MATH1013 Calculus I

Introduction to Functions[1]

Edmund Y. M. Chiang

Department of Mathematics
Hong Kong University of Science & Technology

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Derivatives III (Chapter 4)

[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Horizontal tangents

We first investigate how we could extract useful information from f and $f'(x)$. Consider

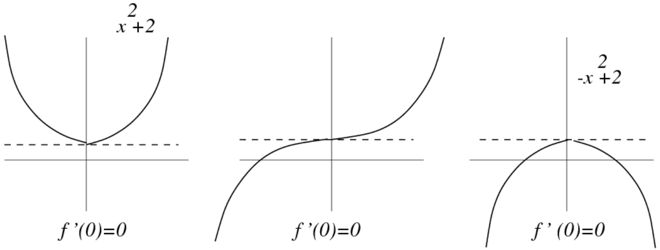
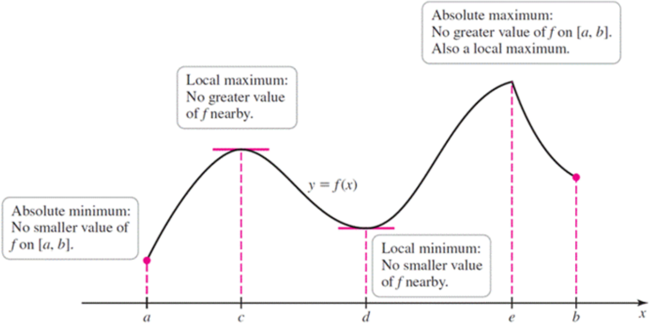


Figure: (Horizontal tangents)

Maximum/Minimum

We see the drawing (p. 233) below that

- At some local maximum/minimum, $f'(x) = 0$.
- $f(x)$ may fail to have derivative at certain local maximum/minimum, such as the point c where $f'(c)$ fails to exist.
- In a finite interval $[a, b]$, f may have global maximum/minimum.



At extrema

- **Definition** We call $x = a$ a *critical point* of f if $f'(a) = 0$.
- If *f has a maximum or a minimum at a*, then $f'(a) = 0$ is a critical point.
- The converse is **not necessarily true**.
  - That is, at a *critical point* a ($f'(a) = 0$) may **not** represent $f(a)$ has either a maximum or minimum there.
  - **Example** $f(x) = x^3 + 2$ has $f'(0) = 0$ but $f(0)$ is neither a maximum nor a minimum.
  - **Example** $f(x) = x^4$ has $f'(0) = 0$ and $f(0)$ is a maximum
- That is, knowing $f'(a) = 0$ is insufficient to decide if $f(a)$ is an extrema.

Critical point examples

Example Find the critical point(s) of

- $f(x) = ax^2 + bx + c$,
  Since $f'(x) = 2ax + b$. So the critical point appears at
  $$2ax + b = 0 \quad \text{or} \quad x = -b/2a.$$

- $f(x) = 4x^3 - 6$,
  Since $f'(x) = 12x^2$. So the critical point appears at $x = 0$.

- $f(x) = 3x^4 - 4x^3 - 12x^2 + 17$.
  We have
  $$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x+1)(x-2).$$
  Hence the critical points are at $x = -1, 0, 2$.

Critical point exercises

Determine the critical points of the following functions:

- $f(x) = 3x^4 - 8x^3 + 6x^2 + 2,$ $\big((0, 2), (1, 3)\big)$
- $f(t) = 2t^3 + 6t^2 + 6t + 5,$ $\big((-1, 3)\big),$
- $f(x) = (x - 1)^5,$ $\big((1, 0)\big),$
- $f(x) = (x^2 - 1)^5,$ $\big((-1, 0), (0, -1), (1, 0)\big)$
- $f(x) = (x^3 - 1)^4,$ $\big((0, 1), (1, 0)\big)$

Absolute extrema example

- **Example** (p. 236) Find the maximum/minimum of $f(x) = x^4 - 2x^3$ on $[-2, 2]$.
  We note that since this is a smooth function, so f' exists at all points in $[-2, 2]$.

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$$

- so that the critical points are at $\{0, \frac{3}{2}\}$.
- But $f(0) = 0$, $f(\frac{3}{2}) = -\frac{27}{16}$, $f(-2) = 32$, $f(2) = 0$ so that
  - $f(\frac{3}{2}) = -\frac{27}{16}$ is both a local and global minimum,
  - while $f(0) = 0$ is neither a max nor a min, and that
  - $f(-2) = 32$ is a global maximum on $[-2, 2]$.
- So f can attend an absolute maximum/minimum at end points of a finite interval rather than at the critial points.

Absolute extrema example (cont.)

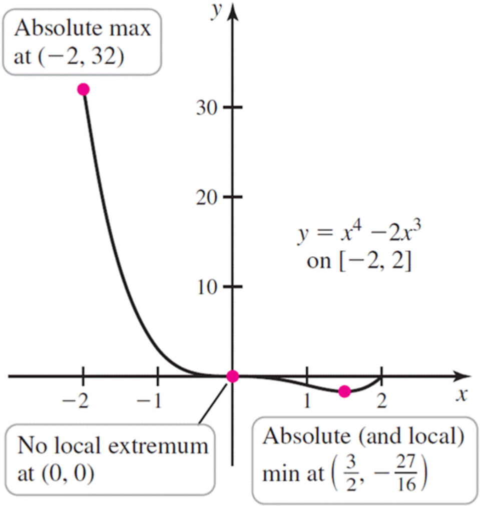


Figure: (Figure 4.11 (p. 236))

Behaviour at extrema

- **Example** Let $y = f(x) = x^2 - 4x + 4 = (x-2)^2$. Sketch the graphs of f and f' on the **same axis** and discuss any findings. The curve of the quadratic $f(x) = (x-2)^2$ has a minimum at $x = 2$. But

$$f'(x) = 2(x-2) = 2x - 4.$$

  is a straight line with gradient 2. It equals to zero when $x = 2$.
- Suppose we don't know in advance that $x = 2$ is a minimum of f, then how do we find out this from f' what happens to f at $x = 2$?

