Solutions to Midterm 1 (White: Version A), Math 005 Fall 2002

Part I: 10 multiple choice questions.

1. If $f(x) = g(x+3)$ and $g(x) = x^2 + 2x + 1$, compute $f(1)$. $\boxed{\textbf{(b)}\ f(1) = g(1+3) = g(4) = (4+1)^2 = 25}$. Observe $g(x) = (x+1)^2$.

2. Jenny has decided to pay off two loans on which she has not made any previous repayment. The first loan is a loan in which she obtained \$9,000 two years ago at an annual interest rate of 8% compounded semiannually. The second is a loan where \$12,500 is due two years from now at an annual interest rate of 16% compounded quarterly. To the nearest dollar, how much is due now?
$\boxed{\textbf{(d)}\ \$9000(1+\frac{0.08}{2})^4 + \$12,500(1+\frac{0.16}{4})^{-8} \ = \ \$19,662}$.

3. The domain of the function defined by the formula $f(x) = \dfrac{x^2-1}{2x^3}$ is: $\boxed{\textbf{(a)}\ \text{All real numbers except } 0}$ since division by 0 is undefined.

4. The derivative of $f(x) = \dfrac{x^2+1}{2} - \dfrac{1}{x^{\frac{3}{2}}}$ is: $\boxed{\textbf{(c)}\ f'(x) \ = \ \frac{2x}{2} - (\frac{-3}{2})x^{-\frac{5}{2}} \ = \ x + \frac{3}{2x^{\frac{5}{2}}}}$.

5. Find the limit $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^2+2x-3}$. $\boxed{\textbf{(d)}\ 0}$ since $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^2+2x-3} = \lim\limits_{x\to 1} \dfrac{(x-1)(x-1)}{(x-1)(x+3)} = \lim\limits_{x\to 1} \dfrac{(x-1)}{(x+3)}$.

6. Determine the tangent line to the graph of the function $y = f(x) = -6x^2 + 3x - 2$ at the point $(2, f(2))$. $\boxed{\textbf{(a)}\ y = -21x + 22}$ since $f(2) = -20$, $f'(2) = -21$ and the tangent line to the graph at $(2, f(2))$ is given by $y = f(2) + [f'(2)](x-2) = -20 - 21(x-2) = -21x + 22$.

7. Compute $g'(4)$ for $g(x) = (31 - 15\sqrt{x})(x^2 - 16)$. $\boxed{\textbf{(e)}\ 8}$. Use the product rule (and the power rule) to compute $g'(x) \ = \ (\frac{-15}{2\sqrt{x}})(x^2-16) \ + \ (31 - 15\sqrt{x})(2x)$, whence $g'(4) = 0 + (31-30)(8) = 8$.

8. Conversion between temperature measured in Fahrenheit F and measured in Celsius C is a linear function. Water freezes at 32F and 0C, and boils at 212F and 100C. The temperature in New York City is 68F. What is this temperature in C? $\boxed{\textbf{(a)}\ 0 + \frac{5}{9}(68-32) = 20}$. Required 'slope' is $\dfrac{100-0}{212-32} = \dfrac{100}{180} = \dfrac{5}{9}$.

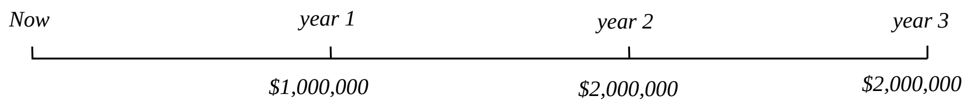
9. Find the maximum/minimum of the function $y = f(x) = 2x^2 + 5x - 3$. $\boxed{\textbf{(a)}\ y = -\frac{49}{8}, \text{ min.}}$ since the parabolic graph opens upwards (**why?**) and its vertex is at $(\frac{-5}{4}, \frac{-49}{8})$.

10. A vertical pole of height 4 metres snaps at a height of x metres above the ground. The upright portion (of x-metres) now forms the *perpendicular* side of a right-angled triangle, the snapped upper portion forms the *hypoteneuse* of the right-angled triangle (with the *base* of the triangle being along the ground). Determine the function $f(x)$ that gives the area (in square metres) of the relevant right-angled triangle. $\boxed{\textbf{(c)}\ f(x) = \frac{x}{2}\sqrt{16-8x}}$. We have a right-angled triangle with the length of its *perpendicular* $= x$ metres and the length of its *hypoteneuse* $= 4 - x$ metres. Thus, using the Pythagorean Theorem, the base has length $= \sqrt{(4-x)^2 - x^2} = \sqrt{16-8x}$. The area is now given by '*half height times base*'.

Part II: 4 long questions.

