Solutions to some long questions.

1. [*8 pts*] A common estimate used by investors is the "Rule of 70": Money invested at an annual interest rate of $r\%$ takes approximately $\dfrac{70}{r}$ years to double.

   (a) Determine the **exact** doubling time T as a function of r. [*4 pts*]

Soln.:

$$(1+\frac{r}{100})^T = 2 \;\Rightarrow\; T\ln(1+\frac{r}{100}) = \ln(2) \;\Rightarrow\; T = \frac{\ln(2)}{\ln(1+\frac{r}{100})} \;.$$

Points: +2 for first equality (correct mult. factor). -1 if r used instead of $\frac{r}{100}$.
+1 for idea of using logs to solve for T.
+1 for final simplifications.

   (b) Using the linear differential/tangent approximation that $\ln(1+x)$ is approximately x near $x=0$, explain the "Rule of 70". [*4 pts*]

Soln.: For r lying in some 'reasonable range' (say $0 \le r \le 20$), $\frac{r}{100}$ is close to 0 and we can use the linear approximation for $\ln(1+x)$, based at $x=0$, to obtain, $\ln(1+\frac{r}{100}) \approx \frac{r}{100}$. Thus,

$$T \;=\; \frac{\ln(2)}{\ln(1+\frac{r}{100})} \;\approx\; \frac{\ln(2)}{\frac{r}{100}} \;\approx\; \frac{(0.693)(100)}{r} \;=\; \frac{69.3}{r} \;.$$

One can think of the "Rule of 70" as a 'rounded-off' version of the above rule. One can even explain the decision to round-off to a number (here 70) greater than $100\ln(2) \approx 69.3$. The linear approximation 'x' over-estimates $\ln(1+x)$. To see this, graph $\ln(1+x)$ and the relevant tangent line, or observe that $\ln(1+x)$ is concave down. Thus the denominator of our 'estimate' is larger than it should be. This can be partly compensated by suitably increasing the numerator. In fact some people use a "Rule of 72" !

Points: +2 for $T = \frac{69.3}{r}$.
+1 for observing '$\frac{r}{100}$ is small'.
+1 for saying why 69.3 is rounded **above**.

2. [*15 pts*] Below you are asked to compute $\displaystyle\int \frac{\ln(5x)\,dx}{x}$ in two different ways and then asked to analyze your work.

   (a) Use the **substitution** $u = \ln(5x)$ to compute $\displaystyle\int \frac{\ln(5x)\,dx}{x}$. [*5 pts*]

Soln.: Using the Chain Rule,

$$u = \ln(5x) \;\Rightarrow\; \frac{du}{dx} = \frac{1}{5x}(5) = \frac{1}{x} \;\Rightarrow\; du = \frac{dx}{x} \;.$$

Thus,

$$\int \frac{\ln(5x)\,dx}{x} \;=\; \int u\,du \;=\; \frac{u^2}{2} + c_1 \;=\; \frac{[\ln(5x)]^2}{2} + c_1 \;,$$

where c_1 is an arbitrary constant.

Points: +2 for du computation. +3 (2+1) overall for getting to $\int udu$.
+1 for correct integration and back substitution. +1 for $+c$.

   (b) Use the identity, $\ln(5x) = \ln(5) + \ln(x)$, to modify the given integral and then compute the integral. *Show all your work for full credit.* [*5 pts*]

Soln.:

$$\int \frac{\ln(5x)\,dx}{x} = \int \frac{\ln(5)\,dx}{x} + \int \frac{\ln(x)\,dx}{x} = \ln(5)\int \frac{dx}{x} + \int \frac{\ln(x)\,dx}{x}.$$

The first integral is a 'standard' anti-derivative and the second can be done via the substitution $v = \ln(x)$ (similar to the computation in part (a) of the question). This should yield

$$\int \frac{\ln(5x)\,dx}{x} = \ln(5)\ln|x| + \frac{[\ln(x)]^2}{2} + c_2\ ,$$

where c_2 is an arbitrary constant.

Points: +2 for each integral and +1 for the $+c$.
-1 if no $|x|$ in $\ln|x|$ AND no mention made of why $x > 0$.

