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Mid-term Examination Solutions

MATH 005 Algebra and Calculus I

25 October 2000

Question 1

(a) Simplify the following expressions:

(i)

$$\sqrt[3]{\frac{27x^3y^6}{125a^9b^3}},$$
$$=\left[\left(\frac{3xy^2}{5a^3b}\right)^3\right]^{1/3}$$
$$=\frac{3xy^2}{5a^3b}.$$

(ii)

$$\frac{x+2y}{2x+y}\sqrt{\frac{2y^2+8xy+8x^2}{2y+x}}$$
$$=\frac{x+2y}{2x+y}\sqrt{\frac{2(y+2x)^2}{2y+x}}$$
$$=\sqrt{2\left(\frac{x+2y}{2x+y}\right)^2\frac{(y+2x)^2}{2y+x}}$$
$$=\sqrt{2(x+2y)}.$$

[5]

(b) Remove the denominators of the following expressions:

(i)

$$\frac{3\sqrt{5}+4\sqrt{2}}{3\sqrt{5}-4\sqrt{2}},$$
$$=\frac{(3\sqrt{5}+4\sqrt{2})^2}{(3\sqrt{5}-4\sqrt{2})(3\sqrt{5}+4\sqrt{2})}$$
$$=\frac{9\cdot 5+24\sqrt{10}+16\cdot 2}{9\cdot 5-16\cdot 2}$$
$$=\frac{77+24\sqrt{10}}{13}.$$

(ii)

$$\frac{a+8b}{a^{1/3}+2b^{1/3}}$$
$$=\frac{(a^{1/3}+2b^{1/3})(a^{2/3}-2a^{1/3}b^{1/3}+4b^{2/3})}{a^{1/3}+2b^{1/3}}$$
$$=a^{2/3}-2a^{1/3}b^{1/3}+4b^{2/3}.$$

[5]

(c) Solve the followings equations:

(i) $x^2 - 2x + 1 - k(x^2 - 1) = 0, (k \neq 1)$ [4]
(Hint: factorization)

Solution Notice that

$$\begin{aligned} 0 &= x^2 - 2x + 1 - k(x^2 - 1) \\ &= (x-1)^2 - k(x-1)(x+1) \\ &= (x-1)\Big[x - 1 - k(x+1)\Big] \\ &= (x-1)\Big[(1-k)x - (1+k)\Big]. \end{aligned}$$

Hence either $x = 1$ or $x = \dfrac{1+k}{1-k}$ provide $k \neq 1$.

(ii) Squaring both sides of

$$\sqrt{x} + \sqrt{x + \sqrt{1-x}} = 1$$

in the form $$\left(\sqrt{x + \sqrt{1-x}}\right)^2 = (1 - \sqrt{x})^2.$$

Hence $$x + \sqrt{1-x} = 1 + x - 2\sqrt{x}.$$

Squaring both sides again yields $$\left(\sqrt{1-x}\right)^2 = \left(1 - 2\sqrt{x}\right)^2,$$

i.e., $$1 - x = 1 - 4\sqrt{x} + 4x,$$

and thus, $$(4\sqrt{x})^2 = (5x)^2,$$

i.e., $$16x = 25x^2.$$

Thus, $x = 0$, or $16/25$. It is easy to check that $x = 0$ is the only solution.

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Question 2

(a) Let $p = g(q)$ be a quadratic function given by

$$p = g(q) = -q^2 - 4q + 46$$

(i) Determine the maximum/minimum value of $p = g(q)$; [1]

Solution Since the leading coefficient is -1, so the quadratic has a maximum at $q = -2$.

(ii) Apply the method of "completing the square" to find the value of q at which the maximum/minimum value of $g(q)$ occurs. Find also the value of $g(q)$ at this q. [5]

Solution

$$\begin{aligned} p = -g(q) &= -q^2 - 4q + 46 \\ &= -[q^2 + 4q] + 46 \\ &= -[(q+2)^2 - 4] + 46 \\ &= -(q+2)^2 + 50. \end{aligned}$$

Thus $p = -(q+2)^2 + 50 \leq 50$ for all choices of q. Moreover, equality holds if and only if $q = -2$ and $p = 50$.