Comparing two graphs

To answer this question, let us plot f and f' against x on the same coordinate axis in the following way:

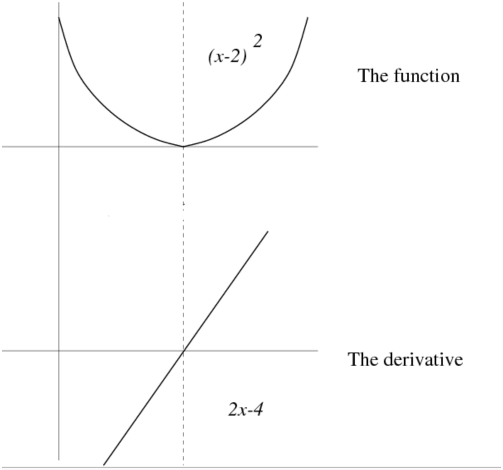


Figure: (Horizontal tangents)

Close up analysis of signs

- We note from the above figure that when f is decreasing in the region $x < 2$, $f'(x) = 2x - 4 < 0$ for $x < 2$. When f is increasing for the region $x > 2$, since $f'(x) = 2x - 4 > 0$ for $x > 2$. And at the critical point, i.e., $x = 2$, f reaches its minimum.
- We summarize the fundings below. Around the local minimum[2] of $f(x)$ at $x = 2$, the behaviour of $f'(x)$ is

$$f'(x) = \begin{cases} = 2x - 4 < 0, & \text{if } x < 2 \\ = 0, & \text{if } x = 2 \\ = 2x - 4 > 0, & \text{if } x > 2. \end{cases}$$

for x close to $x = 2$.

[2] It's a local minimum since the above analysis holds good only around the critical point in general even though it works globally for this particular example under discussion.

Close up analysis of signs: general quadratic

For $y = f(x) = ax^2 + bx + c$ $(a < 0)$, we have the critical point at $f'(x) = 2ax + b = 0$, i.e., $x = -b/(2a)$.
The behaviour of $f'(x)$ is

$$y = f'(x) = \begin{cases} 2ax + b > 0, & \text{if } x < -b/(2a) \\ 0, & \text{if } x = -b/(2a) \\ 2ax + b < 0, & \text{if } x > -b/(2a). \end{cases}$$

for x close to $x = -b/(2a)$.

Observation When $f(x)$ is quadratic, then around a critical point:

- f' increases from negative to positive around the critical point being minimum,
- f' decreases from positive to negative around the critical point being maximum.

Comparing two graphs: general quadratic

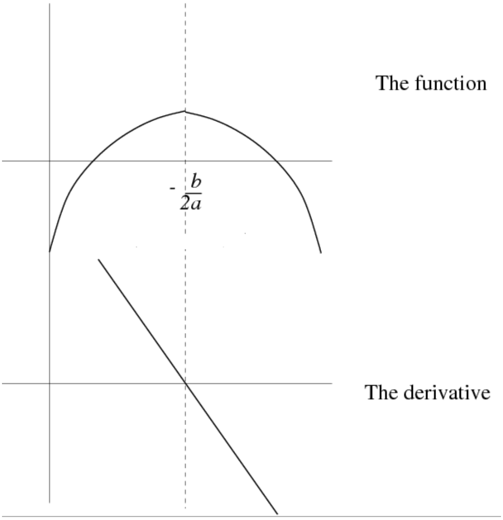


Figure: (Horizontal tangents)

Derivatives and behaviour

- We have seen that when a function $y = f(x)$ reaches a local maximum or local minimum, then $f'(x) = 0$.
- We have seen that if we only know that f has a critical point at a, then the natural of $f(a)$ being a max/min is inconclusive.
- What we want to show next is that a more detailed investigation on the behaviour of $f'(x)$ around the critical point would allow us to decide the nature of the point.
- In fact, we'll ask ourselves a more fundamental question about how does the $f'(x)$ affect the behaviour of $f(x)$.
- This is doen via the so-called first order derivative test. However, a completely vigorous argument will only be given later.

First order approximation

The first order approximation formula can be used to analyse the local behaviour of f. So suppose

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Then we have

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

when h is small. That is

$$f(a+h) - f(a) \approx h\,f'(a) = \begin{cases} > 0, & \text{if } f'(a) > 0; \\ < 0, & \text{if } f'(a) < 0 \end{cases}$$

when $h > 0$ is small. Since h is a positive quantity so the sign of $f(a+h) - f(a)$ depends on the sign of $f'(a)$. Therefore f is increasing <u>around</u> a if $f'(a) > 0$ and f is decreasing <u>around</u> a if $f'(a) < 0$.

First order approximation (cont.)

More precisely,

$$f(a+h) - f(a) = h\,f'(a) + \epsilon(h)$$

where $\epsilon(h)$ denote an error term that is much smaller than h and $\epsilon(h) \to 0$ as $h \to 0$. So we may ignore this error in our consideration.

- If $f'(a) > 0$, and since $h > 0$ then

$$f(a+h) - f(a) = h\,f'(a) + \epsilon(h) > 0$$

  holds as long as $\epsilon(h)$ remains small.
- If $f'(a) > 0$, and since $-h < 0$ then

$$f(a-h) - f(a) = (-h)\,f'(a) + \epsilon(h) < 0$$

  holds as long as $\epsilon(h)$ remains small. This corresponds to the left limit. So we see that f is increasing around the a.
- the analysis for $f'(a) < 0$ is opposite, that f is decreasing around the a.

