MATH1013 Calculus I

Introduction to Functions[1]

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Revision

[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Definition of functions

- **Definition** A function is a rule f that assigns to each x in a set D a *unique* value denoted by $f(x)$. $\mathbb{C}$.
- **Definitions** The set D is called the domain of the function f, and the set of values of $f(x)$ assumes, as x varies over the domain, is called the range of the function $f(x)$.
- $$x \longmapsto f(x), \qquad \text{or} \qquad y = f(x),$$
- One can think of this as a model of

  one input $\rightarrow$ **one output**

- Important point: for **each** x in D, one can find (there exists) **one** value $f(x)$ (or y) that corresponds to it.
- However, depending on the f under consideration, one could have two or more x that correspond to the same $f(x)$.
- This strange looking idea was created to describe dynamical

Different types of functions

- $y = f(x) = x + 1$. For each x there corresponds to one and only one y.
- $y = x^3$. For each x there corresponds to one and only one y
- Where $f(x_1) = f(x_2)$ implies $x_1 = x_2$, or equivalently $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, we say the function f is injective or one-one. So the above two examples are injective functions.
- (Eg revisited) $f(x) = x^2 - 2x$ is not injective, as two different x can correspond to the same $f(x_1) = y = f(x_2)$
- (Non-function) $y^2 = 1 - x^2$. Since for each x input, there always correspond to two outputs of $f(x) = \pm\sqrt{1 - x^2}$ within the domain of f.

Composition

- **Definition** Given two functions f and g, their composition $f \circ g$ is defined, by

$$(f \circ g)(x) = f(u) = f(g(x))$$

for each x in the domain of $f \circ g$. Let $u = g(x)$ and $y = f(u)$, then $f \circ g$ is understood as

$$y = (f \circ g)(x) = f(g(x)) = f(u), \qquad u = g(x),$$

- as shown in

$$x \longmapsto u = g(x) \longmapsto y = f(u)$$

- with g takes the domain of g (range) into (part of) domain of f, and f maps that into (part of) the range of f. The two together thus forms a new function $f \circ g$.

Different classes of functions

- Polynomials, Rational fns.
- Exponential fns $f(x) = 2^x$, $f(x) = e^x$
- Logarithmic fns $f(x) = \ln x$, $f(x) = \log_x$
- trigonometric rations/fns $\sin x$, $\cos x$, $\tan x$, etc
- Periodic properties
- Inverse trigonometric fns

Exponential functions of different bases

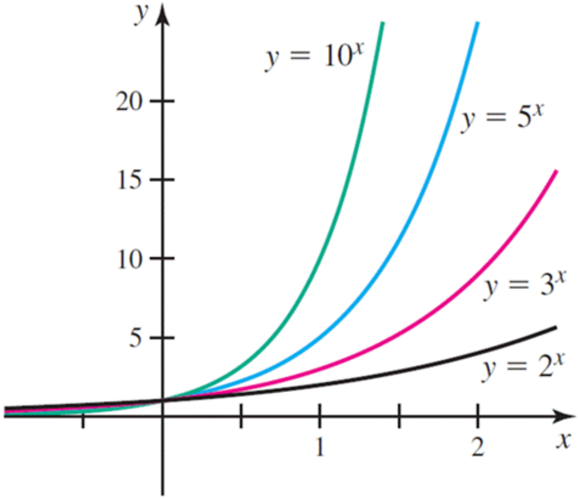
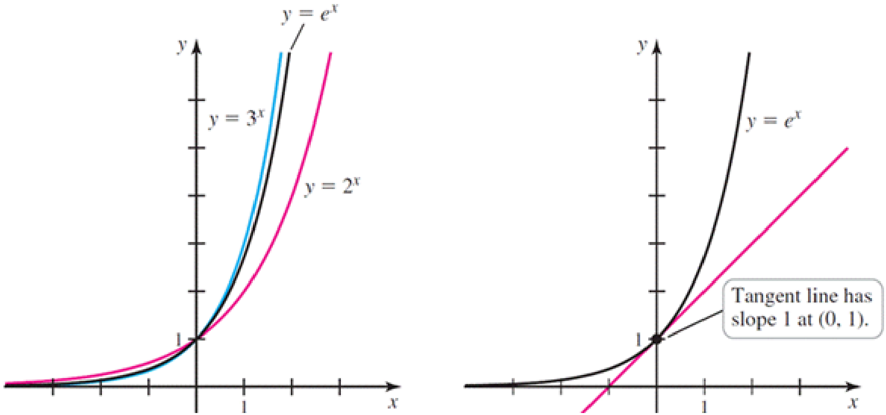


Figure: (Publisher Figure 1.42)

Exponential functions



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Ratios as circular functions: $0 \leq \theta \leq 2\pi$

- Due to the property of similar triangles, it is sufficient that we consider unit circle in extending to $0 \leq \theta \leq 2\pi$ with the ratios augmented with appropriate signs from the $xy-$coordinate axes.

20. Line Values of Functions. About the vertex O of a given angle as center describe a circle of unit radius cutting the initial side in B and the terminal side in P_2 (Fig. 23). We suppose at first that the angle lies in the second quadrant. Draw M_2P_2 perpendicular to the diameter BB'. Then, by the definitions of § 18, taking account of the algebraic signs, we have

$$\sin BOP_2 = \frac{M_2P_2}{OP_2} = \frac{M_2P_2}{1} = M_2P_2,$$

$$\cos BOP_2 = \frac{OM_2}{OP_2} = \frac{OM_2}{1} = OM_2.$$

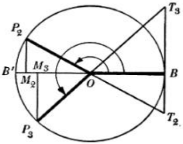


Fig. 23

Line values for the tangent and secant may be obtained by drawing at B a tangent to the circle and producing the terminal side, OP_2, backward till it meets this tangent in T_2.

We have then

$$\tan BOP_2 = \frac{BT_2}{OB} = \frac{BT_2}{1} = BT_2,$$

Ratios beyond 2π

- For sine and cosine functions, if θ is an angle beyond 2π, then $\theta = \phi + k\,2\pi$ for some $0 \le \phi \le 2\pi$. Thus one can write down their meanings from the definitions

$$\sin(\theta) = \sin(\phi + k\,2\pi) = \sin\phi, \quad \cos(\theta) = \cos(\phi + k\,2\pi) = \cos\phi$$

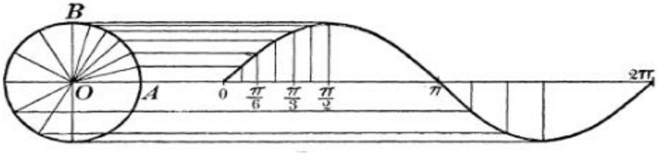
where $0 \le \phi \le 2\pi$, and k is any integer.

