MATH1013 Calculus I

Introduction to Functions[1]

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Integration I (Chapter 4)

[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Primitives

Definition Let $F(x)$ and $f(x)$ be two given functions defined on an interval I. If

$$F'(x) = f(x), \qquad \text{holds for all } x \text{ in } I$$

then we say $F(x)$ is called a *primitive* or an anti-derivative of $f(x)$. We use the notation

$$F(x) = \int f(x)\, dx$$

to denoted that F is a primitive of f, and we call the process of finding a primitive F for f *indefinite integration* (or simply integration).

Examples

Remark We note that if $F(x)$ is a primitive of $f(x)$, then $F(x) + C$, where C is an arbitrary constant, is also a primitive of $f(x)$ since

$$(F(x) + C)' = F'(x) + 0 = f(x).$$

We call C a *constant of integration*. **Examples**

- $\int x \, dx = \frac{1}{2}x^2 + C$,
- $\int 2x \, dx = x^2 + C$,
- $\int x^2 dx = \frac{1}{3}x^3 + C$,
- $\int x^8 dx = \frac{1}{9}x^9 + C$.

Non-uniqueness

Recall from that Theorem 4.11 that if the derivatives of two functions F_1, F_2 agree on I: i.e., $F_1'(x) = F_2'(x)$, then F_1, F_2 differ by a constant. That is,

$$F_1(x) = F_2(x) + k, \text{ holds for all } \quad x \text{ in } I$$

for some constant k.

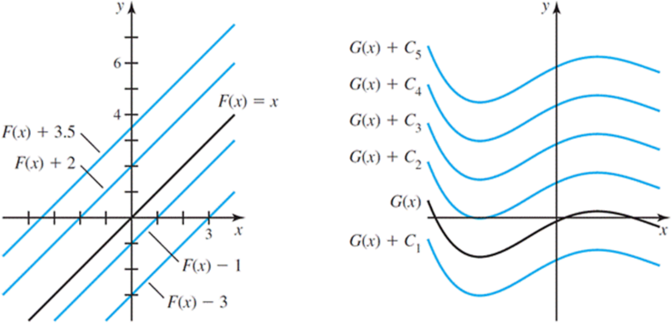


Figure: (Publisher Figure 4.78)

Primitives of monomials x^p

Theorem 4.17 *Let* $p \neq -1$ *be a real number. Then*

$$\int x^p \, dx = \frac{1}{p+1} x^{p+1} + C,$$

for some arbitrary constant C.

Examples

- $\int \sqrt[2]{x}^3 \, dx = \int x^{3/2} dx = \frac{x^{3/2+1}}{\frac{3}{2}+1} + C = \frac{2}{5} x^{5/2} + C$
- $\int \frac{1}{\sqrt[2]{x}^3} \, dx = \int x^{-3/2} \, dx = \frac{x^{-3/2+1}}{-3/2+1} + C = \frac{-2}{\sqrt{x}} + C$
- $\int \frac{1}{x^2} \, dx = \int x^{-2} \, dx = \frac{x^{-2+1}}{-2+1} + C = \frac{-1}{x} + C.$
- $\int \frac{1}{x^4} \, dx = \frac{x^{-4+1}}{-4+1} + C = \int x^{-4} \, dx = \frac{-1}{3x^3} + C.$

Exercises

1. $\displaystyle\int x^{15}\,dx$,
2. $\displaystyle\int x^{-12}\,dx$,
3. $\displaystyle\int x^{-12}\,dx$,
4. $\displaystyle\int 3x^{4}\,dx$,
5. $\displaystyle\int \sqrt{x}\,dx$,
6. $\displaystyle\int \sqrt[4]{x^5}\,dx$,
7. $\displaystyle\int \frac{1}{\sqrt[7]{x^8}}\,dx$
8. $\displaystyle\int \frac{4}{\sqrt[4]{x}}\,dx$.

Remark Differentiate your answers to verify whether they are correct.

Linear combinations

Since

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} \text{ and } \frac{d(kf)}{dx} = k\,\frac{df}{dx},$$

where k is a constant. So we deduce

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx,$$

and

$$\int k\,f(x)\,dx = k\,\int f(x)\,dx,$$

where k is a constant. We can easily generalize the above consideration to linear combination of $\{f_1, \cdots, f_n\}$

$$\int k_1\,f_1(x) + k_2\,f_2(x) + \cdots + k_n\,f_n(x)\,dx$$

$$= k_1 \int f_1(x)\,dx + k_2 \int f_2(x)\,dx + \cdots + k_n \int f_n(x)\,dx$$

where $\{k_1, \cdots, k_n\}$.

Finding primitive examples

1.

$$\int x^{3/2} + \frac{1}{x^{3/2}} + \frac{2}{x^3}\,dx = \int x^{3/2}dx + \int x^{-3/2}dx + 2\int x^{-3}dx$$

$$= \left(\frac{2}{5}x^{5/2} + c_1\right) - 2x^{-\frac{1}{2}} + c_2 + (-x^{-2} + c_3)$$

$$= \frac{2}{5}x^{5/2} - 2x^{-\frac{1}{2}} - x^{-2} + C.$$

2.

$$\int y^{1/2}(1+y)^2\,dy = \int y^{1/2}(1+2y+y^2)\,dy$$

$$= \int y^{1/2}dy + \int 2y^{3/2}dy + \int y^{5/2}dy = \frac{2}{3}y^{3/2} + \frac{4}{5}y^{5/2} + \frac{2}{7}y^{7/2} + C.$$

3.

$$\int \frac{1}{t^{1/2}}(1+t)^2\,dt = \int t^{-1/2}(1+2t+t^2)\,dt$$

$$= \int t^{-1/2}dt + 2\int t^{1/2}dt + \int t^{3/2}\,dt = 2t^{1/2} + \frac{4}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C.$$

Primitives of trigonometric functions

Indefinite Integrals of Trigonometric Functions

1. $\frac{d}{dx}(\sin ax) = a \cos ax \quad \rightarrow \quad \int \cos ax\, dx = \frac{1}{a} \sin ax + C$