Example

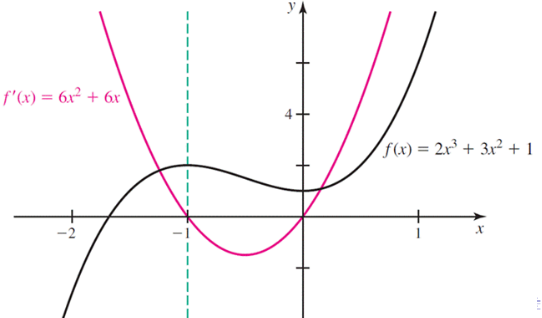
- **Example** Determine the regions on the x-axis where the function $y = f(x) = \dfrac{1}{x^2}$ is increasing and decreasing.

Example (p. 242)

Find the intervals of increase or decrease of $f(x) = 2x^3 + 3x^2 + 1$. Since

$$f'(x) = 6x^2 + 6x = 6x(x+1) = \begin{cases} > 0, & \text{if } x < -1; \\ < 0, & \text{if } -1 < x < 0; \\ > 0. & \text{if } x > 0 \end{cases}$$

so that $f(x)$ is increasing on $(-\infty, -1)$ and $(0, +\infty)$, and decreasing on $(-1, 0)$.

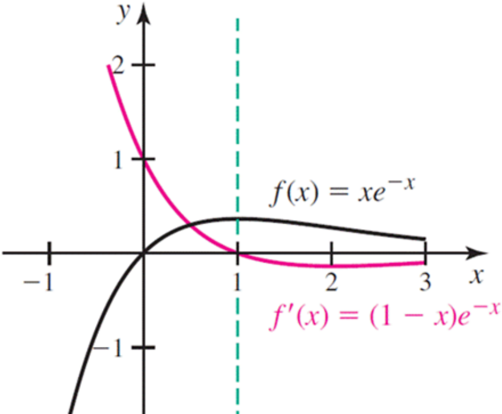


Example (p. 242)

Find the intervals of increase or decrease of $f(x) = xe^{-x}$

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} = \begin{cases} > 0, & \text{if } x < 1; \\ < 0, & \text{if } x > 1 \end{cases}$$

so that $f(x)$ is increasing on $(-\infty, 1)$ and decreasing on $(1, +\infty)$.



Exercises

Find the intervals of increase and decrease for the following functions

- $f(x) = x^2 - 4x + 5$, (f is increasing for $x > 2$; f is decreasing for $x < 2$)
- $f(x) = x^3 - 3x - 4$. (f is increasing for $x < -1, x > 1$; f is decreasing for $-1 < x < 1$)
- $f(x) = x^5 - 5x^4 + 100$ (f is increasing for $x < 0, x > 4$; $f(x)$ is decreasing for $0 < x < 4$)
- $f(x) = \frac{1}{x^3}$. (f is increasing for $x > 0$; f is decreasing for $x < 0$)

First order derivative test

We have seen that around a critical point a being a maximum/minimum, the derivative $f'(x)$ changes signs. That is,

- when $f(a)$ is a local maximum,

$$f'(x) \begin{cases} > 0, & \text{if } x < a; \\ = 0, & \text{if } x = a; \\ < 0, & \text{if } x > a. \end{cases}$$

$f' \downarrow$ that is $\nearrow \longrightarrow \searrow$

- when $f(a)$ is a local minimum,

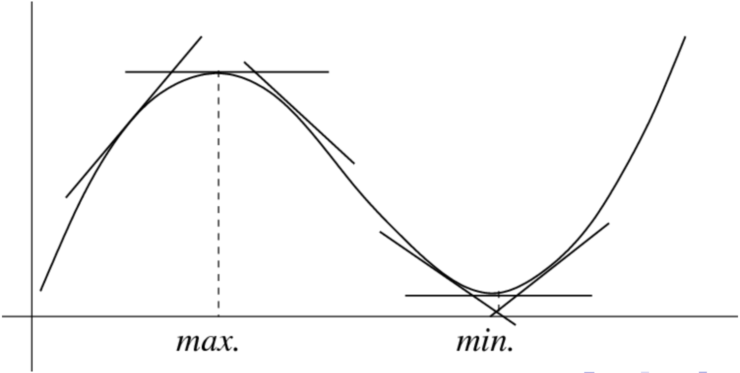
$$f'(x) \begin{cases} < 0, & \text{if } x < a; \\ = 0, & \text{if } x = a; \\ > 0, & \text{if } x > a. \end{cases}$$

$f' \uparrow$ that is $\searrow \longrightarrow \nearrow$

First order derivative test: Converse statements

It is not difficult to see that the converses also hold if $x = a$ is a critical point: $f'(a) = 0$. That is,

- if $f'(x)$ decreases from being positive to being negative, then $f(a)$ is a local maximum;
- if $f'(x)$ increases from being negative to being positive, then $f(a)$ is a local minimum



Example

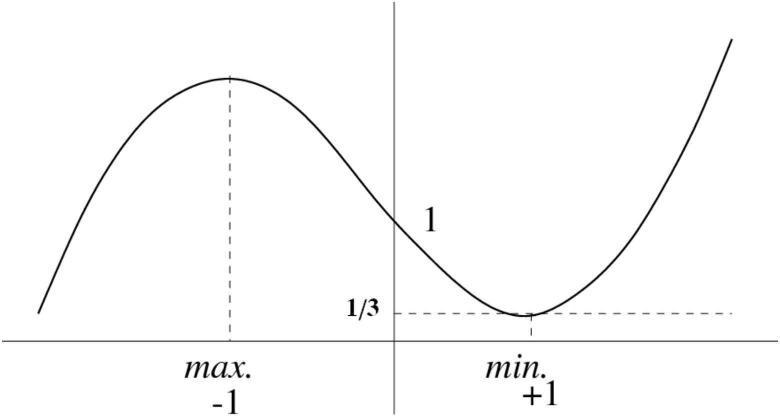
Question Find the intervals of increase/decrease and investigate the nature of the critical points $f(x) = \frac{1}{3}x^3 - x + 1$.
The critical points of f occurs when
$0 = f'(x) = x^2 - 1 = (x+1)(x-1)$. That is, when $x = \pm 1$. We have

$$f'(x) = (x+1)(x-1) = \begin{cases} > 0, & \text{if } x < -1; \\ = 0, & \text{if } x = -1; \\ < 0, & \text{if } -1 < x < 1; \\ = 0, & \text{if } x = 1; \\ > 0, & \text{if } x > 1. \end{cases}$$

Hence intervals of increase are $(-\infty, -1)$ and $(1, \infty)$, and interval of decrease is $(-1, 1)$.