(iii) Sketch the graph of $p = g(q)$ [3]

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Question 2

(b) Suppose that for a certain commodity the supply and demand curves are given by

$$p = f(q) = q^2 + 2q + 10, \quad (1)$$

and

$$p = g(q) = -q^2 - 4q + 46. \quad (2)$$

However, it is not known which equation represents the supply equation and which equation represents the demand curve.

(i) Determine the breakeven point; [3]

Solution At the breakeven point, we have

$$q^2 + 2q + 10 = -q^2 - 4q + 46,$$

i.e., $$q^2 + 3q - 18 = 0,$$

i.e., $$(q+6)(q-3) = 0,$$

and either $q = 3$ or -6. Since q cannot be negative at the breakeven point, so $a = 3$ and $p = -3^2 - 4(3) + 46 = 25$.

(ii) Determine which equation is the supply curve and which equation is the demand curve by sketching the graphs of (1) on the same axes. [7]

Solution Since

$$\begin{aligned} p &= q^2 + 2q + 10 \\ &= (q+1)^2 + 10 - 1^2 \\ &= (q+1)^2 + 9. \end{aligned}$$

Thus $p(q+1)^2 + 9 \geq 9$ where equality holds if and only if $g = -1$ and $p = 9$. Since the leading coefficient is 1, the function has a minimum at $q = -1$.

(iii) Give justification to your answers to (ii).

Since the curve 1 is increasing for $x > 0$, we conclude that it is a supply curve. [1]

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Question 3

(a) A worker wants to deposite \$12,000 into a bank for two years. Bank A offers a saving plan that the deposite is compounded quarterly under an annual interest rate 6%, and Bank B offers a saving plan that the deposite is compounded monthly under an annual interest rate 5%. Which plan will yield an higher return? [7]

Solution

The principal and the interest from Bank A after two years is

$$\$12,000\left(1+\frac{6/4}{100}\right)^8 \approx 12,000(1.1265) \approx \$13,517.91$$

and that from Bank B after two years is

$$\$12,000\left(1+\frac{5/12}{100}\right)^{24} \approx 12,000(1.10494) \approx \$13,259.3.$$

Thus, Bank A's offer has a higher return.

(b) Solve the following equations

$$\begin{aligned} x+2y+3z &= 9, \\ -4x+\ y+6z &= -9, \\ 2x+7y+5z &= 13 \end{aligned}$$ [9]

simultaneously.

Solution

$$\begin{aligned} x+2y+3z &= 9, & (L_1) \\ -4x+\ y+6z &= -9, & (L_2) \\ 2x+7y+5z &= 13. & (L_3) \end{aligned}$$

$$\begin{aligned} & & x+2y+3z &= 9, & (L_1) \\ L_2 &\to 4L_1+L_2 & 9y+18z &= 27, & (L_2) \\ L_3 &\to -2L_1+L_3 & 3y-z &= -5. & (L_3) \end{aligned}$$

$$\begin{aligned} & & x+2y+3z &= 9, & (L_1) \\ & & 9y+18z &= 27, & (L_2) \\ L_3 &\to -\tfrac{1}{3}L_2+L_3 & -7z &= -14. & (L_3) \end{aligned}$$

Thus we obtain $z=2$ from (L_3), and so (L_2) gives $9y=27-18z=27-18(2)=-9$, and thus $y=-1$. Substituting $z=2$ and $y=-1$ into (L_1) yields $x=9-2y-3z=9-2(-1)-3(2)=5$. So the solution is $x=5, y=-1$ and $z=2$.

