Math005 Algebra and Calculus I, Fall 2003
Mid-term Exam Solution

Part I: Multiple Choice Questions

Version A

Question	1	2	3	4	5	6	7	8
Answer	c	e	a	b	c	c	d	R = f(E) = $\frac{8.8930E-20060}{8.950}$

Version B

Question	1	2	3	4	5	6	7	8
Answer	b	a	c	d	a	b	b	R = f(P) = $\frac{12.7590P-18060}{12.8480}$

Solution for Version A MC

1. The *monthly* interest rate is $\frac{12\%}{12} = 1\%$. We compute the present values of the down payment and the 4 payments at 3, 6, 9 and 12 months. We set the sum of these present values equal to US$1,400.00. We get $1400 = D + P(1.01)^{-3} + P(1.01)^{-6} + P(1.01)^{-9} + P(1.01)^{-12}$. The answer is (c).

2. The constant rate of decrease is given by $\dfrac{30000 - 300000}{10} = -27000$. After 6 years, the car would depreciated by $6 \times 27000 = 162000$ dollars. Hence the value of the car after 6 years would be $300000 - 162000 = 138000$ dollars.
Or, in the form of a linear function, the value (y) of the car after x years is given by $y = -27000x + 300000$. Putting in $x = 6$ results in $y = 138000$. The answer is (e)

3. We have $\log_b x + \log_b(x-4) = \log_b 21$, $\log_b[x(x-4)] = \log_b 21$, therefore x satisfies $x(x-4) = 21$. Solving this equation, we obtain $x = 7$ or $x = -3$. We reject the root $x = -3$ because $\log_b(-3)$ doesn't exist. The answer is (a)

4. When moving towards $x = 2$ from the right hand side of $x = 2$, i.e. $x \to 2^+$, $f(x) \to 1$ (found by moving along the graph on the right of $x = 2$). $\lim\limits_{x\to 2^+}([f(x)]^2 + 1) = \left[\lim\limits_{x\to 2^+} f(x)\right]^2 + 1 = 1^2 + 1 = 2$. The answer is (b).

5. The solution is $Pe^{3r} + Se^{-2s}$ (continuous future value + continuous present value). The answer is (c).

6. At $t = 10$th day, $R'(10) = -\frac{10}{3}(10)^2 + 40(10) + 50 = 116.67$. Thus we have
$$R'(10) \;=\; \lim_{h\to 0}\frac{R(10+h) - R(10)}{h} \;\overset{h=1}{\approx}\; R(11) - R(10),$$
or equivalently, by using the equation of the tangent line to the graph of $y = R(x)$ at $t = 10$, i.e., $y = R(10) + R'(10)(x-10)$,
$$R(11) \approx R(10) + R'(10) = 43,400 + 116.67 = 43,516.67.$$
The answer is (c).

7. From the definition $g(x+1) = x \cdot f(x+1) - x$, we see that

$$g(x) = g((x-1)+1) = (x-1)\cdot f((x-1)+1) - (x-1) = x\,f(x) - f(x) - x + 1$$

Take the derivative: $g'(x) = \underbrace{f(x) + x\,f'(x)}_{\text{from product rule}} - f'(x) - 1$. Therefore $g'(1) = f(1) + f'(1) - f'(1) - 1 = f(1) - 1$. The answer is (d).

8.

$$f(E) = \frac{8.8930E - 20060}{8.9500}$$

Part II: Long Questions

9. (a)

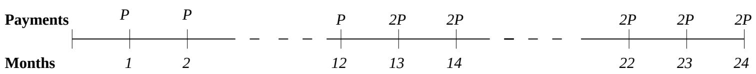


Figure showing the payments over the 24 months

(i) The PV for the first 12 months:

$$PV = P(1+0.005)^{-1} + \cdots + P(1+0.005)^{-12}.$$

(ii) The PV for the last 12 months:

$$PV = 2P\left[(1+0.005)^{-1} + \cdots + (1+0.005)^{-24}\right] - 2P\left[(1+0.005)^{-1} + \cdots + P(1+0.005)^{-12}\right].$$

(iii) The total amount from (i) and (ii) above is

$$\frac{2P}{0.005}\left(1 - \frac{1}{(1+0.005)^{24}}\right) - \frac{P}{0.005}\left(1 - \frac{1}{(1+0.005)^{12}}\right).$$

(iv) We substitute $P = \$2,500$ into the expression obtained in (iii) to obtain $\$83,767$ to the nearest dollar.