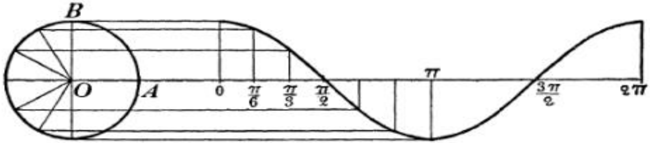
- The case for tangent ratio is slightly different, with the first extension to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and then to arbitrary θ. Thus one can write down

$$\tan(\theta) = \tan(\phi + k\,\pi) = \tan\phi$$

where $\theta = \phi + k\pi$ for some k and $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$.

Periodic sine and cosine functions





Periodic tangent function

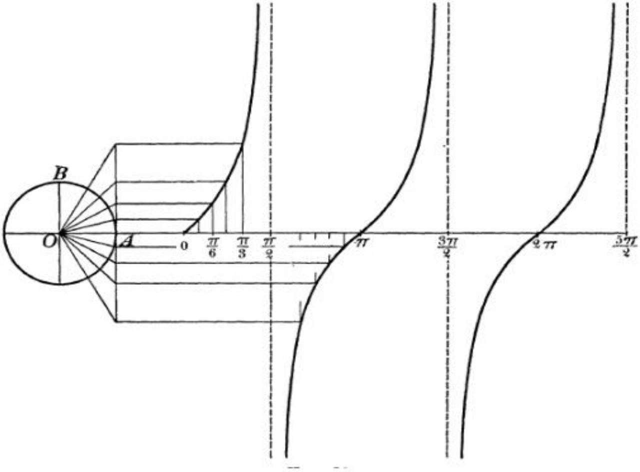


Figure: (Source: tangent, Bocher page 46)

Inverse functions

- **Definition** Let f be a function defined on its domain D. Then a function $^{-1}$ is called an inverse of f if

$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in } D.$$

  That is, $x = f^{-1}(y)$ whenever $y = f(x)$.
- **Remark 1** It follows that the domain of f^{-1} is on the range of f.
- **Remark 2** There is no guarantee that every function has an inverse.
- **Remark 3** If f has two inverse functions, then the two inverse functions must be identically the same.

Inverse function figure

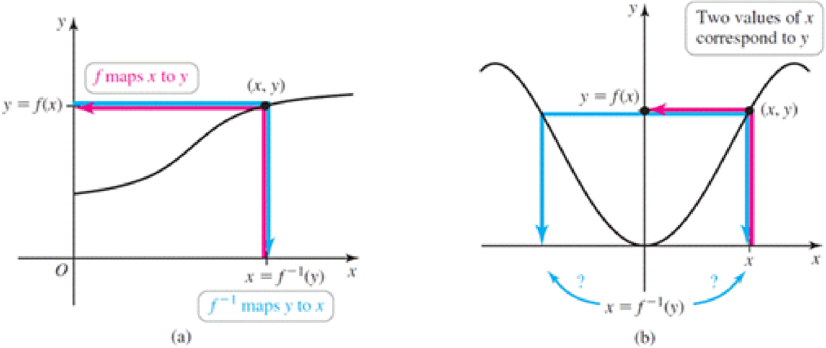


Figure: (Publisher Figure 1.49)

Inverse trigonometric functions

- The sine and cosine functions map the $[k\,2\pi,\ (k+1)\,2\pi]$ onto the range $[-1,\ 1]$ for each integer k. So it is

  many $\longrightarrow$ one

  so an inverse would be possible only if we suitably restrict the domain of either sine and cosine functions. We note that even the image of $[0,\ 2\pi]$ "covers" the $[-1,\ 1]$ more than once.
- In fact, for the sine function, only the subset $[\frac{\pi}{2},\ \frac{3\pi}{2}]$ of $[0,\ 2\pi]$ would be mapped onto $[-1,\ 1]$ exactly once. That is, the sine function is one-one on $[\frac{\pi}{2},\ \frac{3\pi}{2}]$.
- However, it's more convenient to define the inverse $\sin^{-1} x$ on $[-\frac{\pi}{2},\ \frac{\pi}{2}]$.

$$\sin^{-1} x : \quad \Big[-\frac{\pi}{2},\ \frac{\pi}{2}\Big] \longrightarrow [-1,\ 1].$$

- $$\cos^{-1} x : \quad [0,\ \pi] \longrightarrow [-1,\ 1].$$

Logarithm as inverse function

- If we view $y = f(x) = b^x$ as a given function, then its inverse is given by $y = f^{-1}(x) = \log_b x$ since we can check

$$(f^{-1} \circ f)(x) = \log_b(b^x) = x$$

  by the definition of logarithm.
- In fact, even

$$(f \circ f^{-1})(x) = b^{\log_b x} = x$$

  holds trivially.
- The graph of $\log_b x$ is obtained from rotating $y = b^x$ along the line $x = y$ by 180 degrees.

Newton's trouble

- Suppose an object moves according to the rule $S(t) = 20 + 4t^2$ where S measures the distance of the object from the initial position t seconds later.
- We now compute instantaneous velocity of the object at time t: let dt and dS be the virtual time and virtual distance respectively. Then the change of virtual distance is given by $dS = S(t+dt) - S(t)$. So the virtual velocity is

$$\frac{dS}{dt} = \frac{S(t+dt) - S(t)}{dt} = \frac{4(t+dt)^2 - 4t^2}{dt} = 8\,t + 4\,dt.$$

- Newton then delete the last dt:

$$\frac{dS}{dt} = 8\,t + 4\,\cancel{dt} = 8\,t.$$

- So do we have $dt = 0$? If so, then one would have $\frac{dS}{dt} = \frac{0}{0}$. That was the question that Newton could not answer satisfactorily during his life time.