Solution by matrix. Waiting the coefficient matrix by

$$\left(\begin{array}{ccc:c} -1 & 2 & 3 & 9 \\ -4 & 1 & 6 & -9 \\ 2 & 7 & 5 & 13 \end{array}\right) \xrightarrow[R_3\to -2R_1+R_3]{R_2\to 4R_1+R_2} \left(\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & 9 & 18 & 27 \\ 0 & 3 & -1 & -5 \end{array}\right)$$

$$\xrightarrow{R_3\to -\frac{1}{3}R_2+R_3} \left(\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & 9 & 18 & 27 \\ 0 & 0 & -7 & -14 \end{array}\right).$$

That is ,

$$\begin{aligned} x+2y+\ 3z &= 9 \\ 9x+18z &= 27 \\ -7z &= -14, \end{aligned}$$

and we proceed as in the previous method to obtain $x=5, y=-1$ and $z=2$.

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Question 4

(a) Let $g(x)$ be a function defined by

$$g(x) = \begin{cases} |x| - 1, & x \le 0, \\ x - 1, & 0 \le x \le 2, \\ x/2, & 2 \le x. \end{cases}$$

(i) Calculate $g(-3), g(0)$ and $g(3)$; [3]

Solution

$$g(-3) = |-3| - 1 = 2, \quad g(0) = 0 - 1 = -1, \quad g(3) = \frac{1}{2}(3) = 3/2.$$

(ii) sketch the graph of the function g; [4]

(iii) find the domain of g (give justification); [2]

Solution Since g is defined on each of $x \le 0, 0 \le x \le 2$ and $x \ge 2$. Hence the domain of g is the whole x-axis.

(iv) find the range of g (give justification). [2]

Solution We see clearly from (ii) that $g(x) \ge -1$ and with equality only if $x = 0$. Also $g(x)$ can increase without bound. So the range of $g(x)$ is $y \ge -1$.

Question 4 (Cont'd)

(b) Let $h(x) = 1/\sqrt{x}, f(x) = x - 2, f(x) = x - 2$ and $g(x) = \log x$. Find the expressions of the following functions and their domains:

(i) $h\Big(h(x)\Big)$,

Solution $h(h(x)) = \dfrac{1}{\sqrt{h(x)}} = x^{+1/4}$;

the domain is the whole positive real axis,

(ii) $\Big(h(f(x))\Big)^2$,

Solution $\Big(h(f(x))\Big)^3 = \Big(\dfrac{1}{\sqrt{f(x)}}\Big)^3 = \Big(\dfrac{1}{\sqrt{x-2}}\Big)^3 = (x-2)^{-3/2}$;

Since $(x-2)^{-3/2}$ is meaningful only when $x > 2$. Thus, the domain is all those number larger than 2.

(iii) $f\Big(e^{\frac{1}{2}g(x)}\Big)$. [9]

Solution

$$\begin{aligned} f\Big(h(x) + e^{\frac{1}{2}g(x)}\Big) &= h(x) + e^{\frac{1}{2}g(x)} - 2 \\ &= \frac{1}{\sqrt{x}} + e^{\log(1/2)} - 2 \\ &= \frac{1}{\sqrt{x}} + x^{1/2} - 2 \\ &= \frac{1}{\sqrt{x}} + \sqrt{x} - 2. \end{aligned}$$

Thus, the domain is the whole positive x-axis.

(c) Find the inverse of $y = f(x) = x^2 + 4x - 12$. [4]

Solution Since

$$\begin{aligned} y &= x^2 + 4x - 12 \\ &= (x+2)^2 - 12 - 4 \\ &= (x+2)^2 - 16. \end{aligned}$$

Thus $$(x+2)^2 = y + 16,$$

and so $$x + 2 = \pm\sqrt{y+16},$$

i.e., $$x = -2 \pm \sqrt{y+16}.$$

We define $$g(x) = -2 + \sqrt{x+16}.$$

which is the inverse of y defined on $x \geq -16$.