(b)

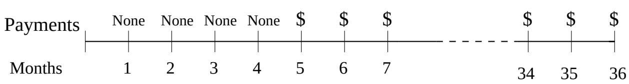


Figure indicating the payments of the sinking fund

We need to come up with $\$37,500 - 8,000 = 29,500$, from the sinking fund. Let Q be the monthly payment for the sinking fund, then

$$\begin{aligned}29500 &= Q + Q(1+0.01) + Q(1+0.01)^2 + \cdots + Q(1+0.01)^{31}\\ &= \frac{Q}{0.01}\left[(1+0.01)^{32} - 1\right].\end{aligned}$$

Hence, we obtain

$$Q = 295/\left[(1+0.01)^{32} - 1\right] = \$786.79,$$

to the two decimal places.

10. (a) The total cost is a linear function: $C(x) = a + bx$, and the fixed cost is $2000 = C(0) = a$. Also, $C(10) = 2800$, i.e. $2000 + 10b = 2800$, and therefore $b = 80$.

    The total cost function is $C(x) = 2000 + 80x$.

    (b) $R(x) = xp(x) = x(200 - x) = 200x - x^2$, and

$$P(x) = R(x) - C(x) = 200x - x^2 - 2000 - 80x = -x^2 + 120x - 2000.$$

    Both $R(x)$ and $P(x)$ have the same domain: $0 \leq x \leq 200$, since $x \geq 0$ and $p(x) = 200 - x \geq 0$.

    (c) (i) The break-even points are the roots of $P(x) = R(x) - C(x) = 0$, that is, $-x^2 + 120x - 2000 = 0$. We have $-x^2 + 120x - 2000 = -(x - 20)(x - 100) = 0$, hence, $x = 20$ or $x = 100$.

    (ii) She gets a profit if $P(x) = -(x-20)(x-100) > 0$, i.e., $x-20$ and $x-100$ must have different signs (one positive, one negative) so that together with the negative sign in front of the two factors, $P(x) > 0$. Need to take $20 < x < 100$.

    (iii) Doris will suffer a loss if

$$P(x) = R(x) - C(x) < 0.$$

    By throwing out from the domain of $P(x)$ the break-even points and the range of x where a profit is possible (part (ii)), the solution is $0 \leq x < 20$ OR $100 < x \leq 200$.

    (d) The maximum profit is equal to the maximum value of $P(x)$. By using completing square, we find that

$$P(x) = -x^2 + 120x - 2000 = -(x - 60)^2 + 1600.$$

    Therefore, the maximum profit is \$1600, with the price $p(60) = 200 - 60 = 140$.

11. (a) The domain is $t \geq 0$.

    (b) (i) $1 + e^{-t} > 1$ since the exponential function e^{-t} only takes on positive values.

    (ii) If $t \geq 0$, then $e^{2t} \geq 1$, so the reciprocal $e^{-2t} \leq 1$. Therefore, $12e^{-2t} \leq 12$ and so $33 + 12e^{-2t} \leq 33 + 12 = 45$.

    (iii) In the expression $MS(t) = \dfrac{33 + 12e^{-2t}}{1 + e^{-t}}$, the top expression is less than or equal to 45 and the bottom expression is greater than 1. Therefore, the division is less than or equal to 45. This means the market share never reaches any precentage 45% or above. The market share never reaches 50%.

    (c) As $t \to \infty$, both $e^{-2t} \to 0$ and $e^{-t} \to 0$. Therefore, $\frac{33+12e^{-2t}}{1+e^{-t}} \to \frac{33+0}{1+0} = 33$. As $t \to \infty$, the market share of the CNB approaches 33%.

    (c) We solve

$$23 \;=\; \frac{33 + 12e^{-2t}}{1 + e^{-t}} \;.$$

    Multiply both sides by $1+e^{-t}$ to get $23+23e-t = 33+12e^{-2t}$. Simplify to get $12e^{-2t} - 23e^{-t} + 10 = 0$. We first solve the quadratic equation $12u^2 - 23u + 10 = 0$, where $u = e^{-t}$, and then t can be found by taking $t = -\ln u$. We factor $12u^2 - 23u + 10 = (4u - 5)(3u - 2)$, so $u = \frac{5}{4}$ or $u = \frac{2}{3}$. For $u = \frac{5}{4}$, we have $t = -\ln(\frac{5}{4}) = -0.2231$ which is not in the domain. For $u = \frac{2}{3}$, we have $t = -\ln(\frac{2}{3}) = 0.4054$. The market share reaches 23% at $t = 0.4054$ years.