2. $\frac{d}{dx}(\cos ax) = -a \sin ax \quad \rightarrow \quad \int \sin ax\, dx = -\frac{1}{a} \cos ax + C$

3. $\frac{d}{dx}(\tan ax) = a \sec^2 ax \quad \rightarrow \quad \int \sec^2 ax\, dx = \frac{1}{a} \tan ax + C$

4. $\frac{d}{dx}(\cot ax) = -a \csc^2 ax \quad \rightarrow \quad \int \csc^2 ax\, dx = -\frac{1}{a} \cot ax + C$

5. $\frac{d}{dx}(\sec ax) = a \sec ax \tan ax \quad \rightarrow \quad \int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax + C$

6. $\frac{d}{dx}(\csc ax) = -a \csc ax \cot ax \quad \rightarrow \quad \int \csc ax \cot ax\, dx = -\frac{1}{a} \csc ax + C$

Figure: (Publisher Table 4.9)

Primitives of various special functions

Other Definite Integrals

$$\frac{d}{dx}(e^{ax}) = ae^{ax} \rightarrow \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \rightarrow \int \frac{dx}{x} = \ln|x| + C \quad (\text{for } x \neq 0)$$

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2 - x^2}} \rightarrow \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (\text{for } |x| \leq |a|, a > 0)$$

$$\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{a}{x^2 + a^2} \rightarrow \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \quad (\text{for all } x \text{ and } a \neq 0)$$

$$\frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2 - a^2}} \rightarrow \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C \quad (\text{for } |x| \geq a > 0)$$

Figure: (Publisher Table 4.10)

Examples

- (p. 320) $\int (\sin 2y + \cos 3y)\, dy$
- (p. 315) $\int (\sec^2 3x + \cos \frac{x}{2})\, dx$
- (p. 316) $\int (e^{-10x} + e^{x/10})\, dx$
- (p. 316) $\int \frac{4}{\sqrt{9 - x^2}}\, dx$,
- (p. 316) $\int \frac{1}{16t^2 + 1}\, dt$.

Initial value problems

The simplest differential equation of first order is of the form

$$f'(x) = G(x), \qquad \text{where } G(x) \text{ is a given function;}$$
$$f(a) = b, \qquad \text{where } a, b \text{ are given initial condition.}$$

The $f'(x) = G(x)$ which is called a first order differential equation, together with the initial value condition is called an initial value problem (IVP).

An example of IVP

Example (p. 317) Solve the IVP

$$f'(x) = x^2 - 2x,$$
$$f(1) = \frac{1}{3}.$$

So a simple integration yields

$$f(x) = \int (x^2 - 2x)\, dx = \frac{1}{3}x^3 - x^2 + C.$$

But $\frac{1}{3} = f(1) = \frac{1}{3} \cdot 1^3 - 1^2 + C$ implies that $C = 1$. So

$$f(x) = \frac{1}{3}x^3 - x^2 + 1.$$

Sketch of the last IVP

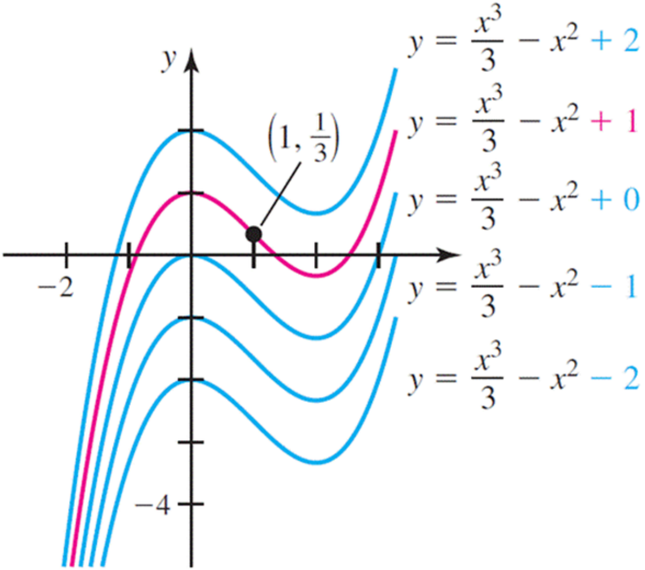


Figure: (Publisher Figure 4.87)

Example 6 (p. 318)

Race runner A begins at the point $s(0) = 0$ and runs with velocity $v(t) = 2t$. Runner B starts at the point $S(0) = 8$ and runs with velocity $V(t) = 2$. Find the positions of the runner for $t \geq 0$ and determine who is ahead at $t = 6$.
We have two IVP here. Namely,

$$\frac{ds}{dt} = v(t) = 2t, \qquad s(0) = 0.$$

with solution $s(t) = t^2$ and

$$\frac{dS}{dt} = V(t) = 2, \qquad S(0) = 8$$

with solution $S(t) = 2t + 8$. Therefore, the two runners meet when $s(t) = S(t)$, meaning that $t^2 - 2t - 8 = 0$. That is, $t = 4$.

Example 6 (p. 318) figure

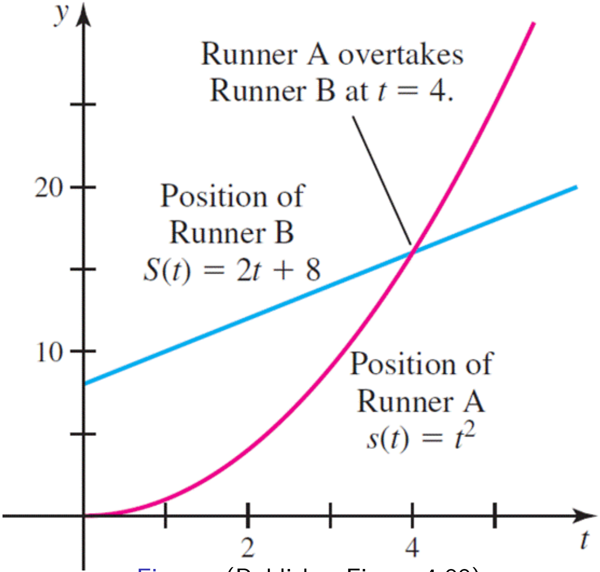


Figure: (Publisher Figure 4.88)

Example 7 (p. 319)

Neglecting air resistance, the motion of an object moving vertically near Earth's surface is determined by the acceleration due to gravity, which is approx. $9.8\ m/s^2$. Suppose a stone is thrown vertically upward at $t = 0$ with a velocity of $40\ m/s$ from the edge of a cliff that is 100 m above a river.