Example (cont.)

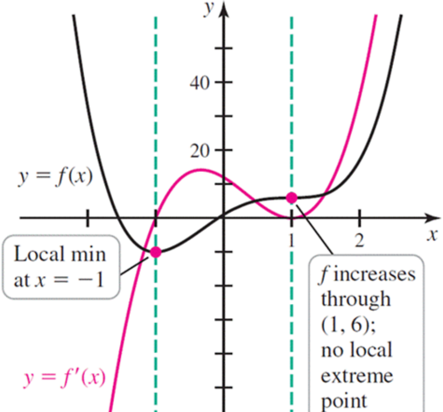
We can also apply the first order test to conclude that the critical point $x = -1$ is a local maximum and the other critical point $x = 1$ is a local minimum.



Example (p. 244 publisher)

Let $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. Find the intervals of increase/decrease and any local extrema of f.

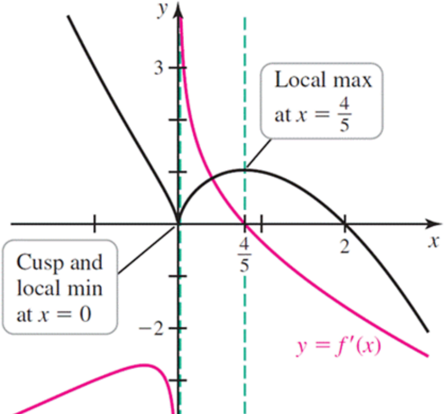
$$f'(x) = 12(x+1)(x-1)^2 = \begin{cases} < 0, & \text{if } x < -1; \\ > 0, & \text{if } -1 < x < 1; \\ > 0, & \text{if } x > 1. \end{cases}$$



Example (p. 245, publisher)

Let $f(x) = x^{2/3}(2 - x)$. Find the intervals of in increase/decrease and any local extrema of f.

$$f'(x) = \frac{4}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{4-5x}{3x^{1/3}} = \begin{cases} < 0, & \text{if } x < 0; \\ > 0, & \text{if } 0 < x < 4/5; \\ < 0, & \text{if } x > 4/5. \end{cases}$$



Example

Find the maximum(s) and minimum(s) of
$f(x) = x^3 - 2x^2 - 4x + 6$.

- critical points: $0 = f'(x) = 3x^2 - 4x - 4 = (3x + 2)(x - 2)$, which gives the *possible* $x = \{-\frac{3}{2}, 2\}$ for local maximums/minimums or neither.
- Determine critical points'nature:

$$f'(x) = (3x + 2)(x - 2) = \begin{cases} > 0, & \text{if } x < -2/3,\ x > 2; \\ = 0, & \text{if } x = -2/3 \text{ or } 2; \\ < 0, & \text{if } x = -2/3 < x < 2. \end{cases}$$

  So f is increasing on $(-\infty, -2/3)$ and $(2, \infty)$, and f is decreasing on $(-2/3, 2)$.
- We conclude that $f(-2/3) \approx 7.48$ is a local maximum and $f(2) = -2$ is a local minimum. Note also that $f(0) = 6$.

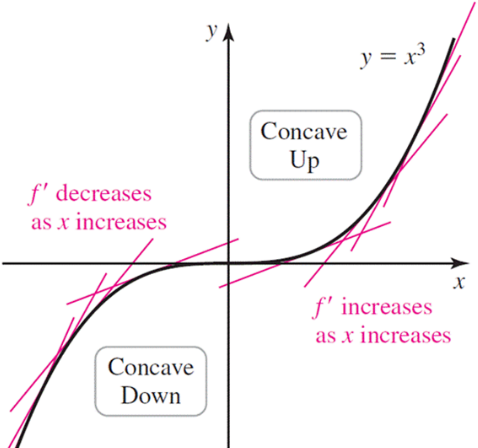
Nature of critical point exercises

Determine the nature of the critical points of the following functions:

- $f(x) = 3x^4 - 8x^3 + 6x^2 + 2$, (0, 2) relative minimum; (1, 3) neither
- $f(t) = 2t^3 + 6t^2 + 6t + 5$, (−1, 3) neither
- $f(x) = (x - 1)^5$, (1, 0) neither,
- $f(x) = (x^2 - 1)^5$, (−1, 0) neither; (0, −1) relative minimum; (1 , 0) neither
- $f(x) = (x^3 - 1)^4$, (0, 1) neither, (1, 0) relative minimum.

Concavity I (publisher)

- **Definition** A differentiable function f is concave up over an interval I if f' is increasing over I.
- **Definition** A differentiable function f is concave down over an interval I if f' is decreasing over I.