(c) Your solutions in parts (a) and (b) above should look different. Explain this apparent difference given that both solutions compute the **same** indefinite integral. *Justify!* [*5 pts*]

Soln.: First off, in this problem, $\ln|x| = \ln(x)$ since x must be positive for the original integrand $\ln(5x)$ to be well defined. Second, using the logarithmic identity mentioned earlier and then using the expansion $(a+b)^2 = a^2 + 2ab + b^2$, we obtain

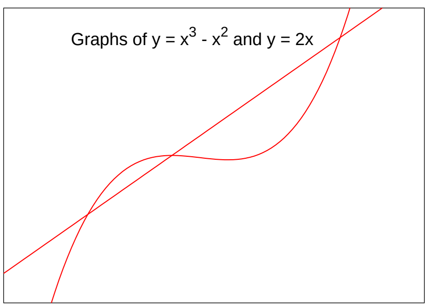
$$\frac{[\ln(5x)]^2}{2} = \frac{[\ln(5) + \ln(x)]^2}{2} = \frac{[\ln(5)]^2}{2} + \ln(5)\ln(x) + \frac{[\ln(x)]^2}{2}.$$

Thus, the two 'different' families of anti-derivatives are the same and this can be made explicit by choosing $c_2 = \frac{[\ln(5)]^2}{2} + c_1$.
Another way of explaining the 'difference' is to observe that two anti-derivatives for the same function can differ by a constant, which is exactly the case here. They both yield the **same** indefinite integral, which is the corresponding infinite family of anti-derivatives (obtained by adding an arbitrary constant).

Points: +2 for meaningful explanation.
+2 for supporting computation (either above manipulation or differentiation of both anti-derivatives).
-1 if $\ln|x|$ converted to $\ln(x)$ with no explanation. Note that a student will not loose more than 1 point for this on the entire question.
Atmost +2 (depending on reasoning used) if a student has incorrectly got two anti-derivatives in parts (a) and (b) that actually differ, and s/he says something meaningful.
ZERO if someone obviously fudges to hide an error.

3. [*16 pts*] We are given two curves, defined by $y = x^3 - x^2$ and $y = 2x$. See graph.

(a) Find the coordinates of the intersection points of the two curves. [*5 pts*]



Soln.: The intersection points are obtained via solving $x^3 - x^2 = 2x$, or, $x^3 - x^2 - 2x = 0$.

$$x^3 - x^2 - 2x \; = \; x(x^2 - x - 2) \; = \; x(x+1)(x-2) \; = \; 0$$

has solutions $x = -1, 0, 2$. The corresponding intersection points are $(-1,-2)$, $(0,0)$, $(2,4)$.

Points: +3 for correct factorization. (-2 for each missing factor?)
+1 for $x = -1, 0, 2$ and +1 for the intersection point coordinates.

(b) Write down an expression using definite integrals that computes the total area of the finite regions enclosed between the line $y = 2x$ and the curve $y = x^3 - x^2$. *[6 pts]*

Soln.:

$$\int_{-1}^{0} [(x^3 - x^2) - 2x] dx \; + \; \int_{0}^{2} [2x - (x^3 - x^2)] dx \; .$$

Points: -3 for missing one of the integrals, -1 for any incorrect limit, -2 for sign error in height.
A student should get atmost 4 out of 11 (on parts (b) and (c)) if s/he only has a single integral from -1 to 2.

(c) Find the total area of the finite regions enclosed between the line $y = 2x$ and the curve $y = x^3 - x^2$. *[5 pts]*

Soln.: Computing the above integrals we obtain

$$\left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \; + \; \left[x^2 - \frac{x^4}{4} + \frac{x^3}{3}\right]_{0}^{2} \; = \; (-\frac{1}{4} - \frac{1}{3} + 1) + (4 - 4 + \frac{8}{3}) \; = \; \frac{5}{12} + \frac{8}{3} = \frac{37}{12}.$$

Points: -1 for each incorrect anti-derivative.
-1 for significant algebra error.
-2 for leaving a **NEGATIVE** answer unexplained.
-1 if ‘second’ summand is smaller than first one and left unexplained.