1. Find the velocity $v(t)$ of the object, for $t \geq 0$, and in particular, when the object starts to fall back down.
2. Find the position $s(t)$ of the object, for $t \geq 0$.
3. Find the maximum height of the object above the river.
4. With what speed does the object strike the river?

Example 7 (p. 319)

We measure the height function $s(t)$ from the sea level and adopt the upward direction to be our positive direction. Thus the initial height is $s(0) = 100$.

(1) The acceleration $\frac{dv}{dt}$ due to gravity pointing to the centre of Earth, which is therefore negative. In fact we have the IVP:

$$\frac{dv(t)}{dt} = v'(t) = -9.8, \qquad v(0) = 40$$

Solving the DE gives $v(t) = -9.8t + C$. Thus

$$40 = v(0) = -9.8(0) + C,$$

giving $C = 40$. Hence $v(t) = -9.8t + 40$.

The object starts to fall back down after it reached the maximum height, where $v(t) = 0$, that is, when $v(t) = -9.8t + 40 = 0$, giving $t \approx 4.1$ s.

Example 7 (p. 319)

(2) The height $s(t)$ satisfies the IVP

$$\frac{ds(t)}{dt} = v(t) = -9.8t + 40, \qquad s(0) = 100.$$

Solving the DE yields

$$s(t) = -4.9t^2 + 40t + C.$$

The initial condition $s(0) = 100$ implies that $100 = s(0) = C$. So

$$s(t) = -4.9t^2 + 40t + 100.$$

Example 7 (p. 319)

(3) As a result the maximum height is reached when the parabolic $s(t)$ is at its critical point:

$$0 = \frac{ds}{dt} = v(t) = -9.8t + 40,$$

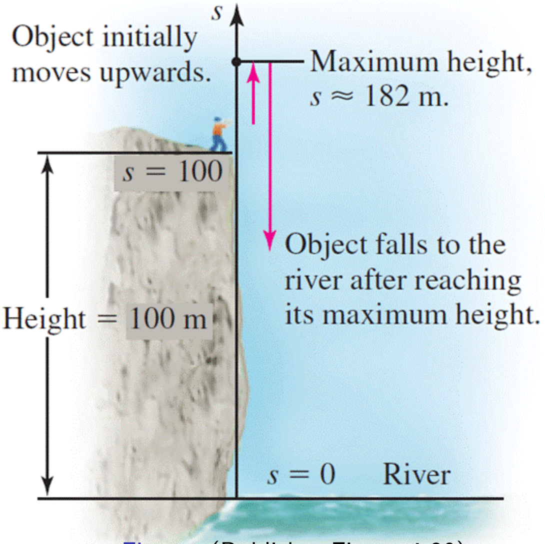
That is, when $t \approx 4.1$ s. Thus the maximum height is

$$s(4.1) \approx 182 \text{ m}.$$

(4) The object hits the sea when $s(t) = 0$. Solving the quadratic Eqn $s(t) = 0$ gives us two roots, namely $t \approx -2$ (which is to be discarded) and $t \approx 10.2$. So the velocity of the object when it strike the sea is given by

$$v(10.2) \approx -9.8(10.2) + 400 = -59.96 \approx -60.$$

Example 7 (p. 319)



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Example 7 (p. 319)

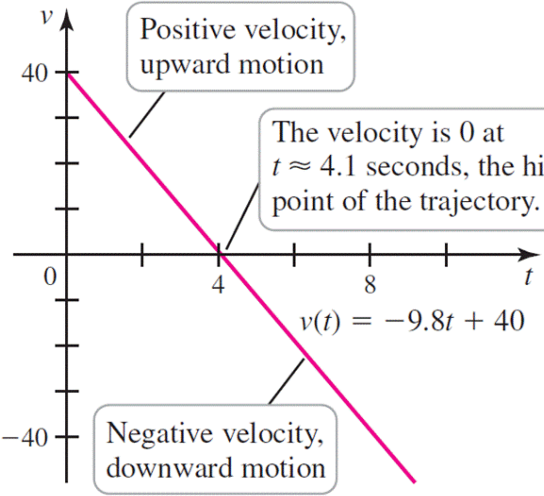


Figure: (Publisher Figure 4.90)

Example 7 (p. 319)

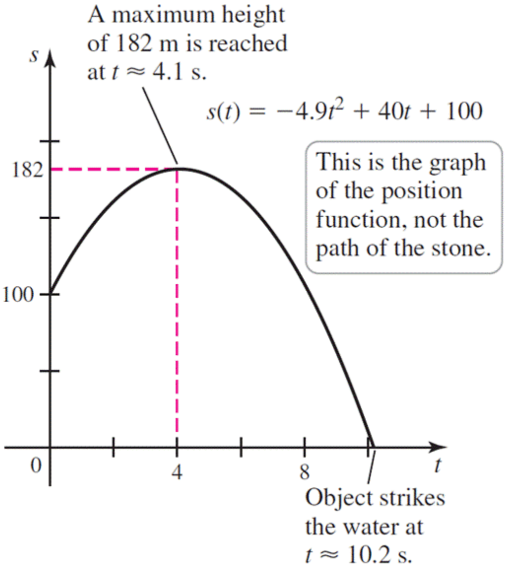
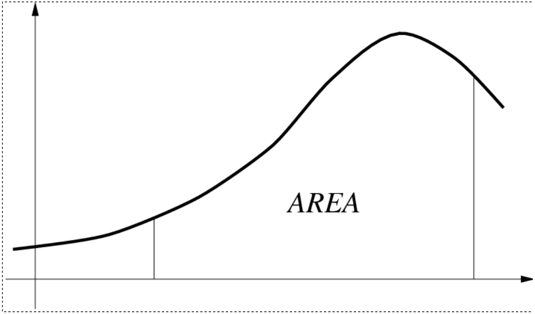


Figure: (Publisher Figure 4.91)

Area under curve

We shall consider continuous functions defined on a closed interval only. The aim is to develop a theory that can be used to find area of the region under the given function.



