HKUST

MATH005 ALGEBRA AND CALCULUS I

Final Examination (Version A) Name:____________________

20th December 2002 Student I.D.:____________________

16:30–19:30 Tutorial Section:____________________

Directions:

- Do NOT open the exam until instructed to do so.
- Please write your name, ID number, and Section in the space provided above.
- Answer ALL questions.
- This is a closed book examination.
- No graphical calculators are allowed.
- You may write on both sides of the examination papers.
- Once you are allowed to open the exam, please check that you have 7 pages of questions in addition to the cover page.
- You must show the working steps of your answers in order to receive full marks.
- All mobile phones and pagers should be switched off during the examination.
- Cheating is a serious offense. Students who commit this offense may receive zero mark in the examination. However, more serious penalty may be imposed.

Question No.	Points	Out of
Q. 1-14		70
Q. 15		29
Q. 16		28
Q. 17		15
Q. 18		16
Q. 19		16
Q. 20		18
Q. 21		8
Total Points		200

Part I: Answer each of the following 14 multiple choice questions.
Each is worth 5 points. No partial credit.

1. A debt of a company is to be paid off in 16 months in four installments. The repayment is set at \$ 12,335 at the beginning of every four-month period. If the interest on the debt is calculated at an annual rate of 7.5% compounded every four months then the present value of the debt is

   (a) $\$12,335\left[1+\left(1+\frac{0.075}{12}\times 4\right)^1+\left(1+\frac{0.075}{12}\times 4\right)^2+\left(1+\frac{0.075}{12}\times 4\right)^3\right]$

   (b) $\$12,335\left[1+\left(1+0.075\times\frac{1}{3}\right)^{-1}+\left(1+0.075\times\frac{1}{3}\right)^{-2}+\left(1+0.075\times\frac{1}{3}\right)^{-3}\right]$

   (c) $\$11,335$

   (d) $\$110,335$

   (e) $\$12,335\left[(1.025)^{-1}+(1.025)^{-2}+(1.025)^{-3}+(1.025)^{-4}\right]$

2. John has started a saving plan in which a fixed amount of money will be deposited into an account at the end of every month in the coming 36 months. The annual interest rate on the account is 6%, compounded monthly. If John wants to have a total sum of \$ 36,000 at the end of the 36 month period, what is the amount of the monthly deposit John would need to make, rounded to the nearest dollar?

   (a) \$917 (b) \$951 (c) \$1,051 (d) \$915 (e) \$1,030

3. The derivative of $y = 13^x$ equals

   (a) $\frac{1}{13^x}$ (b) $x13^{x-1}$ (c) $\ln(13)\,13^x$ (d) 13^x (e) $\frac{13^{x+1}}{x+1}$

4. Which of the following is the equation of the tangent line to the graph of the function $y = x(x-2)^2$ at the point $(1, 1)$?

   (a) $y = -2x + 3$ (b) $y = -x + 1$ (c) $y = -3x + 4$
   (d) $y = 2x - 1$ (e) $y = -x + 2$

5. For which of the following value of k will the function $f(x) = kx^2 - \frac{1}{x^2}$ have a point of inflection at $x = 1$?

(a) 3 (b) 4 (c) 5 (d) 1 (e) 2

6. The manager of a hotel estimates that the daily demand of rooms at the hotel is given by the function $q(p) = 1280 - 2p$, where q gives the number of rooms rented when each room is rented at p dollars per day. What is the maximum possible daily revenue?

(a) \$409,600 (b) \$204,000 (c) \$208,000 (d) \$204,800 (e) \$206,080

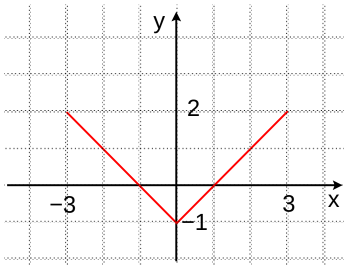
7. The definite integral $\int_0^{1000} te^{-t}\,dt$ equals

(a) $1 - e^{-1000}$ (b) 1 (c) $1 - 1000e^{-1000}$ (d) $999e^{-1000} + 1$ (e) $1 - 1001e^{-1000}$

8. Suppose $C(q)$ is the cost function to produce q units of a certain product. If the marginal cost function is given by $MC(q) = C'(q) = 0.2q^2 + 4q + 500$ (\$/unit), determine the exact cost needed to increase the production level from $q = 90$ to $q = 150$.