11. **For both parts (a) and (b)**, assume that the interest rate is 8% per year compounded every 6 months. Show your calculations and write down any formula you use.

(a) Company C has just received a loan and will repay it with three payments. The first payment, due one year from now, will be $\$1,000,000$. The second and third payments, due two and three years from now, will be $\$2,000,000$ each. Determine the loan amount. (7 points)



Solution: The loan amount $\$L$ is given by

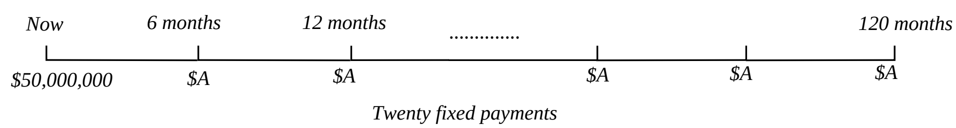
$$L \;=\; 1,000,000\Big[1(1+\frac{.08}{2})^{-2} \;+\; 2(1+\frac{.08}{2})^{-4} \;+\; 2(1+\frac{.08}{2})^{-6}\Big].$$

We are essentially bringing all the amounts to the present. Note that the interest being compounded every six months results in an interest rate of $\frac{.08}{2}$ (per 6 month period) and that 1 year equals 2 periods.

Calculation should lead to $\boxed{\$L = \$4,214,793.65}$ where we have rounded the answer to 2 decimal places. Note that not retaining 'enough' decimal places during the calculation can lead to round-off error. e.g., in this example, if you wished to have an answer accurate to 2 decimal places then each of the three summands should be accurate to 3 decimal places. This requires that the multiplicative factors be accurate to 10 decimal places in order to account for all the 'millions'.

CAUTION! On obtaining an answer please don't immediately move on. Check to see if it atleast makes sense. e.g., calculating a loan amount greater than 5 million dollars is definitely wrong. **Why?** In such cases you should try to locate your error and rectify it or atleast indicate that some error has been made (this could get you a higher score!).

(b) Company B has a debt of $\$50,000,000$ and wishes to repay it with twenty equal payments of $\$A$, with the first payment due in 6 months and then every 6 months afterwards. Determine A. (8 points)



Solution: The essential equation is

$$50,000,000 \;=\; A\Big[(1.04)^{-1} + (1.04)^{-2} + \ldots + (1.04)^{-20}\Big].$$

One can then calculate, or use a formula for, this geometric sum and simplify (or directly use the formula for the present value of an annuity) to obtain

$$50,000,000 \;=\; \frac{A}{0.04}\big(1 - (1.04)^{-20}\big).$$

Solving for A should lead to $\boxed{\$A = \$3,679,087.52}$ (rounded to 2 decimal places). Once again one needs to be careful about not introducing round-off error early in the computation (see note for part (a) above).

Aside: Without much calculation one should be able to see that an answer where A is less than 2.5 million is wrong. Why? Can you come up with a quick estimate for an obvious upper bound for A?

12. Maggie's Candy Store makes chocolate at a cost of \$15/kg, and sells it at a price of \$30/kg. Customers have been consuming 500 kgs of chocolate each day at the \$30/kg price. Maggie wishes to increase her profit by raising her chocolate price, but estimates that for each \$1/kg increase in the price, 20 less kgs will be sold each day. Let x be the selling price (in \$/kg) of Maggie's chocolate.

   (a) Express the amount of chocolate purchased each day as a linear function of x. (2 points)

   Solution: We are given enough data to write the equation of a line in 'point-slope' form. The amount of chocolate purchased, $A(x)$, in kgs is given by

$$A(x) \;=\; 500 \;-20(x-30) \;=\; 1100-20x.$$

   With the given units, Point on line: $(30, 500)$; Slope of line: -20.

   (b) Express Maggie's profit as a function of x. (4 points)

   Solution: Since Profit = Revenue - Cost = Amount(Selling Price - Cost Price), the Profit function $P(x)$ (in dollars) is given by

$$P(x) \;=\; (1100-20x)(x-15) \;=\; 20(55-x)(x-15).$$

   It is not essential to expand this quadratic polynomial but if you wish to do so then

$$P(x) \;=\; 20(-x^2+70x-825).$$

   (c) Determine the price x that yields Maggie's maximum profit. What is the maximum profit? Explain your reasoning. (7 points)