Example

We consider the problem of finding the area under the straight line $f(x) = x$ for the interval $0 \leq x \leq 1$.

We divide the interval $[0, 1]$ into five subintervals of equal width. By a *partition* of $[0, 1]$ with five points: $\{x_0, x_1, x_2, x_3, x_4, x_5\}$ of $[0, 1]$. So we have the subintervals:

$$
\begin{aligned}
[x_0,\ x_1] &= [0,\ 1/5] \\
[x_1,\ x_2] &= [1/5,\ 2/5] \\
[x_2,\ x_3] &= [2/5,\ 3/5] \\
[x_3,\ x_4] &= [3/5,\ 4/5] \\
[x_4,\ x_5] &= [4/5,\ 5/5]
\end{aligned}
$$

Upper sum

We consider the maximum values attained by f in each of the above intervals:

$$\begin{aligned} f(x_1) &= f(1/5) = 1/5, \\ f(x_2) &= f(2/5) = 2/5, \\ f(x_3) &= f(3/5) = 3/5, \\ f(x_4) &= f(4/5) = 4/5, \\ f(x_5) &= f(5/5) = 5/5 = 1. \end{aligned}$$

Let us sum the areas of the five rectangles each of which has the maximum height in $[x_{i-1}, x_i]$ and with base $1/n$. Thus we have

$$\begin{aligned} \overline{S_5} &= f\left(\frac{1}{5}\right)\frac{1}{5} + f\left(\frac{2}{5}\right)\frac{1}{5} + f\left(\frac{3}{5}\right)\frac{1}{5} + f\left(\frac{4}{5}\right)\frac{1}{5} + f\left(\frac{5}{5}\right)\frac{1}{5} \\ &= \frac{1}{25}(1+2+3+4+5) = \frac{15}{25} = \frac{3}{25}, \end{aligned}$$

is called an *upper sum* of f over $[0, 1]$ with respect to the above partition. The approximation $3/25$ is larger than the actual area.

Lower sum

We consider the minimum values attained by f in each of the above intervals:

$$f(x_0) = f(0/5) = 0/5,$$
$$f(x_1) = f(1/5) = 1/5,$$
$$f(x_2) = f(2/5) = 2/5,$$
$$f(x_3) = f(3/5) = 3/5,$$
$$f(x_4) = f(4/5) = 4/5.$$

Let us sum the areas of the five rectangles each of which has the minimum height in $[x_{i-1}, x_i]$ and with base $1/n$. Thus we have

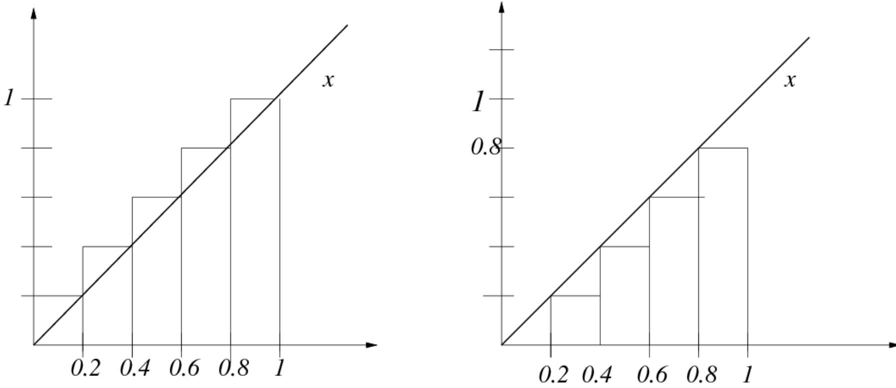
$$\underline{S}_5 = f\left(\frac{0}{5}\right)\frac{1}{5} + f\left(\frac{1}{5}\right)\frac{1}{5} + f\left(\frac{2}{5}\right)\frac{1}{5} + f\left(\frac{3}{5}\right)\frac{1}{5} + f\left(\frac{4}{5}\right)\frac{1}{5}$$
$$= \frac{1}{25}(0+1+2+3+4) = \frac{10}{25} = \frac{2}{5},$$

is called an *lower sum* of f over $[0, 1]$ with respect to the above partition. The approximation $2/25$ is smaller than the actual area.

Upper and Lower sums figues

Suppose the value of the area that we are to find is A. Then we clearly see from the figure below that the following inequalities must hold:

$$\frac{2}{5} = \underline{S_5} \leq A \leq \overline{S_5} = \frac{3}{5}.$$



Upper and lower sums: genearal case

It is also clear that the above argument that we divide $[0, 1]$ into five equal intervals is nothing special. Hence the same argument applies for any number of intervals. Hence we must have

$$\underline{S}_n \leq A \leq \overline{S}_n, \tag{2}$$

where n is any positive integer. We partition $[0, 1]$ into n equal subintervals:

$$\{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\}.$$

and the width of each of the subintervals of $1/n$.