Re-assessing the problem

- Let us begin with the above example about the movement of the object P. Since we are interested to know the magnitude of the average velocity of P near 2, so let us rewrite the expression in the following form:

$$g(x) = \frac{S(2+x) - S(2)}{x}.$$

- This is a function g depends on the variable x, which can be made as close to 16 as we wish by chooesing t close to 2.
- That is, $g(x)$ approaches the value 16 as x approaches 0. On the other hand, we **cannot** put $x = 0$ in the function $g(x)$, since both the numerator $S(2+x) - S(x)$ and the denominator x would be zero.
- We say that the function g has limit equal to 16 as x approaches 0 abbreviated as

$$\lim_{x \to 0} g(x) = 16.$$

Limit definition

- Note that the above statement is merely an abbreviation for the statement: *The function g can get as close to 16 as possible if we let x approach 0 as close as we wish.*
- It is important to note that we are not allowed to put $x = 0$ above
- **Definition** *Let a and l be two real numbers. If the value of the funciton $f(x)$ approaches l as close as we wish as x approaches a, then we say the limit of f is equal to l as x tends to a. The statement is denoted by*

$$\lim_{x \to a} f(x) = l.$$

Alternatively, we may also write

$$f(x) \to l \quad \text{as} \quad x \to a.$$

Examples

- Find $\lim_{x\to 2} \frac{x^3-8}{x-2}$.
- Note that we can not substitute $x = 2$ in the expression. For then both the numerator and denominator will be zero. Consider

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2}$$
$$= \lim_{x\to 2}(x^2+2x+4) = 12.$$

- The above is an abbreviation of the expression:

$$\frac{x^3-8}{x-2} = \frac{(x-2)(x^2+2x+4)}{x-2} = x^2+2x+4$$

tends to the value 12 as x tends to 2.

- or more briefly

$$\frac{x^3-8}{x-2} = x^2+2x+4 \mapsto 12, \qquad \text{as } x \mapsto 2.$$

Newton's thought

- So he simply considers that is a virtual distance dS traveled by the object in a virtual time dt. He considers both to be infinitesimal small quantities.
- So do we have $dt = 0$? If so, then one would have $\frac{dS}{dt} = \frac{0}{0}$. That was the question that Newton could not answer satisfactorily during his life time.
- To put the question differently, is an infinitesimal quantity equal to zero? If dt is infinitely small then it would have to be less than any positive quantity, and we conclude it must be equal to zero. For suppose $dt \neq 0$ then $dt > 0$. Hence $dt = r > 0$ is an actual positive quantity. But then we could find $r/2 < dt$, contradicting the fact that dt is smaller then any positive quantity. Hence $dt = 0$.
- Newton was actually attacked by many people, and among them was the Bishop Berkeley. But he method of calculation of instantaneous velocity has been used by other since then.

A function that has no limit at 0

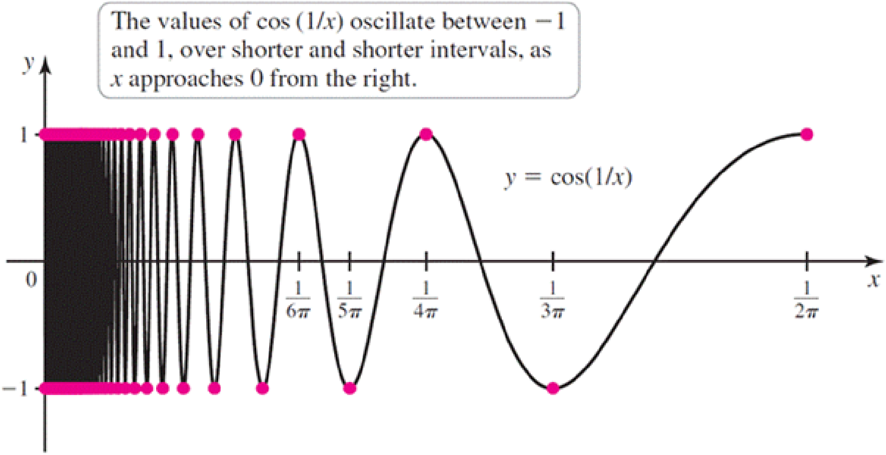


Figure: (Publisher Figure 2.14)

Limit laws

- Suppose $\lim_{x\to a} f(x) = \ell$, $\lim_{x\to a} g(x) = m$ both exist. Let c be a constant, then the following hold:
- $$\lim_{x\to a}\Big(f(x)+g(x)\Big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = \ell + m$$
- $$\lim_{x\to a}\Big(c\,f(x)\Big) = c\,\lim_{x\to a} f(x) = c\ell$$
- $$\lim_{x\to a}\Big(f(x)g(x)\Big) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) = \ell m$$
- $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\ell}{m} \quad \text{provided } m \neq 0.$$

The real difficulty

- We recall that in terms of $\varepsilon - \delta$ language $\lim_{x\to a} f(x) = \ell$ really means
  Given an arbitrary $\varepsilon > 0$, one can find a $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon, \qquad \text{whenever } 0 < |x - a| < \delta.$$

- So for $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$, one needs to show, assuming that $\lim_{x\to a} g(x) = s$
  Given an arbitrary $\varepsilon > 0$, one can find a $\delta > 0$ such that

$$\big|[f(x) + g(x)] - (\ell + s)\big| < \varepsilon, \qquad \text{whenever } 0 < |x - a| < \delta.$$

  with the given assumption.
- This is slightly not easy. Some other laws are more difficult to verify using this language. So this explains why one needs to state these seemingly simple laws as separate entities.

Limits at infinity

- **Definitions** Let ℓ and a be real numbers. If f tends to ℓ as x becomes arbitrary large and positive, we say f has the limit ℓ at positive infinity, written as

$$\lim_{x\to+\infty} f(x) = \ell \qquad (f \to \ell, \quad \text{as} \quad x \to +\infty).$$

- Similarly, if f tends to ℓ as x becomes arbitrary large and negative, we say f has the limit ℓ at negative infinity, written as

$$\lim_{x\to-\infty} f(x) = \ell \qquad (f \to \ell, \quad \text{as} \quad x \to -\infty).$$

Rate of change

- We may consider rate of change of a given function $f(x)$ not necessarily refereed to time, distance and velocity.
- **Definition** Let $f(x)$ be a function of x, then f is differentiable at x if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists.

- The limit is called the derivative of f at x or the rate of change of f with respect to x, and it is denoted by $f'(x)$.
- Other notations are

$$\frac{df(x)}{dx} \quad \text{or} \quad \left.\frac{df}{dx}\right|_x \quad \text{or} \quad \frac{df}{dx}$$

- If $y = f(x)$, we also write $\frac{dy}{dx}$. We treat this notation as an operator instead of quotient of infinitesimal quantities.
- However, we shall see later that they are interchangeable.