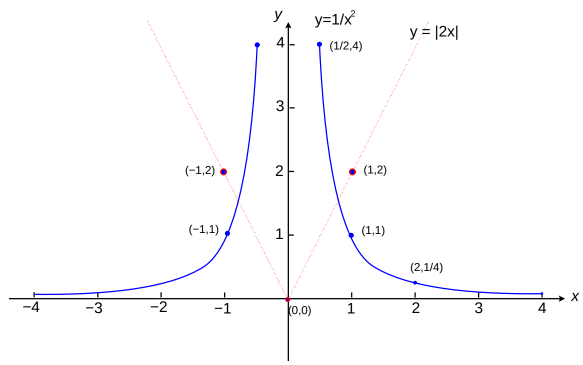
   Solution: The graph $y = P(x)$ is a downward opening parabola (**why?**) and thus the vertex will correspond to the maximum of the function $P(x)$. This would be a straightforward means of justifying one's answer. It is not enough to give the vertex as we also need to justify why it yields a **maximum**. We thus get a $\boxed{\text{maximum profit of \$8000}}$ for an $\boxed{\text{optimal selling price of } x = 35 \text{ (in \$/kg)}}$ as the vertex is given by $(35, 8000)$.

   The vertex can be obtained by using a formula or one can use the factored form of $P(x)$ to recognize that the 'zeros' of the quadratic function are at $x = 15$ and $x = 55$ and the x-coordinate of the vertex is exactly midway between these values ($x = 35$). Then the maximum profit is given by $P(35) = 20(55-35)(35-15)$.

   Caution! One could also use calculus to determine the desired x value by solving for $P'(x) = 0$. But by itself this does not tell us if we have a maximum, minimum, or neither at $x = 35$.

13. Consider the functions $f(x) = \dfrac{1}{x^2}$ and $g(x) = |2x|$.

(a) Sketch the graphs of $f(x)$ and $g(x)$ using the coordinate axes below, and indicate accurately the coordinates of at least three points on each of the graphs you sketch. (5 points)



(b) State the limit definition of the derivative of a function at a point x. (3 points)

$$u'(x) = \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}.$$

(c) **Using the limit definition**, find the derivative of $f(x) = \dfrac{1}{x^2}$. (5 points)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2-(x+h)^2}{(x+h)^2x^2}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2-x^2-2hx-h^2}{(x+h)^2x^2}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{h(-2x-h)}{(x+h)^2x^2}\right] \\
&= \lim_{h\to 0} \left[\frac{-2x-h}{(x+h)^2x^2}\right] \\
&= \left[\frac{-2x-0}{(x+0)^2x^2}\right], \text{ for } x \neq 0 \\
&= \frac{-2}{x^3}, \ x \neq 0.
\end{aligned}$$

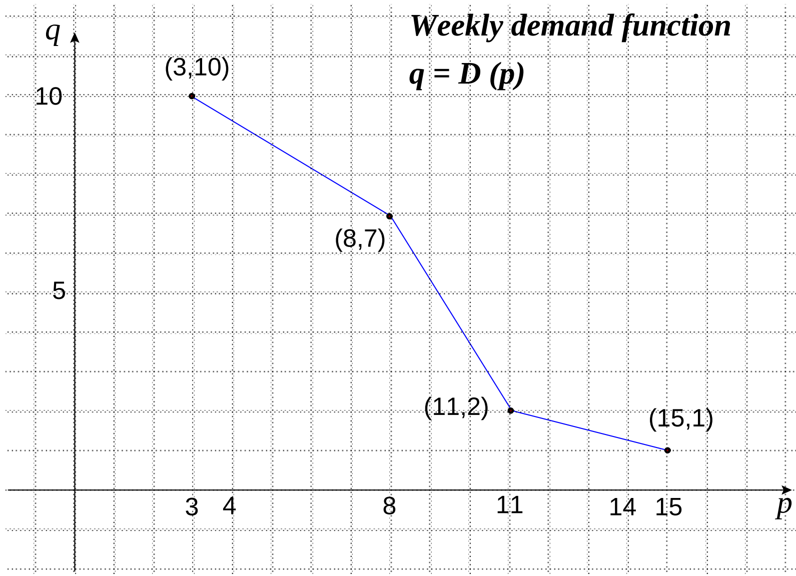
For evaluating the limit at the end, we use the fact that for a function $u(t)$ which is continuous at $t = a$, $\lim_{h\to a} u(h) = u(a)$.

(d) Can the above functions $f(x)$ and $g(x)$ have the same rate of change (i.e., derivative) at some value of x? Explain your reasoning. $(1 + 2$ points)

Solution: No! Rather than work algebraically with derivative formulae, notice from the graphs in part (a) that for $x < 0$, $f(x)$ is increasing while $g(x)$ is decreasing. Thus one will have a positive derivative and the other a negative one, and they cannot be equal for any $x < 0$. Similarly, for $x > 0$, $f(x)$ is decreasing while $g(x)$ is increasing. Once again the derivatives cannot be equal for any $x > 0$. At $x = 0$ neither function has a derivative so we cannot even talk about equating them!