(a) \$352,000 (b) \$148,800 (c) \$235,200 (d) \$198,000 (e) \$125,000

9. The picture below is the graph of a function $f(x)$, over the interval $-3 \leq x \leq 3$.

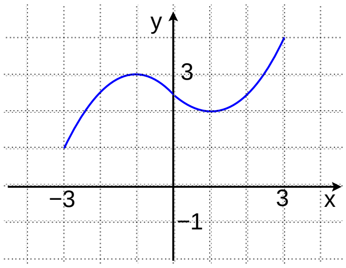


A new function $A(x)$ on the interval $[-3, 3]$ is defined, via a definite integral, by

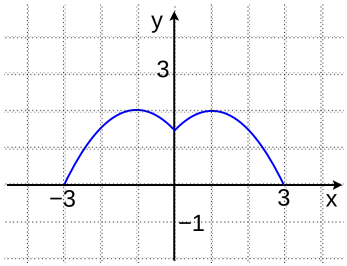
$$A(x) = \int_{-3}^{x} f(t)dt.$$

Which of the following pictures is the graph of $A(x)$?

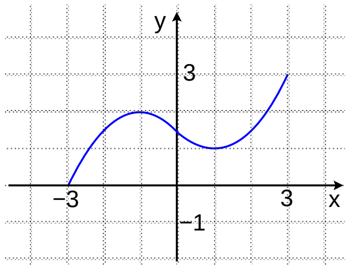
(a)



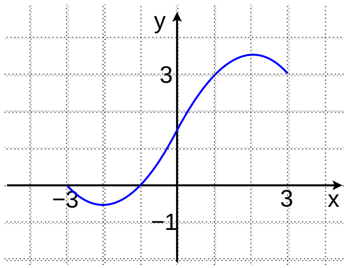
(b)



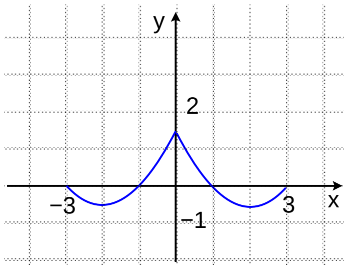
(c)



(d)



(e)



10. The definite integral $\int_0^1 (1+x)e^{-x^2-2x}dx$ equals

(a) $-\frac{1}{2}e^{-3} + \frac{1}{2}$ (b) $2e^{-3} + 1$ (c) $\frac{3}{2}e^{-3}$ (d) $\frac{1}{2}e^{-3} - \frac{1}{2}$ (e) $2e^{-3} - 1$

11. Money is transferred continuously into a new account at the constant rate of \$2,400 per year. The account also earns interest at the annual rate of 6% compounded continuously. Set up a definite integral that gives the amount of money accumulated in the account at the end of 4 years.

(a) $\int_1^5 2400te^{0.06(4-t)}\,dt$ (b) $\int_0^4 2400e^{0.06(4-t)}\,dt$ (c) $\int_0^4 2400\,te^{0.06t}\,dt$

(d) $\int_0^4 2400e^{0.06t}\,dt$ (e) $\int_0^4 2400\,te^{0.06(4-t)}\,dt$

12. The definite integral $\int_0^1 \frac{dx}{2+x}$ equals

(a) $\ln(3)$

(b) $\lim_{n\to+\infty}\left(\frac{1}{2+(\frac{0}{n})}+\frac{1}{2+(\frac{1}{n})}+\frac{1}{2+(\frac{2}{n})}+\cdots+\frac{1}{2+(\frac{n-1}{n})}\right)$

(c) $e = 2.71828...$

(d) $\lim_{n\to+\infty}\frac{1}{n}\left(\frac{1}{2+(\frac{0}{n})}+\frac{1}{2+(\frac{1}{n})}+\frac{1}{2+(\frac{2}{n})}+\cdots+\frac{1}{2+(\frac{n-1}{n})}\right)$

(e) $\lim_{n\to+\infty}\frac{1}{n}\left(\frac{0}{2+(\frac{0}{n})}+\frac{1}{2+(\frac{1}{n})}+\frac{2}{2+(\frac{2}{n})}+\cdots+\frac{n-1}{2+(\frac{n-1}{n})}\right)$

13. The magnitude M of an earthquake and its energy E are related by the equation

$$1.5M = \log_{10}\left(\frac{E}{2.5\times 10^{11}}\right),$$

where M is given in terms of Richter's scale. Express E in terms of M.