Differentiation rules

- **Theorem** Let c be a constant and that a function f is differentiable at x. Then

$$\frac{d(c\,f)}{dx} = c\,\frac{df}{dx} \qquad \text{or} \qquad (c\,f)'(x) = c\,f'(x).$$

Proof Consider the following limit

$$\lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} = \lim_{\Delta x \to 0} c\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$
$$= c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Differentiation rules

- **Theorem** Let $f(x)$ and $g(x)$ be differentiable at x. Then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{df}{dx} + \frac{dg}{dx}.$$

Proof We have

$$\frac{d}{dx}(f(x)+g(x)) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)+g(x+\Delta x)-\big(f(x)+g(x)\big)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)+\big(g(x+\Delta x)-g(x)\big)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x\to 0} \frac{g(x+\Delta x)-g(x)}{\Delta x}$$

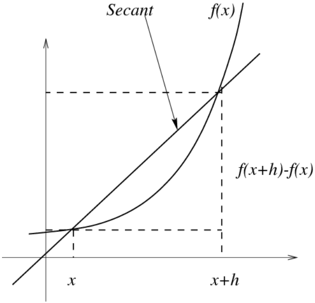
$$= \frac{df}{dx} + \frac{dg}{dx}.$$

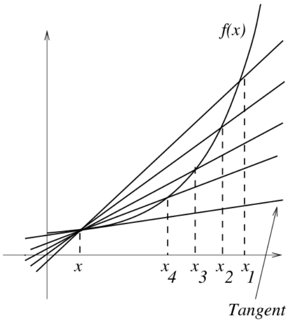
Graphical Interpretation

- Draw a straight line called secant passing through the pair of points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$. Then the

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

gives the gradient (slope) of the above secant. See the diagram.





Graphical Interpretation (cont.)

- We choose $\Delta x_1, \Delta x_2, \Delta x_3, \ldots$ with magnitudes decreasing to zero. Then we have a sequence of secants passing through the points $\big((x, f(x)),\ (f(x + \Delta x_i)\big)$.
- The corresponding gradients of the secants are given by

$$m_i = \frac{f(x + \Delta x_i) - f(x)}{\Delta x_i}.$$

- Suppose we already know that f has a derivative at x , we conclude that the sequence of gradients $\{m_i\}$ must tend to $f'(x)$ as Δx_i tends to zero.
- The point $\big(x + \Delta x, f(x + \Delta x)\big)$ is getting closer and closer to $\big(x, f(x)\big)$, as $\Delta x \to 0$, they *eventually coincide* to become a single point.
- It follows that the corresponding secants are *tending* to a straight line with only *one* point of contact with f at x. See the diagram from last slide. This line is called the *tangent* to f *at* x.

An example has no tangent

- **Example** The $|x|$ is not differentiable at 0.
- Consider

$$\lim_{\Delta x\to 0+}\frac{|0+\Delta x|-|0|}{\Delta x}=\lim_{\Delta x\to 0+}\frac{|\Delta x|}{\Delta x}=\lim_{\Delta x\to 0+}\frac{\Delta x}{\Delta x}=1.$$

- On the other hand, we have, according to the definition of $|x|$ that

$$\lim_{\Delta x\to 0-}\frac{|0+\Delta x|-|0|}{\Delta x}=\lim_{\Delta x\to 0-}\frac{|\Delta x|}{\Delta x}=\lim_{\Delta x\to 0-}\frac{-\Delta x}{\Delta x}=-1.$$

- So the left and right limits are not the same, and we conclude that $|x|$ does not have a derivative at 0 (however, it has derivatives at all other points). It is important to understand the corresponding situation on its graph drawn on last slide (right figure).

Last example's sketch

It is instructive to plot the curve of P and P' on the same coordinate axis. See the diagram on the left below.

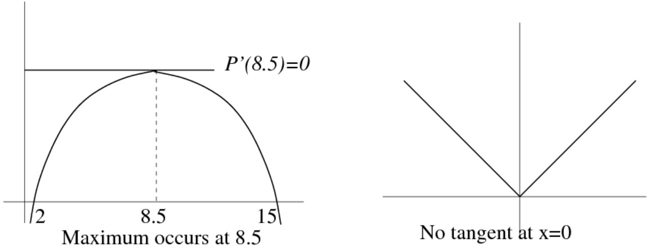


Figure: (Left: Profit function; Right: $|x|$ has no tangent at 0)

How does composition change?

- Suppose that $y = g(u)$ and $u = f(x)$. i.e., is y is a function of u and u is a function of x.
- When we compose to get $y = g(f(x))$, which is now a function of x, written as $y = h(x)$.
- If x is changed to $x + \Delta x$, there's a corresponding change in u,

$$u + \Delta u = f(x + \Delta x).$$

- As a result, it will induce a further change in y. Hence

$$y + \Delta y = g(u + \Delta u).$$

Example

- **Example** Let $y = u^3 + 1$ and $u = 2x - 4$. Find the increases of y and u due to an increase of x from x to $x + \Delta x$ where Δx is a small increment of x. When x is increased to $x + \Delta x$ the change in u is

$$\begin{aligned} u(x + \Delta x) - u(x) &= [2(x + \Delta x) - 4] - (2x - 4) \\ &= 2\Delta x. \end{aligned}$$

$$\begin{aligned} \Delta u &= u(x + \Delta x) - u(x) \\ &= 2\Delta x. \end{aligned}$$

$$\begin{aligned} y(x + \Delta x) - y(x) &= y(u + \Delta u) - y(u) \\ &= [(u + \Delta u)^3 + 1] - [u^3 + 1] \\ &= (u^3 + 3u^2(\Delta u) + 3u(\Delta u)^2 + 1) - (u^3 + 1) \\ &= 3u^2(\Delta u) + 3u(\Delta u)^2 \\ &= 3(2x - 4)^2(2\Delta x) + 3(2x - 4)(2\Delta x)^2 \end{aligned}$$