Upper sum: genearal case

Hence the upper sum is

$$\begin{aligned}\overline{S}_n &= f\Big(\frac{1}{n}\Big)\,\frac{1}{n} + f\Big(\frac{2}{n}\Big)\,\frac{1}{n} + \cdots + f\Big(\frac{n-1}{n}\Big)\,\frac{1}{n} + f\Big(\frac{n}{n}\Big)\frac{1}{n} \\ &= \frac{1}{n}\,\frac{1}{n} + \frac{2}{n}\,\frac{1}{n} + \cdots + \frac{n-1}{n}\,\frac{1}{n} + \frac{n}{n}\,\frac{1}{n} \\ &= \frac{1}{n^2}\big(1+2+3+\cdots+(n-1)+n\big) \\ &= \frac{1}{n^2}\,\frac{n(n+1)}{2} \\ &= \frac{1}{2}\Big(1+\frac{1}{n}\Big).\end{aligned}$$

Note that we have used the fundamental formula that

$$1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}$$

Lower sum: genearal case

Hence the lower sum is

$$\begin{aligned}\underline{S}_n &= f\Big(\frac{0}{n}\Big)\,\frac{1}{n} + f\Big(\frac{1}{n}\Big)\,\frac{1}{n} + \cdots + f\Big(\frac{n-2}{n}\Big)\,\frac{1}{n} + f\Big(\frac{n-1}{n}\Big)\,\frac{1}{n}\\ &= \frac{1}{n}\,\frac{0}{n} + \frac{1}{n}\,\frac{1}{n} + \cdots + \frac{n-2}{n}\,\frac{1}{n} + \frac{n-1}{n}\,\frac{1}{n}\\ &= \frac{1}{n^2}\Big(0+1+2+\cdots+(n-2)+n-1\Big)\\ &= \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{1}{2}\Big(1-\frac{1}{n}\Big).\end{aligned}$$

We deduce that

$$\frac{1}{2}\Big(1-\frac{1}{n}\Big) \le A \le \frac{1}{2}\Big(1+\frac{1}{n}\Big), \qquad \text{for all } n.$$

Letting $n \to +\infty$ gives that $\frac{1}{2} \le A \le \frac{1}{2}$. That is $A = 1/2$ and

$$\lim_{n\to+\infty} \overline{S}_n = 1/2 = \lim_{n\to+\infty} \underline{S}_n.$$

Riemann sums

We consider a closed interval $[a, b]$ and let

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}, \quad \Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}, \ i = 1, \cdots, n$$

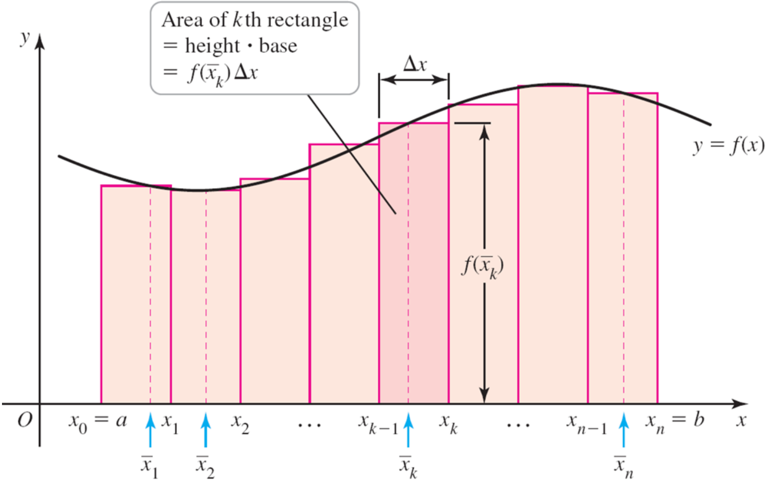
to be any points lying inside $[a, b]$. Let $f(x)$ be a continuous function defined on $[a, b]$. Then we define a Riemann sum of $f(x)$ over $[a, b]$ with respect to partition P to be the sum

$$f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n$$

where x_i^* is an arbitrary point lying in $[x_{i-1}, x_i]$, $i = 1, 2, \cdots, n$. Depending on the choices of x_i^*, we have

1. Left Riemann sum if $x_i^* = x_{i-1}$;
2. Right Riemann sum if $x_i^* = x_i$;
3. Mid-point Riemann sum if $x_i^* = (x_{i-1} + x_i)/2$.

Riemann sum figure



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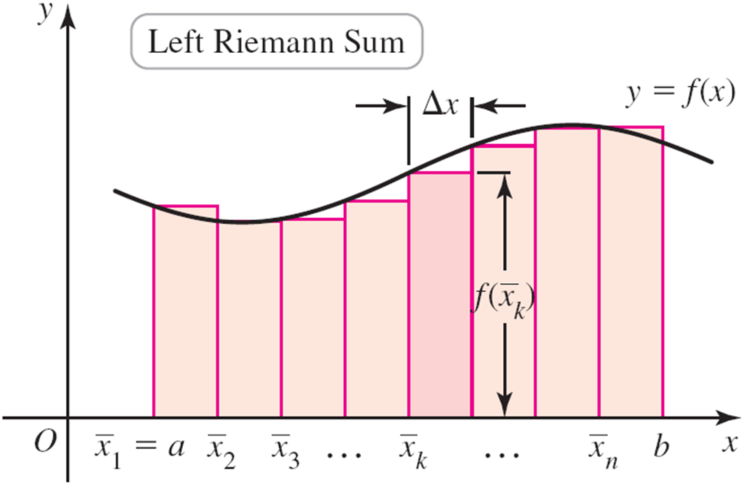
Left Riemann sum figure



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Right Riemann sum figure

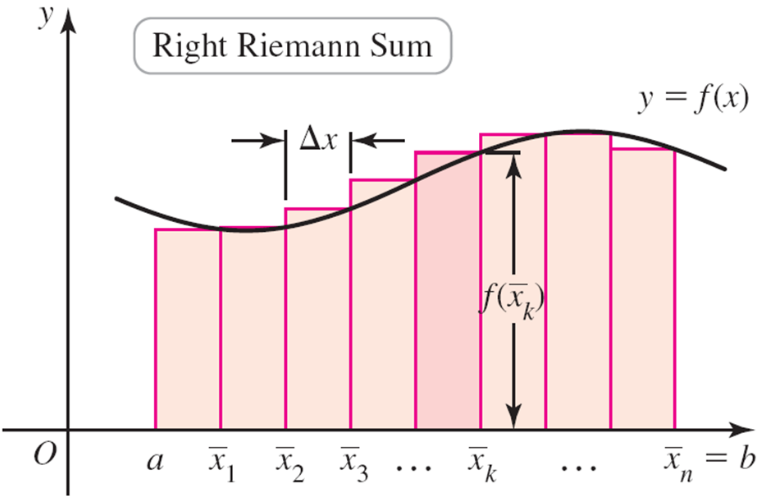


Figure: (Publisher Figure 5.10)