If you are still not familiar with the graphs of the functions $y = e^t$, $y = e^{-t}$, and other exponential functions, see p.97-100, and Ex 2-2, 1, p.107.

12. (a) The derivative of a function $F(x)$ at a point x, denoted by $F'(x)$ or $\frac{dF}{dx}$, is defined via a limiting process by

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} \;,$$

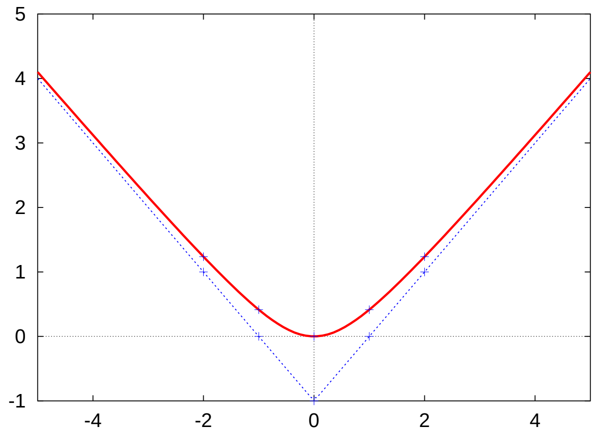
    whenever the limit exists.

(b) By the limit definition of derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[\sqrt{(x+h)^2+1}-1\right]-\left[\sqrt{x^2+1}-1\right]}{h}\\
&= \lim_{h\to 0}\frac{\left(\sqrt{(x+h)^2+1}-\sqrt{x^2+1}\right)\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}\\
&= \lim_{h\to 0}\frac{[(x+h)^2+1]-(x^2+1)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}\\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2+1-x^2-1}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} = \lim_{h\to 0}\frac{h(2x+h)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}\\
&= \lim_{h\to 0}\frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} = \frac{2x}{\sqrt{x^2+1}+\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}
\end{aligned}
$$

(c) The graphs of the functions $f(x)=\sqrt{x^2+1}-1$ and $g(x)=|x|-1$:

x	$f(x)=\sqrt{x^2+1}-1$	$g(x)=\lvert x\rvert-1$
-2	$\sqrt{5}-1\approx 1.246$	1
-1	$\sqrt{2}-1\approx 0.414$	0
0	0	-1
1	$\sqrt{2}-1\approx 0.414$	0
2	$\sqrt{5}-1\approx 1.236$	1



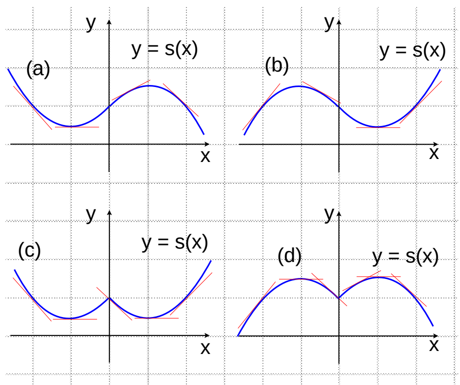
(d) Graph (b).

Note that if $s'(x)=g(x)$, then $g(x)$ should be the slope of the tangent line to the graph of $s(x)$ at $(x, s(x))$. By inspecting the tangents to the graph of $s(x)$, only (b) is possible.

More precisely, for $x>0$, $g(x)=|x|-1=x-1$ is increasing from negative values to positive values, so must the slopes of the tangent lines to $s(x)$. Hence (a) and (d) are out.

For $x<0$, $g(x)=|x|-1=-x-1$ is decreasing from positive values to negative values, so must the slopes of the tangent lines to the graph of $s(x)$. (c) is also out.

Only (b) can match the requirement.



These four graphs are for part (d)

Or, directly from $g(x)=\begin{cases} x-1 & \text{if } x>0\\ -x-1 & \text{if } x<0\end{cases}$ *, one may pick*

$$s(x)=\begin{cases} \frac{1}{2}x^2-x+1 & \text{if } x>0\\ -\frac{1}{2}x^2-x+1 & \text{if } x<0\end{cases}$$

and then easily check that $s(0)=1$*, and* $s'(x)=g(x)$*. That's how graph (b) was produced.*