Chain Rule

- **Theorem** *Let* $y = g(u)$, $u = f(x)$ *and* $b = f(a)$, $c = g(b)$. *Suppose that* g *is differentiable at* $u = b$, *and that* f *is differentiable at* $x = a$. *Then the function* $y = h(x) = g(f(x))$ *is also differentiable at* $x = a$, *and the relationship is given by*

$$\left.\frac{dh}{dx}\right|_{x=a} = \left.\frac{dg}{du}\right|_{u=b} \times \left.\frac{df}{dx}\right|_{x=a},$$

or

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=b} \times \left.\frac{du}{dx}\right|_{x=a},$$

or simply

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

First Order Approximation

- When f is differentiable at x, the quotient

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is very close to $y'(x)$, provided that Δx is taken to be small. See the diagram. Hence we have

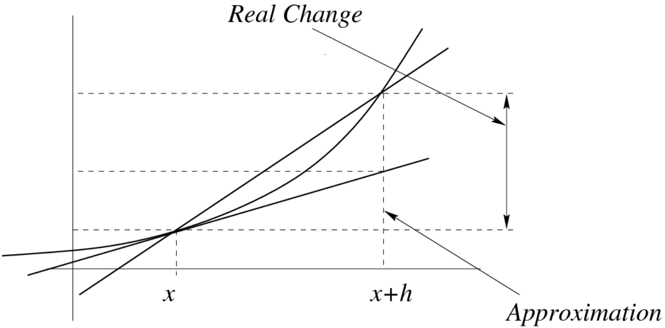
$$\frac{\Delta f}{\Delta x} \approx \frac{dy}{dx} = f'(x).$$

In other words, we have

$$\Delta f = f(x+\Delta x) - f(x) \approx \frac{dy}{dx}\Delta x = f'(x)\Delta x,$$

i.e., the change in f due to a small change Δx at x can be approximated by $f'(x)\Delta x$.

$$\Delta f \approx f'(x)\Delta x,$$



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- In particular, when $\Delta x = 1$, we have

$$\Delta f \approx f'(x),$$

- That is, the $f'(x)$ approximates the change of $f(x)$ when x is increased by one unit.

First Order Approximation

- **Example** Suppose $y = f(x) = x^2$. Find an approximate change of $f(x)$ when x is increased from 2 to 2.5.
- We have $f'(x) = 2x$. And $f'(2) = 2(2) = 4$. Hence

$$\Delta y = y(2 + 0.5) - y(2) \approx y'(2)\,(0.5) = 4\,(0.5) = 2.$$

  Note that the real change of y can be computed directly by $y(2.5) - y(2) = 2.25$. The approximation will become more accurate if we involve changes much smaller than 0.5.
- **Exercise** Repeat the above example when x is increased from 2 to 2.005. How accurate is it?
- **Exercise** Without using the calculus find an approximate value of $3.98^{1/2}$.

More Rules for Differentiation

- **Theorem** Suppose f is differentiable at a then f must also be continuous at a.
- When a function is differentiable at x, i.e., $f'(x)$ exists, it means that the curve of f has a tangent at x. For it is not difficult to see that f must be *nice* there. That is, f is continuous at x.
- **Proof** We need to show $\lim_{x\to a} f(x) = f(a)$. Consider

$$\begin{aligned}\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)\\ &= f'(a)\cdot 0 = 0.\end{aligned}$$

- So $\lim_{x\to a} f(x) = f(a)$.

Logarithmic functions

- Recall that the natural logarithmic function $\log x$ is defined to be the inverse function of the exponential function $y = e^x$. That is $x = \ln(e^x)$ and $x = e^{\ln x}$.
- **Theorem**. We have, for any $x > 0$,

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

$$\begin{aligned}
\frac{\ln(x+h) - \ln x}{h} &= \frac{1}{h}\ln\left(1 + \frac{h}{x}\right) \\
&= \ln\left(1 + \frac{h}{x}\right)^{1/h} \\
&= \ln\left(1 + \frac{1/x}{k}\right)^{k} \\
&\to \ln(e^{1/x}) \\
&= \frac{1}{x}
\end{aligned}$$

as $k \to +\infty$ (equivalent to $h \to 0$).

Different bases

- **Theorem** Let a be any positive real number. Then

$$\frac{d}{dx}\log_a x = \frac{1}{\ln a}\frac{1}{x}.$$

  Similarly, if u is a function of x, then

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}.$$

- **Example** Find the derivative of $y = \log_3(x^2+1)$

$$\frac{dy}{dx} = \frac{2x}{(x^2+1)\ln 3}$$

Implicit Differentiation

We have learned to find derivatives of functions in the form $y = f(x)$, i.e., y can be expressed as a function of x only. However, this is not always the case:

$$xe^y + ye^x = y.$$

if there is a change of x by Δx then there must be a corresponding change in y by a certain amount Δy say, in order the keep the equality. So how can we find $\frac{dy}{dx}$? We illustrate the method called *implicit differentiation* by the following example.

Inverse Sine function

(pp. 209-210) Here is another application of implicit differentiation.
Consider $y = \arcsin x$ on the domain $[-1, 1]$ has range $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. On the other hand $\sin y = x$. Differentiating this equation on both sides yields

$$1 = \frac{dx}{dx} = \frac{d}{dx} \sin y = \cos y \frac{dy}{dx}.$$

Notice that we have $1 - x^2 = \cos^2 y$. So

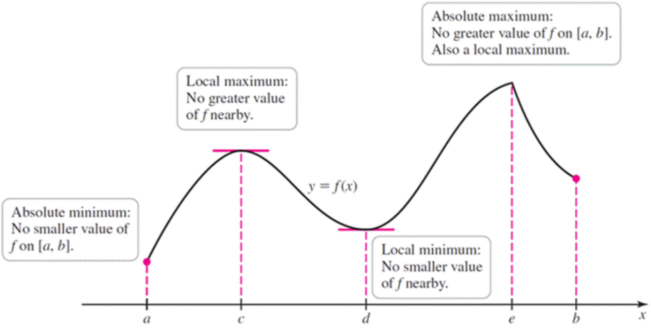
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

on $(-1, 1)$. Note that we have chosen the positive branch of $\pm\sqrt{1 - x^2}$ since $\cos y \geq 0$ on $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
Note that $y' \to +\infty$ as $x \to \pm 1$.

Maximum/Minimum

We see the drawing (p. 233) below that

- At some local maximum/minimum, $f'(x) = 0$.
- $f(x)$ may fail to have derivative at certain local maximum/minimum, such as the point c where $f'(c)$ fails to exist.
- In a finite interval $[a, b]$, f may have global maximum/minimum.