Mid-point Riemann sum figure

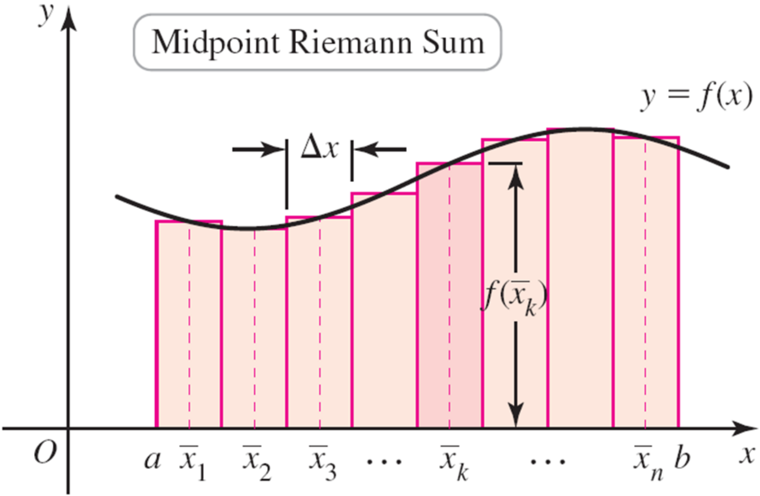


Figure: (Publisher Figure 5.11)

Riemann sum of Sine

We compute various Riemann sums under the sine curve from $x = 0$ to $x = \pi/2$ with six intervals. So we have

$$\Delta x = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}.$$

- Left Riemann sum gives

$$f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n \approx 0.863$$

- Right Riemann sum gives

$$f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n \approx 1.125$$

- Mid-point Riemann sum gives

$$f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n \approx 1.003$$

So we have $0.863 \leq 1.003 \leq 1.125$.

Partition of Riemann sum of Sine

Table 5.2

x	$f(x)$
0	1
0.5	3
1.0	4.5
1.5	5.5
2.0	6.0

Figure: (Publisher Figure 5.2)

Partition of Riemann sum of Sine

THEOREM 5.1 Sums of Positive Integers

Let n be a positive integer.

Sum of a constant c: $$\sum_{k=1}^{n} c = cn$$

Sum of the first n integers: $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Sum of squares of the first n integers: $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of cubes of the first n integers: $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Figure: (Publisher Theorem 5.1)

Example 5 (p. 334)

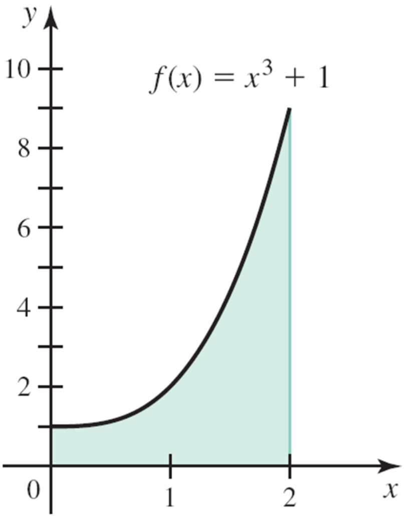


Figure: (Publisher Figure 5.15)

Example 5 (p. 334)

After choosing a partition that divides [0, 2] into 50 subintervals:

$$\Delta x_k = x_k - x_{k-1} = \frac{2-0}{50} = \frac{1}{25} = 0.04.$$

The Right Riemann Sum is given by

$$\sum_{k=1}^{50} f(x_k^*)\,\Delta x_k = \sum_{k=1}^{50} f(x_k)\,(x_k - x_{k-1})$$

$$= \sum_{k=1}^{50} f\left(\frac{k}{25}\right)(0.04) = \sum_{k=1}^{50} \left[\left(\frac{k}{25}\right)^3 + 1\right](0.04)$$

$$= \left[\frac{1}{25^3}\left(\frac{50 \cdot 51}{2}\right)^2 + 50\right](0.04) = 6.1616.$$

Example 5 (p. 334) II

The Left Riemann Sum is given by

$$\sum_{k=0}^{49} f(x_k^*)\,\Delta x_{k+1} = \sum_{k=0}^{49} f(x_k)\,(x_{k+1} - x_k)$$
$$= \sum_{k=0}^{49} f(\frac{k}{25})\,(0.04) = \sum_{k=0}^{49} \left[(\frac{k}{25})^3 + 1\right](0.04)$$
$$= \left[\frac{1}{25^3}(\frac{49 \cdot 50}{2})^2 + 50\right](0.04) = 5.8416.$$

Example 5 (p. 334) III

After choosing a partition that divides $[0, 2]$ into n subintervals:

$$\Delta x_k = x_k - x_{k-1} = \frac{2-0}{n} = \frac{2}{n}.$$

The Right Riemann Sum is given by

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k &= \sum_{k=1}^{n} f(x_k)\,\frac{2}{n} \\
&= \frac{2}{n}\sum_{k=1}^{n}\left[\left(\frac{2k}{n}\right)^3 + 1\right] = \frac{2}{n}\left(\frac{2^3}{n^3}\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 1\right) \\
&= \frac{2}{n}\left(\frac{2^3}{n^3}\cdot\frac{n^2(n+1)^2}{4} + n\right) \\
&= 2\left[2\left(1+\frac{1}{n}\right)^2 + 1\right] \to 6,
\end{aligned}$$

as $n \to \infty$.

Example 5 (p. 334) III

The Left Riemann Sum is given by

$$\sum_{k=0}^{n-1} f(x_k^*)\,\Delta x_{k+1} = \sum_{k=0}^{n-1} f(x_k)\cdot\frac{2}{n}$$

$$= \frac{2}{n}\sum_{k=0}^{n-1}\left[\left(\frac{2k}{n}\right)^3 + 1\right] = \frac{2}{n}\left(\frac{2^3}{n^3}\sum_{k=0}^{n-1} k^3 + \sum_{k=0}^{n-1} 1\right)$$

$$= \frac{2}{n}\left(\frac{2^3}{n^3}\cdot\frac{n^2(n-1)^2}{4} + n\right) = 2\left[2\left(1-\frac{1}{n}\right)^2 + 1\right] \to 6,$$

In fact, we have

$$4\left(1-\frac{1}{n}\right)^2 + 2 \le A \le 4\left(1+\frac{1}{n}\right)^2 + 2$$

to hold for every integer n. So $A = 6$.