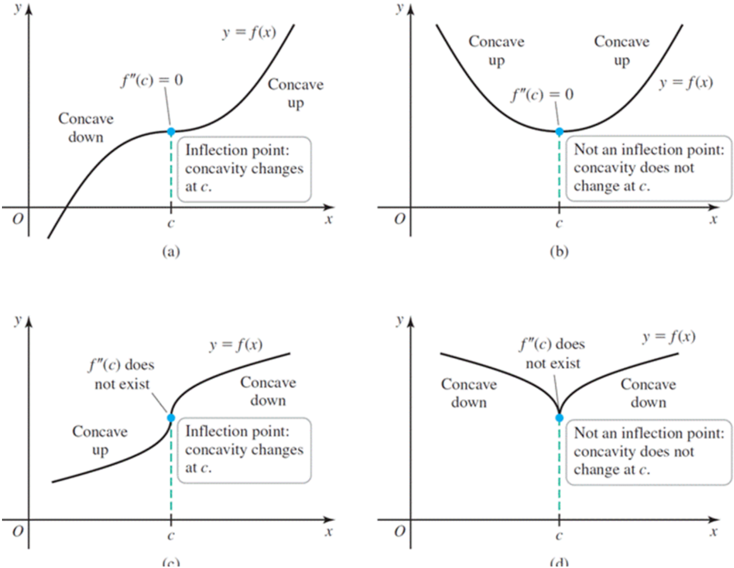
Concavity II

- **Theorem 4.6** Suppose that $f''(x)$ exists over an interval I.
  1. If $f''(x) > 0$, then f is concave up over I;
  2. If $f''(x) < 0$, then f is concave down over I.
- Although the signs of second derivative being positive/negative can determine the nature of concavity, i.e., , it is a sufficient condition for concavity, it is , however, not necessary.
- **Example** $f(x) = \dfrac{1}{x}$ is concave down over $(-\infty, 0)$ and concave up over $(0, \infty,)$.
- **Example** $f(x) = x^4$ is concave up over $(-\infty, \infty)$ and yet it has $f''(0) = 0$.

Inflection point I

- **Definition** A point c is called a point of inflection for a function $f(x)$ if there is a change of concavity or $f''(z)$ is undefined.
- Suppose $f''(x) < 0$ for $x < c$ so concave down and $f''(x) > 0$ and so concave up for $x > c$, then there is a change of concavity at the inflection point $x = c$. We must have $f''(c) = 0$.
- Similarly, if $f''(x) > 0$ for $x < c$ so concave up and $f''(x) < 0$ and so concave down for $x > c$, then there is also a change of concavity at the inflection point $x = c$. Hence $f''(c) = 0$.
- **Example** $f(x) = x^3$ has an inflection point at $x = 0$ since there is a change of concavity and $f''(0) = 0$.
- **Example** $f(x) = x^4$ is concave up over $(-\infty, \infty)$ and yet it has $f''(0) = 0$.
- The next slide shows that f'' is undefined at a point of inflection.

Inflection point II (publisher)



Example (p. 248) I

Identify the intervals of concave up/down of $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$.
We have already computed

$$f'(x) = 12(x+1)(x-1)^2 = \begin{cases} < 0, & \text{if } x < -1; \\ > 0, & \text{if } -1 < x < 1; \\ > 0, & \text{if } x > 1. \end{cases}$$

$$f''(x) = 12(x-1)(3x+1) \begin{cases} > 0, & \text{if } x < -1/3 \text{ or } x > 1; \\ = 0, & \text{if } x = -1/3 \text{ or } x = 1; \\ < 0, & \text{if } -1/3 < x < 1; \end{cases}$$

We deduce that the critical points are $\{-1, 1\}$ and the inflection points are $\{-1/3, 1\}$.

Example (p. 248) II

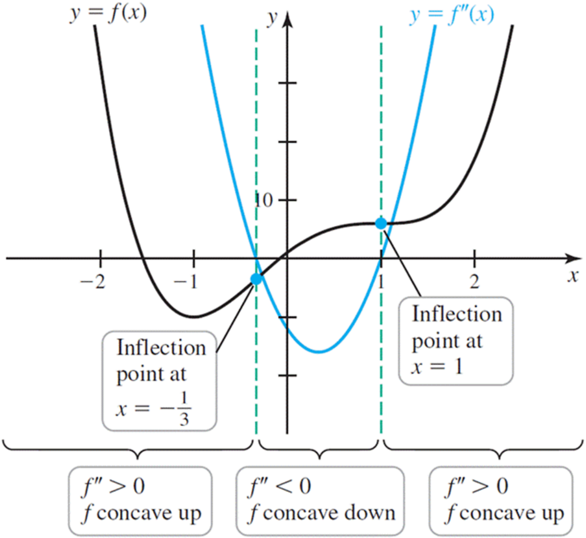
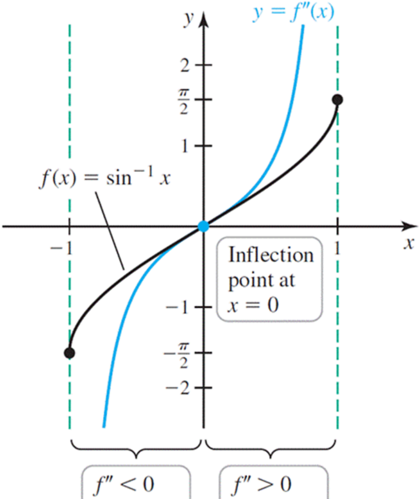


Figure: (Figure 4.31 (publisher))

Example (p. 248) III

Identify the intervals of concave up/down of $f(x) = \arcsin(x)$.

$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad f''(x) = \frac{x}{\sqrt{(1-x^2)^3}}.$$



Example I

Sketch the graph of $f(x) = 3x^4 - 4x^3 - 12x^2 + 17$.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x+1)(x-2)$$
$$= \begin{cases} < 0, & \text{if } x < -1,\ 0 < x < 2; \\ = 0, & \text{if } x = -1,\ 0,\ 2 \\ > 0, & \text{if } -1 < x < 0,\ x > 2. \end{cases}$$

$$f''(x) = 12(3x^2 - 2x - 2)$$
$$= 36\left(x - \frac{1+\sqrt{7}}{3}\right)\left(x - \frac{1-\sqrt{7}}{3}\right)$$
$$= \begin{cases} > 0, & \text{if } x < (1-\sqrt{7})/3, \text{ or } x > (1+\sqrt{7})/3 \\ = 0, & \text{if } x = (1-\sqrt{7})/3 \text{ or } (1+\sqrt{7})/3 \\ < 0, & \text{if } (1-\sqrt{7})/3 < x < (1+\sqrt{7})/3 \end{cases}$$

Inflection points $(1-\sqrt{7})/3 \approx -0.55$ and $(1+\sqrt{7})/3 \approx 1.27$.