At extrema

- **Definition** We call $x = a$ a *critical point* of f if $f'(a) = 0$.
- If *f has a maximum or a minimum at a*, then $f'(a) = 0$ is a critical point.
- The converse is **not necessarily true**.
  - That is, at a *critical point a* ($f'(a) = 0$) may **not** represent $f(a)$ has either a maximum or minimum there.
  - **Example** $f(x) = x^3 + 2$ has $f'(0) = 0$ but $f(0)$ is neither a maximum nor a minimum.
  - **Example** $f(x) = x^4$ has $f'(0) = 0$ and $f(0)$ is a maximum
- That is, knowing $f'(a) = 0$ is insufficient to decide if $f(a)$ is an extrema.

First order approximation

The first order approximation formula can be used to analyse the local behaviour of f. So suppose

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Then we have

$$f'(a) \approx \frac{f(a+h)-f(a)}{h}$$

when h is small. That is

$$f(a+h)-f(a) \approx h\,f'(a) = \begin{cases} > 0, & \text{if } f'(a) > 0; \\ < 0, & \text{if } f'(a) < 0 \end{cases}$$

when $h > 0$ is small. Since h is a positive quantity so the sign of $f(a+h)-f(a)$ depends on the sign of $f'(a)$. Therefore f is increasing around a if $f'(a) > 0$ and f is decreasing around a if $f'(a) < 0$.

First order approximation (cont.)

More precisely,

$$f(a+h) - f(a) = h\,f'(a) + \epsilon(h)$$

where $\epsilon(h)$ denote an error term that is much smaller than h and $\epsilon(h) \to 0$ as $h \to 0$. So we may ignore this error in our consideration.

- If $f'(a) > 0$, and since $h > 0$ then

  $$f(a+h) - f(a) = h\,f'(a) + \epsilon(h) > 0$$

  holds as long as $\epsilon(h)$ remains small.
- If $f'(a) > 0$, and since $-h < 0$ then

  $$f(a-h) - f(a) = (-h)\,f'(a) + \epsilon(h) < 0$$

  holds as long as $\epsilon(h)$ remains small. This corresponds to the left limit. So we see that f is increasing around the a.
- the analysis for $f'(a) < 0$ is opposite, that f is decreasing around the a.

First order derivative test

We have seen that around a critical point a being a maximum/minimum, the derivative $f'(x)$ changes signs. That is,

- when $f(a)$ is a local maximum,

$$f'(x) \begin{cases} > 0, & \text{if } x < a; \\ = 0, & \text{if } x = a; \\ < 0, & \text{if } x > a. \end{cases}$$

$f' \downarrow$ that is $\nearrow \longrightarrow \searrow$

- when $f(a)$ is a local minimum,

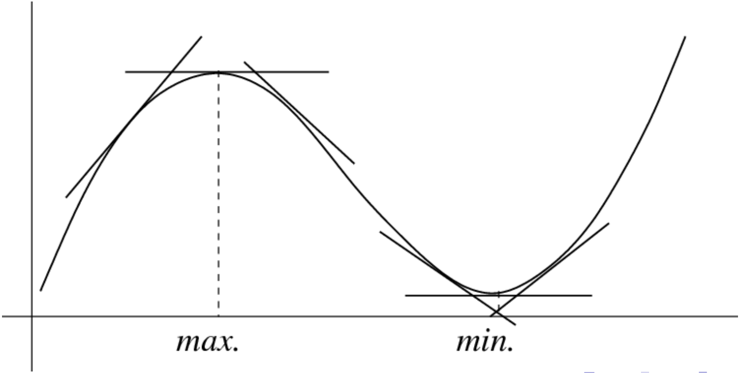
$$f'(x) \begin{cases} < 0, & \text{if } x < a; \\ = 0, & \text{if } x = a; \\ > 0, & \text{if } x > a. \end{cases}$$

$f' \uparrow$ that is $\searrow \longrightarrow \nearrow$

First order derivative test: Converse statements

It is not difficult to see that the converses also hold if $x = a$ is a critical point: $f'(a) = 0$. That is,

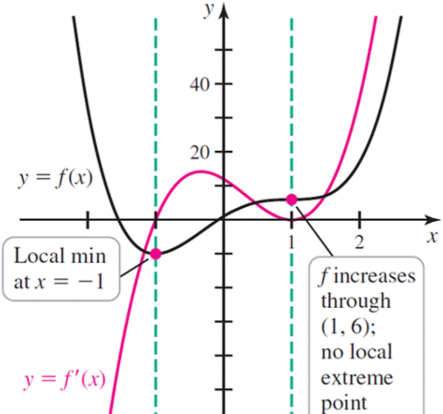
- if $f'(x)$ decreases from being positive to being negative, then $f(a)$ is a local maximum;
- if $f'(x)$ increases from being negative to being positive, then $f(a)$ is a local minimum



Example (p. 244 publisher)

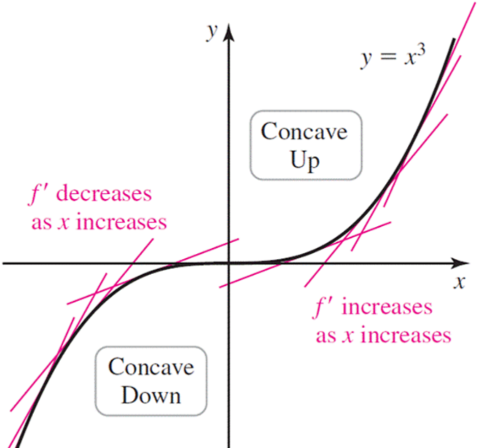
Let $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. Find the intervals of increase/decrease and any local extrema of f.

$$f'(x) = 12(x+1)(x-1)^2 = \begin{cases} < 0, & \text{if } x < -1; \\ > 0, & \text{if } -1 < x < 1; \\ > 0, & \text{if } x > 1. \end{cases}$$



Concavity I (publisher)

- **Definition** A differentiable function f is concave up over an interval I if f' is increasing over I.
- **Definition** A differentiable function f is concave down over an interval I if f' is decreasing over I.