Exercises

- (p. 338) Write the following sum in summation notation:

$$4 + 9 + 14 + \cdots + 44.$$

- (p. 339) Given that $\sum_{k=1}^{4} f(1+k)\cdot 1$ is a Riemann sum of a certain function f over an interval $[a, b]$ with a partition of n subdivisions. Identify the f, $[a, b]$ and n.
- Let $f(x) = x^2$ and let A be the area under f over the interval $[0, 1]$. Show that the following inequalities

$$\frac{1}{3}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right) \le A \le \frac{1}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right).$$

Then show that the area $A = \frac{1}{3}$.

Definite Integrals

DEFINITION Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x_k$ exists (over all partitions of $[a, b]$ and all choices of $\bar{x}_k$ on a partition). This limit is the **definite integral of *f* from *a* to *b***, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x_k.$$

Figure: (Publisher Figure p. 344)

Definite Integral notation

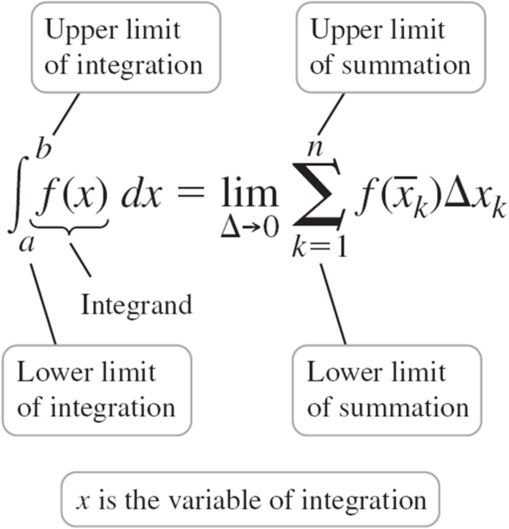


Figure: (Publisher Figure 5.21)

Integrable functions

Theorem 5.2 *Let f be a continuous function except on a finite number of discontinuities over the interval $[a, b]$. Then f is integrable on $[a, b]$. That is,*

$$\lim_{\delta x \to 0} \sum_{k=1}^{n} f(x_k^*) \, \Delta x_k = \int_a^b f(x) \, dx,$$

exists irrespective to the x_k^* and the partition $[x_{k-1}, x_k]$ chosen.

So

- Since $f(x) = x^2$ is continuous over $[0, 1]$ so it is integrable and $\int_0^1 x^2 \, dx = \frac{1}{2}$ according to a previous calculation.
- Since $f(x) = x^3$ is continuous over $[0, 1]$ so it is integrable and $\int_0^1 x^3 \, dx = \frac{1}{3}$ according to a previous calculation.

Piecewise continuous functions

The following function has a finite number of discontinuities and so is integrable. However, we note that part 2 of the area is negative:

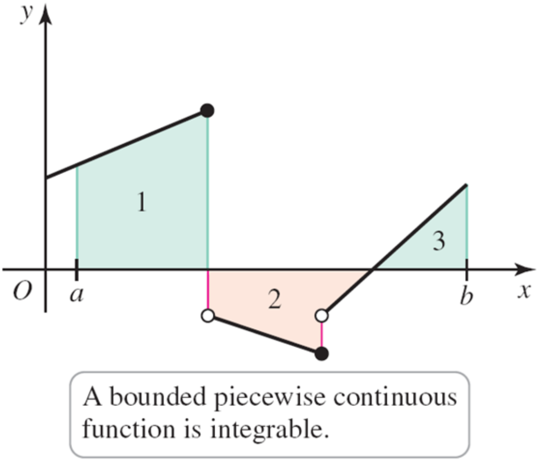


Figure: (Publisher Figure 5.23)

Negative area

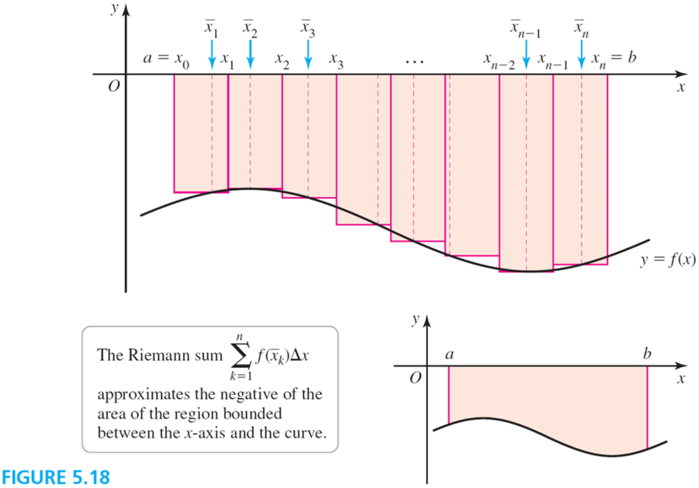


Figure: (Publisher Figure 5.18)

Negative area

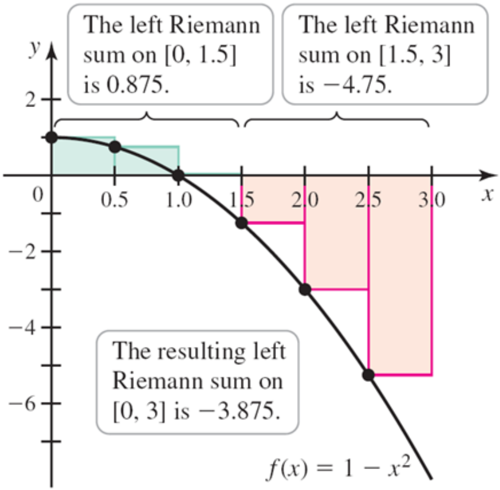


Figure: (Publisher Figure 5.17)