Example II

Sketch the graph of $y = f(x) = x + \dfrac{4}{x+1}$.

- The easiest is to find the where f intersects with with the axes. Suppose $f(x) = 0$, i.e., $0 = x + \frac{4}{x+1}$ or $x^2 + x + 4 = 0$

$$x = \frac{-1 \pm \sqrt{1^4 - 4 \cdot 1 \cdot 4}}{2},$$

  which has no solution since $1^2 - 16 < 0$. So f will never be zero, and so f will never intersect the $x-$axis. Besides, $f(0) = 4$.
- The next step is to consider $x \to +\infty$ and $x \to -\infty$. When x is large and positive $f(x) - x$ is approaching zero. i.e.,

$$\lim_{x \to +\infty} \Big(f(x) - x \Big) = \lim_{x \to +\infty} \left(\frac{4}{x+1} \right) = 0.$$

  Similarly,

$$\lim_{x \to -\infty} \Big(f(x) - x \Big) = \lim_{x \to -\infty} \left(\frac{4}{x+1} \right) = 0.$$

Example II (cont.)

- That is f is "essentially" like x when $x \to \pm\infty$.
  - In fact, since $\frac{4}{x+1} > 0$ as $x \to +\infty$, f approaches $y = x$ from above,
  - $\frac{4}{x+1} < 0$ when $x \to -\infty$, so f tends to $y = x$ from below.
- The third step is to note that $\dfrac{4}{x+1}$ is meaningless when $x = -1$. We have

$$\lim_{x\to(-1)+} f(x) = \lim_{x\to(-1)+} \left(x + \frac{4}{x+1}\right) = +\infty,$$

and

$$\lim_{x\to(-1)-} f(x) = \lim_{x\to(-1)-} \left(x + \frac{4}{x+1}\right) = -\infty,$$

Example II (cont.)

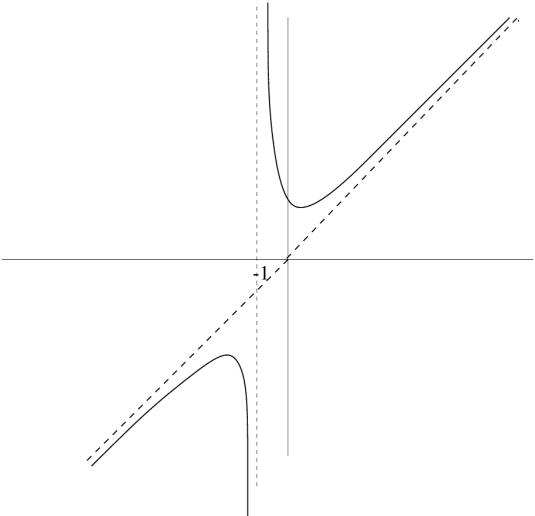
- The fourth step is to identify the critical points.

$$f'(x) = 1 - \frac{4}{(x+1)^2} = \frac{(x-1)(x+3)}{(x+1)^2}$$
$$= \begin{cases} > 0, & \text{if } x < -3; \\ < 0, & \text{if } -3 < x < -1; \\ < 0, & \text{if } -1 < x < 1 \\ > 0, & \text{if } x > 1. \end{cases}$$

We deduce f has a maximum at $x = -3$ and a minimum at $x = 1$. In fact, f is increasing on the intervals $x < -3$ and $x > 1$, and decreasing on the intervals $-3 < x < -1$ and $-1 < x < 1$.

Example II (cont.)

- Now the concavity $f''(x) = \dfrac{8}{(x+1)^3} = \begin{cases} > 0, & \text{if } x > -1; \\ < 0 & \text{if } x < -1. \end{cases}$



Example III (p. 258)

Sketch the curve of $f(x) = \dfrac{10x^3}{x^2-1}$

- Clearly $f(0) = 0$ and f is undefined on $x = \pm 1$.
- Asymptotes: $f(x) = 10x + \dfrac{10x}{x^2-1}$
- Derivatives:

$$f'(x) = \frac{10x^2(x^2-3)}{(x^2-1)^2} = \begin{cases} > 0, & \text{if } x > \sqrt{3} \text{ or } x < -\sqrt{3}; \\ = 0, & \text{if } x = \pm\sqrt{3}; \\ < 0, & \text{if } -\sqrt{3} < x < \sqrt{3}. \end{cases}$$

and

$$f''(x) = \frac{20x(x^2+3)}{(x^2-1)^3} = \begin{cases} < 0, & \text{if } x < -1, \text{ or } 0 < x < 1; \\ = 0, & \text{if } x = 0; \\ > 0, & \text{if } -1 < x < 0, \text{ or } x > 1. \end{cases}$$

So we see that $x = -\sqrt{3}$ is a local maximum and $x = \sqrt{3}$ is a local minimum. Moreover, there is a change of concavity, so $x = 0$ is a point of inflection.

Example (p. 258) (cont.)