(a) $2.5\times 10^{11+1.5M}$ (b) $2.5\times e^{11+1.5M}$ (c) $2.5\times 10^{11-1.5M}$

(d) $2.5\times e^{11-1.5M}$ (e) $1.5E\times\ln[M/(2.5\times 10^{11})]$

14. We are given that $f(x)$ is a smooth and increasing function on the entire real line, and that $f(x)$ has $y=2$ as a horizontal asymptote as $x\to+\infty$. Which of the following choices must be true for $f(x)$?

(a) $f''(x)$ is always positive

(b) as $x\to-\infty$, $f(x)$ is concave up

(c) as $x\to+\infty$, $f(x)$ is concave up

(d) $f(x)$ has a point of inflection

(e) as $x\to+\infty$, $f'(x)$ is decreasing

Part II: Answer each of the following 7 long questions.

15. (29 points) Given functions $f(x) = (x+3)(2-x)$, $g(x) = \sqrt{x+1}$, and $h(x) = \frac{1}{x}$.

(a) Write down explicit expressions for the following composed functions: [4]

$(h \circ f)(x) = h(f(x)) =$

$(g \circ f)(x) = g(f(x)) =$

(b) Find the domain of the function $\sqrt{f(x)} = \sqrt{(x+3)(2-x)}$ in the form of explicit interval(s) on the $x-$axis. [5]

(c) Compute the derivative of the function $\sqrt{(x+3)(2-x)}$. [5]

(d) Determine the inputs of x which yield the absolute maximum and absolute minimum of the function $\sqrt{(x+3)(2-x)}$. *Justify your answer for full credit.* [6]

(e) State the limit definition of the derivative of a function $F(x)$ at a point x. [3]

(f) **Use the limit definition** to find the derivative of $g(x) = \sqrt{x+1}$. [6]

16. (28 points) Maggie's Candy Store has just replaced its Pure Chocolate Bar, which has a 10% market share of the candy bar market, with the new Chocolate and Nuts Bar (CNB). Maggie predicts that the CNB market share percentage, as a function of time, is

$$S(t) = \frac{50}{2 + 3e^{-t}} \qquad t \geq 0 .$$

Here, t is measured in years, and $S(0) = 10$ means the CNB has initial market share 10%.

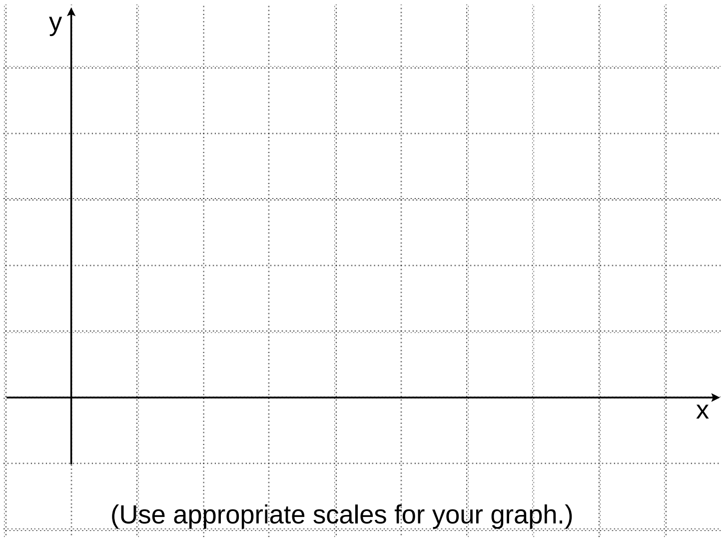
(a) When does the CNB reach a market share of 20%? [4]

(b) Graph the market share function $S(t)$. Indicate the exact coordinates of at least two points on the graph as well as any critical points, inflection points, and asymptotes. [12]

For your information:

$$S'(t) = \frac{150\, e^{-t}}{(2+3e^{-t})^2}$$

$$S''(t) = \frac{150\,(3e^{-2t} - 2e^{-t})}{(2+3e^{-t})^3}$$



(c) Will the CNB ever obtain a 40% market share? *Justify your answer for full credit.* [4]

(d) How will the rate of change of the CNB market share behave as $t \to +\infty$? [4]

(e) At what time will the CNB market share be increasing most rapidly? *Justify your answer for full credit.* [4]

17. (15 points) Below you are asked to compute $\int \frac{\ln(5x)\,dx}{x}$ in two different ways and then to analyze your work.