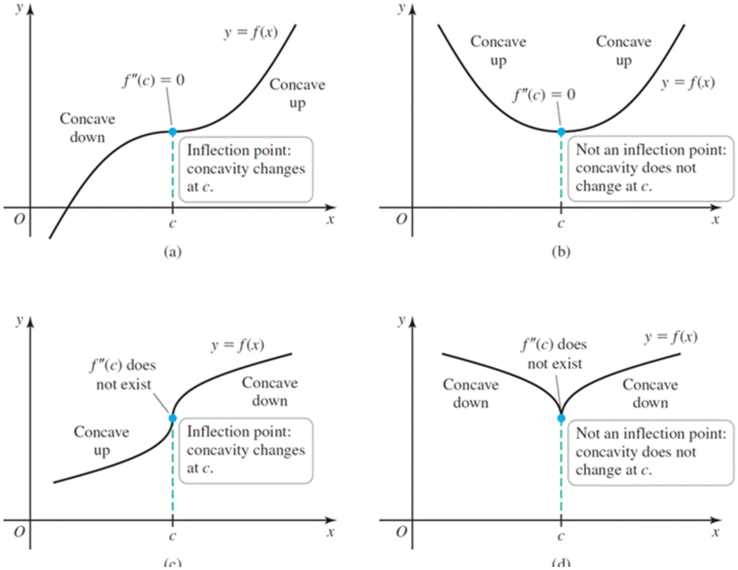
Concavity II

- **Theorem 4.6** Suppose that $f''(x)$ exists over an interval I.
  1. If $f''(x) > 0$, then f is concave up over I;
  2. If $f''(x) < 0$, then f is concave down over I.
- Although the signs of second derivative being positive/negative can determine the nature of concavity, i.e., , it is a necessary condition for concavity, it is , however, not sufficient.
- **Example** $f(x) = \dfrac{1}{x}$ is concave down over $(-\infty, 0)$ and concave up over $(0, \infty,)$.
- **Example** $f(x) = x^4$ is concave up over $(-\infty, \infty)$ and yet it has $f''(0) = 0$.

Inflection point I

- **Definition** A point c is called a point of inflection for a function $f(x)$ if there is a change of concavity.
- Suppose $f''(x) < 0$ for $x < c$ so concave down and $f''(x) > 0$ and so concave up for $x > c$, then there is a change of concavity at the inflection point $x = c$. We must have $f''(c) = 0$.
- Similarly, if $f''(x) > 0$ for $x < c$ so concave up and $f''(x) < 0$ and so concave down for $x > c$, then there is also a change of concavity at the inflection point $x = c$. Hence $f''(c) = 0$.
- **Example** $f(x) = x^3$ has an inflection point at $x = 0$ since there is a change of concavity and $f''(0) = 0$.
- **Example** $f(x) = x^4$ is concave up over $(-\infty, \infty)$ and yet it has $f''(0) = 0$.
- The next slide shows that f'' is undefined at a point of inflection.

Inflection point II (publisher)



Example (p. 248) I

Identify the intervals of concave up/down of
$f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$.
We have already computed

$$f'(x) = 12(x+1)(x-1)^2 = \begin{cases} < 0, & \text{if } x < -1; \\ > 0, & \text{if } -1 < x < 1; \\ > 0, & \text{if } x > 1. \end{cases}$$

$$f''(x) = 12(x-1)(3x+1) \begin{cases} > 0, & \text{if } x < -1/3 \text{ or } x > 1; \\ = 0, & \text{if } x = -1/3 \text{ or } x = 1; \\ < 0, & \text{if } -1/3 < x < 1; \end{cases}$$

We deduce that the critical points are $\{-1, 1\}$ and the inflection points are $\{-1/3, 1\}$.

Example (p. 248) II

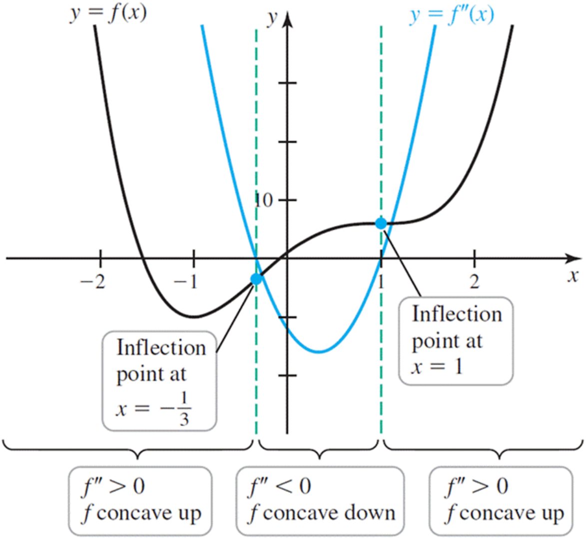


Figure: (Figure 4.31 (publisher))

Example II

Sketch the graph of $y = f(x) = x + \dfrac{4}{x+1}$.

- The easiest is to find the where f intersects with with the axes. Suppose $f(x) = 0$, i.e., $0 = x + \frac{4}{x+1}$ or $x^2 + x + 4 = 0$

  $$x = \frac{-1 \pm \sqrt{1^4 - 4 \cdot 1 \cdot 4}}{2},$$

  which has no solution since $1^2 - 16 < 0$. So f will never be zero, and so f will never intersect the x−axis. Besides, $f(0) = 4$.
- The next step is to consider $x \to +\infty$ and $x \to -\infty$. When x is large and positive $f(x) - x$ is approaching zero. i.e.,

  $$\lim_{x \to +\infty} \left(f(x) - x \right) = \lim_{x \to +\infty} \left(\frac{4}{x+1} \right) = 0.$$

  Similarly,

  $$\lim_{x \to -\infty} \left(f(x) - x \right) = \lim_{x \to -\infty} \left(\frac{4}{x+1} \right) = 0.$$

Example II (cont.)

- That is f is "essentially" like x when $x \to \pm\infty$.
  - In fact, since $\frac{4}{x+1} > 0$ as $x \to +\infty$, f approaches $y = x$ from above,
  - $\frac{4}{x+1} < 0$ when $x \to -\infty$, so f tends to $y = x$ from below.
- The third step is to note that $\dfrac{4}{x+1}$ is meaningless when $x = -1$. We have

$$\lim_{x \to (-1)+} f(x) = \lim_{x \to (-1)+} \left(x + \frac{4}{x+1}\right) = +\infty,$$

and

$$\lim_{x \to (-1)-} f(x) = \lim_{x \to (-1)-} \left(x + \frac{4}{x+1}\right) = -\infty,$$

Example II (cont.)