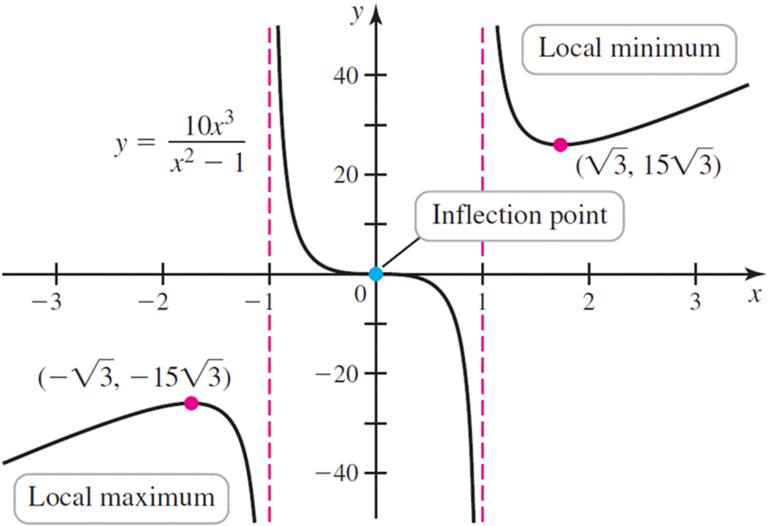


Figure: (Figure 4.45 (publisher))

Normal distribution (p. 260)

Sketch the curve of $f(x) = e^{-x^2}$.

- $f(0) = 1$, and $f(x) > 0$ for all x.
- $\lim_{x\to\pm\infty} f(x) = 0$ from above the $x-$axis.
- $$f'(x) = -2xe^{-x^2} = \begin{cases} < 0, & \text{if } x > 0; \\ = 0, & \text{if } x = 0 \\ > 0, & \text{if } x < 0, \end{cases}$$

  and that $x = 0$ is a local maximum by the first order derivative test.
- $$f''(x) = 2(2x^2-1)e^{-x^2} = \begin{cases} < 0, & \text{if } x < -1/\sqrt{2} \text{ or } x > 1/\sqrt{2}; \\ = 0, & \text{if } x = \pm 1/\sqrt{2} \\ > 0, & \text{if } -1/\sqrt{2} < x < 1/\sqrt{2}, \end{cases}$$

  so that $x = \pm 1/\sqrt{2}$ are points of inflection.

Normal distribution (p. 261) II

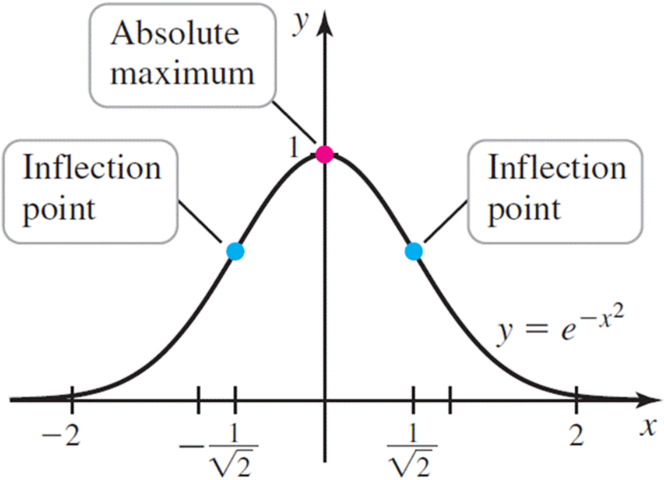
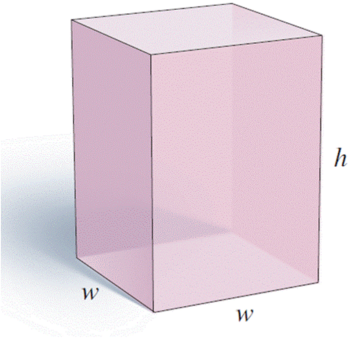


Figure: (Figure 4.47 (publisher))

Optimization I (p. 267 a)

Suppose an airline policy states that all baggage must be box-shapted with a sum of length, width and height not exceeding 64 in. What are the dimenstions and volume of a square-based box with the greatest volume under these condition?



Objective function: $V = w^2h$
Constraint: $2w + h = 64$

Optimization I (p. 267 b)

We want to maximize the volume of a rectangular box under a constraint. Let

$$V = w^2\, h, \qquad 2w + h = 64, \quad (0 \leq w \leq 32)$$

where w is the width and h is the height of the box. That is,

$$V = w^2\, h = w^2\,(64 - 2w) = 64w^2 - 2w^3.$$

Assuming V has a maximum, then we have

$$0 = V'(w) = 128\, w - 6\, w^2 = 2w(64 - 3w),$$

which holds only when $w = 0,\, 64/3 \approx 21.3$. These are the critical points. $V''(w) = 128 - 12w$ so that

$$V''\left(\frac{64}{3}\right) = 128 - 12\left(\frac{64}{3}\right) < 0.$$

This implies that $V\left(\frac{64}{3}\right) \approx 9,709$ is a local maximum. Since V is a smooth function,so we need check the end points:

$V(0) = 0,\qquad V(32) = 0.$ So $V\left(\frac{64}{3}\right) \approx 9,709$ is the absolute

Optimization I (p. 267 c)

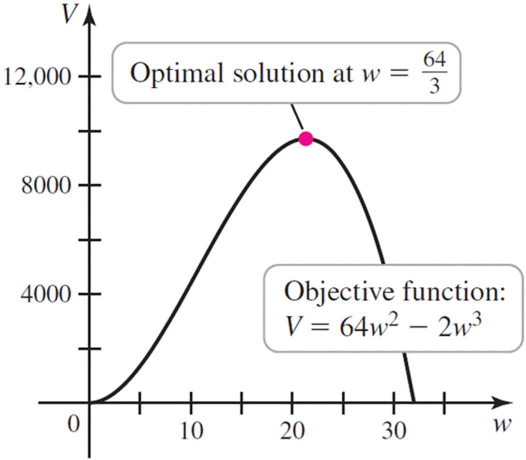
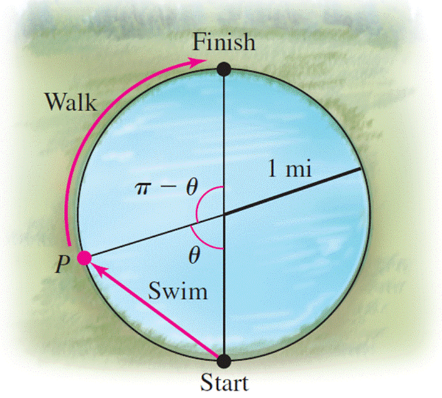


Figure: (Figure 4.54 (publisher))

Optimization II (p. 268 a)

Suppose one is standing on the shore of a circular pond with a radius of 1 mile and to get to a point on the shore directly opposite, first by swimming to a point P with speed 2 mile/hr and then walk along the shore with speed 3 mile/hr. Choose the point P to minimize the travel time.