Caution! The derivative of $g(x)$ is **not** $|2|$ $(|2| = 2)$! $g'(x) = -2$ for $x < 0$ and $g'(x) = +2$ for $x > 0$.

14. The figure below is the graph of a piecewise linear function, which is used to model the weekly demand $q = D(p)$ of ground beef sold in BBQ Supermarket. The variable p is the price (in \$/kg) and q is the quantity (in kgs) of beef sold.



(a) Write down the explicit formula for the demand function $q = D(p)$, on each of the linear pieces shown in the above figure. (7 points) **Solution:**

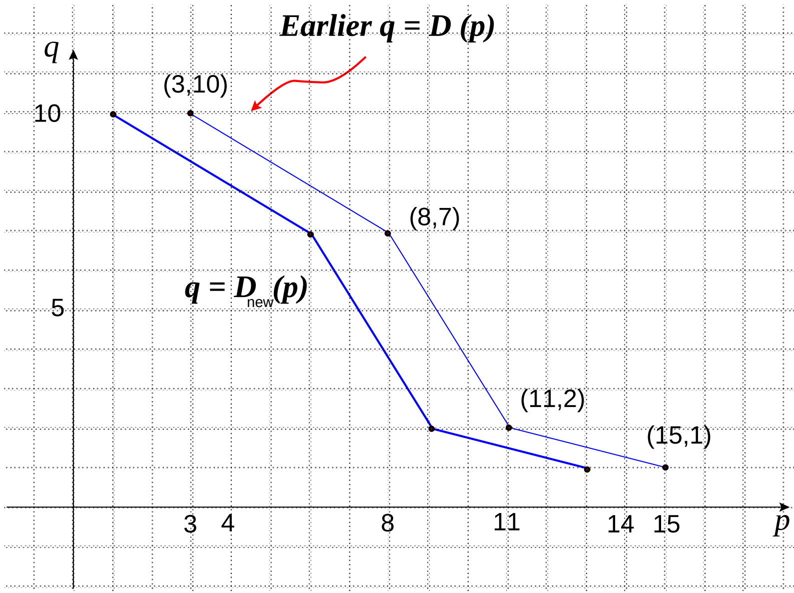
$$\begin{aligned} q &= 10 - \tfrac{3}{5}(p-3), \quad 3 \le p \le 8; \\ q &= 7 - \tfrac{5}{3}(p-8), \quad 8 \le p \le 11; \\ q &= 2 - \tfrac{1}{4}(p-11), \quad 11 \le p \le 15; \end{aligned}$$

One is essentially using the 'point-slope' form or the 'two-point' form for the equation of a line here. The required data can be read from the graph. Further, one could use strict inequalities for describing some of the domain subintervals, e.g. $3 \le p < 8$; $8 \le p < 11$; $11 \le p \le 15$; or some other valid variation. As the function is continuous on its entire domain, one can use more than one of the pieces to define q at $p = 8$ and at $p = 11$.

(b) At what price(s) p within the interval $4 < p < 14$ will the derivative of the demand function not exist? Explain your reasoning. (4 points)

Solution: $q = D(p)$ will not have a derivative at $p = 8$ and $p = 11$. This is because the left-hand limit of the slope is not equal to the right-hand limit of the slope at these points (the derivative at a point a is defined as a limit and we must get the same value when approaching a from either side). One can readily observe this from the graph as the graph has 'kink's at those p-values.

(c) The outburst of the mad cow disease has changed the weekly demand function. People are now paying two dollars less for the same amount of beef than in the old days. Sketch the graph of the new weekly demand function $q = D_{\text{new}}(p)$ in the figure below. (3 points)



(d) Express $q = D_{\text{new}}(p)$ in terms of $D(p)$. (2 points)

Solution: We obtain the graph $q = D_{\text{new}}(p)$ by translating the original graph of $q = D(p)$ **two units to the left**. This is because the new q-value (amount purchased) for p dollars is the amount earlier purchased for $p + 2$ dollars. e.g. $q = D_{\text{new}}(13) = D(13 + 2) = D(15) = 1$. So we *translate* the output $q = 1$ from above $p = 15$ to above $p = 13$. i.e., the point $(15, 1)$ on the graph of $q = D(p)$ moves to the point $(13, 1)$ on the graph of $q = D_{\text{new}}(p)$.

This also gives us the formula

$$q = D_{\text{new}}(p) \;\; = \;\; D(p + 2), \;\; 1 \leq p \leq 13.$$

Note that the domain of the new function also goes through the corresponding shift.