f'(e) = \frac{1}{e} + \frac{1}{2e} e' = \frac{1}{e} + 1/2.$$

Thus, if $y = ax + b$ is the tangent equation at $x = e$, then

$$e + 1 = \left(\frac{1}{e} + \frac{1}{2}\right)(e) + b.$$

Hence $b = e/2$ and $y = \left(\dfrac{1}{e} + \dfrac{1}{2}\right)x + e/2$. $\Big($or $2ey = (3+2)x + e^2$. $\Big)$

Question 8 (16 marks)

Let y be a function of x given by

$$y = f(x) = -x^4 - 3x^2 + 1.$$

(i) Write down the y coordinate when $x = 0$. [1]

Given that $y = f(x) = -x^4 - 3x^2 + 1$.

(i) $f(0) = 1$,

(ii) Evaluate the limits $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. [2]

$\lim_{x\to+\infty} -x^4\Big(1 + \frac{3}{x^2} - \frac{1}{x^4} = -\infty,$

$\lim_{x\to-\infty} -x^4\Big(1 + 3/x^2 - \frac{1}{x^4}\Big) = -\infty.$

More precisely, we have

$$\frac{f(x)}{x^4} + 1 = -\frac{3}{x^2} + \frac{1}{x^4} \sim 0$$

as $x \to \pm\infty$. That is, $f(x) \sim -x^4$ as $x \to \pm\infty$.

(iii) Determine the regions on the x-axis that f is increasing/decreasing. [2]

critical points
$f(x) = -4x^3 - 6x = -2x(2x^2 + 6)$. Thus the only critical point occurs only if $x = 0$ since $2x^2 + 6$ is never zero for any x.

Also

$$f'(x) = -2x(2x^2 + 6) = \begin{cases} < 0, & x > 0, \\ = 0, & x = 0 \\ > 0, & x < 0 \end{cases}$$

Thus f is increasing when $x < 0$ and f is decreasing when $x > 0$.

(iv) Hence find the critical points of $f(x)$ and determine the nature of the critical points. [7]

The analysis in (iii) also implies that $x = 0$ is a maximum point for f. i.e., $(0, 1)$ is a local minimum for f.

(v) Sketch the graph of $f(x)$. [4]