Optimization II (p. 268 b)

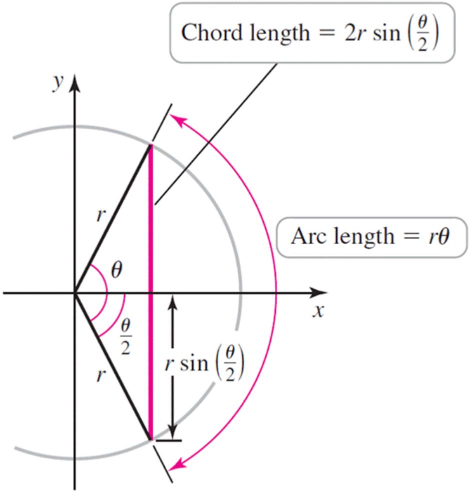


Figure: (Figure 4.56 (publisher))

Optimization II (p. 268 c)

We see that the chord length is $2r\sin(\theta/2)$ and the arc length is $r(\pi-\theta)$. Note that the radius is $r=1$ mile. Thus the travel time is given by

$$\begin{aligned} T(\theta) &= \frac{2\sin(\theta/2)}{2} + \frac{\pi-\theta}{3} \\ &= \sin\left(\frac{\theta}{2}\right) + \frac{\pi-\theta}{3}, \qquad (0 \le \theta \le \pi). \end{aligned}$$

The critical point(s) is given by

$$0 = \frac{dT}{d\theta} = \frac{1}{2}\cos\frac{\theta}{2} - \frac{1}{3}.$$

That is, when $\cos\theta/2 = 2/3$, or $\theta = \arccos(2/3) \approx 1.68 \text{ rad} = 96^\circ$. The end points give $T(0) = \pi/3 \approx 1.05$ hr, and $T(\pi) \approx 1$ hr. But $T(1.68 \text{ rad}) \approx 1.23$ hr.

Optimization II (p. 268 d)

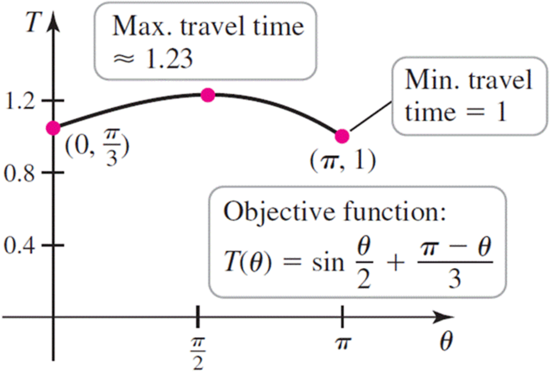
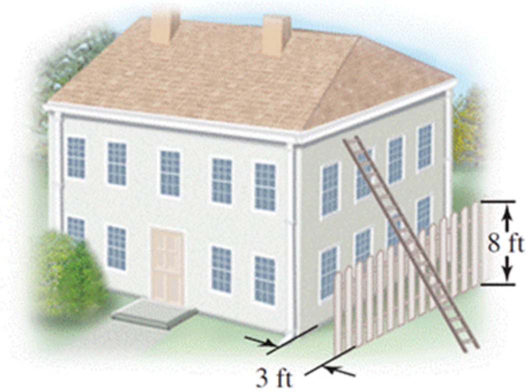


Figure: (Figure 4.57 (publisher))

Optimization III (p. 268 a)

An 8 ft height fence runs parallel to the side of a house 3 ft away. What is the lenght of the shortest ladder that clears the fence and reaches the house? Assume that the vertical wall of the house and the horizontal ground have infinite extent.



(a)

Optimization III (p. 268 b)

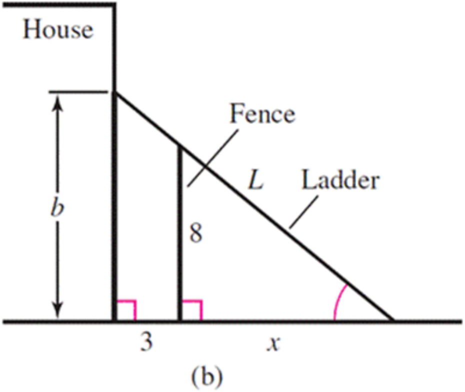


Figure: (Figure 4.58b (publisher))

Optimization III (p. 268 b)

Let L be the length of the ladder, x be the distance of the from the foot of the ladder to the foot of the fence, and let b be the height of the house. It follows from the last slide that we apply Pythagoras theorem to obtain

$$L^2 = (x+3)^2 + b^2.$$

But similar triangles consideration yield $8/x = b/(3+x)$ so that L is a function of x only and its domain is $x > 0$:

$$L^2 = (x+3)^2 + \left(\frac{8(x+3)}{x}\right)^2 = (x+3)^2\left(1+\frac{64}{x^2}\right)$$

It is easy to check

$$\frac{d}{dx}L^2 = \frac{2(x+3)(x^3-192)}{x^3}$$

which equals zero if $x^3 = 192$ or $x \approx 5.77$. First order test implies that $L(5.77) \approx 15$ ft is the minimum length.