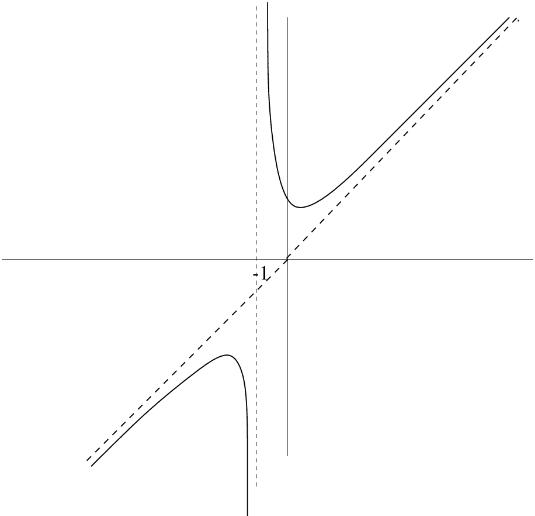
- The fourth step is to identify the critical points.

$$f'(x) = 1 - \frac{4}{(x+1)^2} = \frac{(x-1)(x+3)}{(x+1)^2}$$
$$= \begin{cases} > 0, & \text{if } x < -3; \\ < 0, & \text{if } -3 < x < -1; \\ < 0, & \text{if } -1 < x < 1 \\ > 0, & \text{if } x > 1. \end{cases}$$

We deduce f has a maximum at $x = -3$ and a minimum at $x = 1$. In fact, f is increasing on the intervals $x < -3$ and $x > 1$, and decreasing on the intervals $-3 < x < -1$ and $-1 < x < 1$.

Example II (cont.)

- Now the concavity $f''(x) = \dfrac{8}{(x+1)^3} = \begin{cases} > 0, & \text{if } x > -1; \\ < 0 & \text{if } x < -1. \end{cases}$



Extreme Value theorem

- **Theorem** A function $f(x)$ continuous on a closed interval $[a, b]$ attains its absolute maximum/minimum on $[a, b]$. That is, there exist c, d in $[a, b]$ such that

$$f(x) \geq f(c), \qquad f(x) \leq f(d) \qquad \text{for all } x \text{ in } [a, b].$$

- This result looks very trivial is in fact a deep result in elementary mathematical analysis. It is proved vigorously in chapter 5 (Theorem 5.3) of my supplementary notes on Mathematical Analysis course found in my web site of this course.
- What we will do in the following sides is to show the Extreme Value theorem does not hold when any one of the hypotheses fails to hold.

Mean Value theorem

Theorem If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is a c in $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

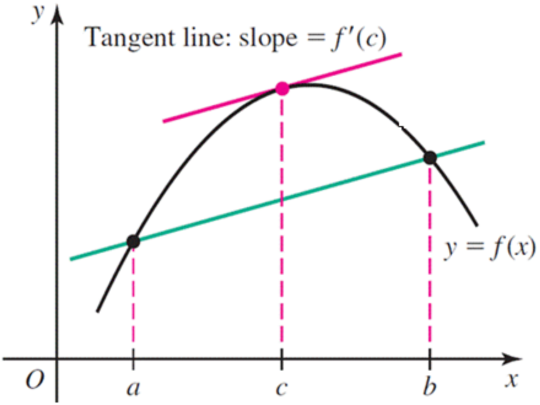


Figure: (Publisher Figure 4.68)

L'Hôpital's Rule 0/0 form

- **Theorem (p.290)** *Suppose* f *and* g *are differentiable on* (a, b) *and* c *lies in* (a, b) *such that* $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ *and* $g'(x) \neq 0$ $(x \neq c)$. *Then*

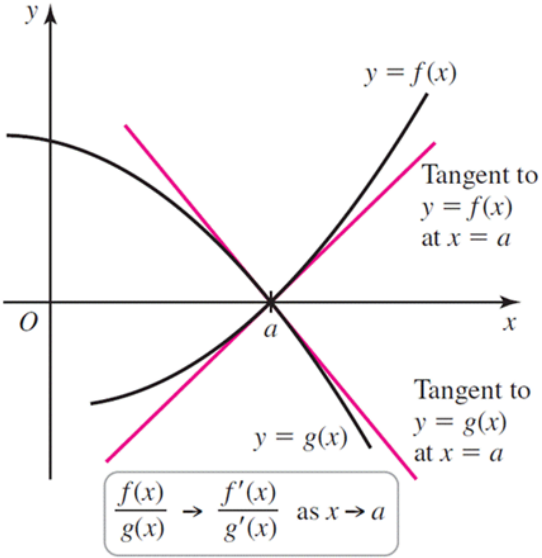
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the last limit exists, including $\pm\infty$.

- The above also holds for $x \to \pm\infty$ or $x \to a\pm$.
- **Example**

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} 1} = 1.$$

Geometric reason (p. 291)



"∞/∞" form

Theorem (p.293) *Suppose f and g are differentiable on (a, b) and c lies in (a, b) such that $\lim_{x\to c} f(x) = \pm\infty = \lim_{x\to c} g(x)$ and $g'(x) \neq 0$ $(x \neq c)$. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the last limit exists, including $\pm\infty$.

Remark The above also holds for $x \to \pm\infty$ or $x \to a\pm$.

- **Example** $\lim_{x\to\infty} \dfrac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$
- **Example** $\lim_{x\to\pi/2-} \dfrac{1 + \tan x}{\sec x}$

Fundamental theorem of calculus (Issac Newton)

Theorem 5.3 *Suppose that f is continuous over $[a, b]$ and that f has a primitive (antiderivative) $F(x)$ over $[a, b]$, that is, $F'(x) = f(x)$ on $[a, b]$. Then*

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Example Find $\int_2^3 x^2\,dx$ by Newton's Fundamental theorem of calculus.

Since $F(x) = \frac{1}{3}x^3$ is a primitive of $f(x) = x^2$, so

$$\int_2^3 x^2\,dx = F(3) - F(2) = \frac{1}{3}3^3 - \frac{1}{3}2^3 = \frac{19}{3}.$$

Examples of Fundamental theorem of calculus

- $$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}. \quad (F(x) = x^2/2)$$

- $$\int_{-1}^3 3x^2 - x + 6\,dx = x^3 - \frac{1}{2}x^2 + 6x\Big|_{-1}^3 = 48.$$

- $$\int_a^b 2x - 5x^2\,dx = x^2 - \frac{5}{3}x^3\Big|_a^b = (b^2 - a^2) + \frac{5}{3}(a^3 - b^3).$$