Net area

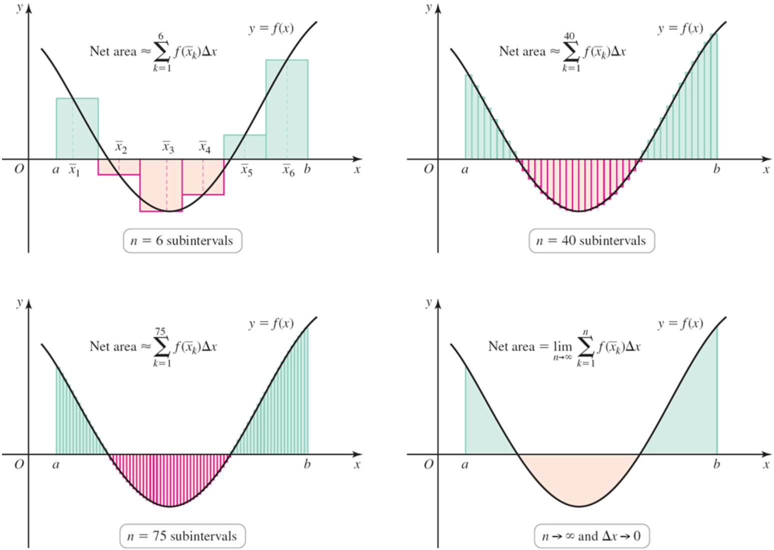
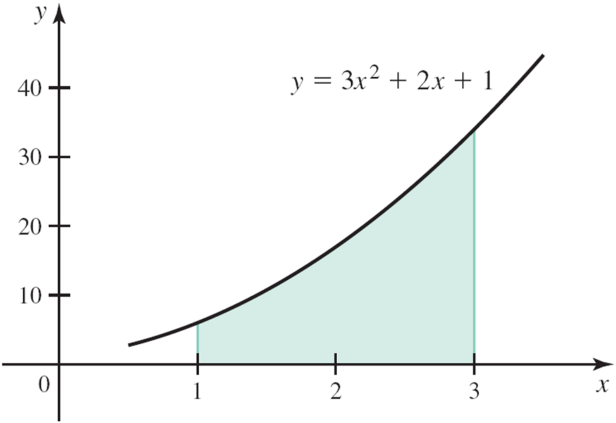


Figure: (Publisher Figure 5.20)

Recognizing integral



$$\lim_{\Delta\to 0}\sum_{k=1}^{n}(3\bar{x}_k^{\,2}+2\bar{x}_k+1)\Delta x_k=\int_1^3(3x^2+2x+1)\,dx$$

Figure: (Publisher Figure 5.24)

Computing net area

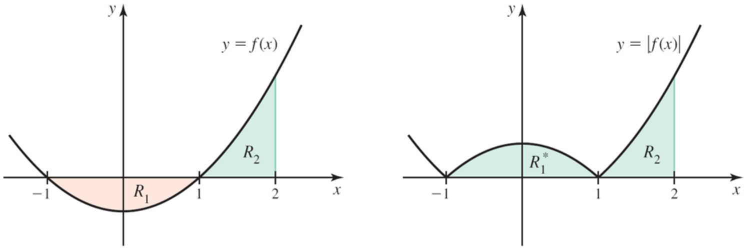


Figure: (Publisher Figure 5.31)

Exercises

1. Write down the right Riemann sum for $\int_0^2 \sqrt{4-x^2}\,dx$;
2. Interpret the sum $\lim_{\Delta\to 0} \sum_{k=1}^{n} \frac{3k}{n(1+\frac{3k}{n})}$ as a certain Riemann integral.
3. Let

$$f(x) = \begin{cases} 2x-2, & \text{if } x \le 2; \\ -x+4, & \text{if } x > 2. \end{cases}$$

Compute both the net area and actual area of $\int_0^5 f(t)\,dt$.

Hints to Exercises

The net area of the last example is given by

$$\int_0^2 (2x-2)\,dx + \int_2^5 (-x+4)\,dx = (x^2-2x)\Big|_0^2 + (-x^2/2+4x)\Big|_2^5$$
$$= (2^2-2\cdot 2) + \frac{1}{2}(2^2-5^2) + (20-8) = 0 + -\frac{1}{2}\cdot 21 + 12 = \frac{3}{2}.$$

The actual area is given by

$$\int_0^1 (2x-2)\,dx + \left|\int_1^2 (2x-2)\,dx\right| + \int_2^4 (-x+4)\,dx + \left|\int_4^5 (-x+4)\,dx\right|$$
$$= \left|(x^2-2x)\Big|_0^1\right| + (x^2-2x)\Big|_1^2 + (-x^2/2+4x)\Big|_2^4 + \left|(-x^2/2+4x)\Big|_4^5\right|$$
$$= |-1| + 1 + 2 + |-\frac{1}{2}| = \frac{9}{2}.$$

Properties of Definite Integral

Let f and g be integrable functions on an interval that contains a, b, and c.

1. $\int_a^a f(x)\,dx = 0$ Definition

2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition

3. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ For any constant c

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

6. The function $|f|$ is integrable and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

Figure: (Publisher Table 5.4)

Sum of integrals

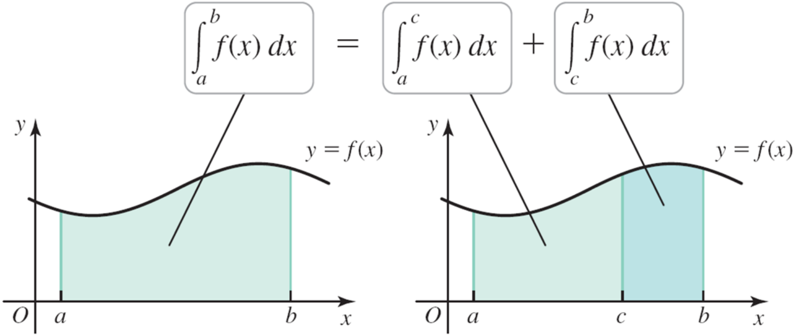


Figure: (Publisher Figure 5.29)