(a) Use the **substitution** $u = \ln(5x)$ to compute $\int \frac{\ln(5x)\,dx}{x}$. *Show all your work for full credit.* [5]

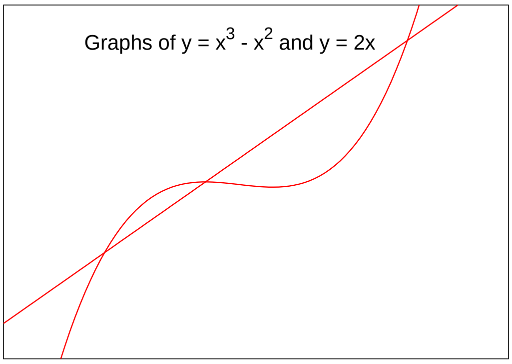
(b) Use the identity, $\ln(5x) = \ln(5) + \ln(x)$, to modify the given integral and then compute the integral. *Show all your work for full credit.*

$$\int \frac{\ln(5x)\,dx}{x} = \int \frac{\ln(5)\,dx}{x} + \int \frac{\ln(x)\,dx}{x}$$ [5]

(c) Your solutions in parts (a) and (b) above should look different. Explain this apparent difference given that they both compute the **same** indefinite integral. *Justify your answer for full credit.* [5]

18. (16 points) We are given two curves, defined by $y = x^3 - x^2$ and $y = 2x$. See graph.

(a) Find the coordinates of the intersection points of the two curves. [6]



(b) Write down an expression using definite integrals that computes the total area of the finite regions enclosed between the line $y = 2x$ and the curve $y = x^3 - x^2$. [6]

(c) Find the total area of the finite regions enclosed between the line $y = 2x$ and the curve $y = x^3 - x^2$. [4]

19. (16 points) A university merger rumor is started by one person at a university. There are 15,001 people in the university community, and the rumor spreads in the community according to the differential equation

$$\frac{dy}{dt} = (15001 - y).$$

Here, $y(t)$ is the number of people who know the rumor t days after it is started, and $y(0) = 1$.

(a) Solve the above differential equation and determine $y(t)$. [8]

(b) Determine how long it takes for the rumor to spread to $10,001$ people. [6]

(c) A **different** model for the spread of the rumor is that the rumor spreads at a rate that is proportional to the product of the number of people who know the rumor and the number of people who do not. Write down (**BUT DO NOT SOLVE!**) the differential equation for this model.

[2]

20. (18 points) A **continuous** piecewise defined function f has domain $0 \le x \le 3$. On the two pieces $0 \le x < 1$ and $1 < x \le 3$, the function is

$$f(x) = \begin{cases} x & \text{for} \quad 0 \le x < 1 \\ ax^2 + bx + c & \text{for} \quad 1 < x \le 3 \end{cases}$$

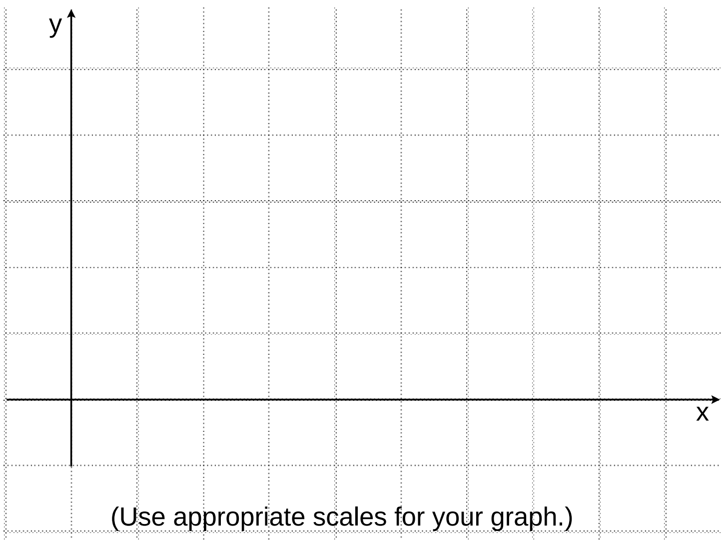
The function satisfies the additional conditions:

$$f'(1) = 1 \qquad \text{and} \qquad f'(2) = 0\ .$$

(a) What is $f(1)$? *Justify your answer for full credit.* [4]

(b) Determine the coefficients a, b, and c. [8]

(c) Draw the graph of f over its domain $0 \le x \le 3$ and indicate its precise value at 2. [6]



21. (8 points) A common estimate used by investors is the "Rule of 70": Money invested at an annual interest rate of $r\%$ takes approximately $70/r$ years to double.

(a) Determine the **exact** doubling time T as a function of r. [4]

(b) Using the linear differential/tangent approximation that $\ln(1 + x)$ is approximately x near $x = 0$, explain the "Rule of 70". [4]