MATH1013 Calculus I

Introduction to Functions[1]

Edmund Y. M. Chiang

Department of Mathematics
Hong Kong University of Science & Technology

March 14, 2013

Derivatives II (Chapter 3)

[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Exponential revisited

In 1748, Euler computed

$$e \approx 2.718281828459045235$$

He also shows that

$$e = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = \lim_{k\to\infty}\overbrace{\left(1+\frac{1}{k}\right)\cdots\left(1+\frac{1}{k}\right)}^{k}$$

We now know that even more is true:

$$e^x = \lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k$$

The first ten thousand digits:

2.7182818284590452353602874713526624977572470936999595749669676
62772407663035354759457138217852516642742746639193200305992181
41359662904357290033429526059563073813232862794349076323382988
7531952510190115738341879307021540891499348841675092447614...

Logarithmic functions

- Recall that the natural logarithmic function $\log x$ is defined to be the inverse function of the exponential function $y = e^x$. That is $x = \ln(e^x)$ and $x = e^{\ln x}$.
- **Theorem**. We have, for any $x > 0$,

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

$$\begin{aligned}
\frac{\ln(x+h) - \ln x}{h} &= \frac{1}{h} \ln\Big(1 + \frac{h}{x}\Big) \\
&= \ln\Big(1 + \frac{h}{x}\Big)^{1/h} \\
&= \ln\Big(1 + \frac{1/x}{k}\Big)^{k} \\
&\to \ln(e^{1/x}) \\
&= \frac{1}{x}
\end{aligned}$$

as $k \to +\infty$ (equivalent to $h \to 0$).

Examples

Example Find the derivative of

- $y = x \ln x$,

$$\frac{dy}{dx} = 1 + \ln x$$

- $y = (x-1)(x-2)(x-3)(x-4)^3$

$$\frac{dy}{dx} = y\Big(\frac{1}{x-2} + \frac{1}{x-2} + \frac{1}{x-3} + \frac{3}{x-4}\Big)$$

- $y = \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{2x}{x^2+1}$$

Examples

Example Find the derivative of

- $y = \ln \dfrac{(x^{1/2}+1)(x-1)}{4x-1}$

$$\frac{dy}{dx} = y\left[\frac{x^{-1/2}}{2(x^{1/2}+1)} + \frac{1}{x-1} - \frac{4}{4x-1}\right]$$

- $y = \sqrt{\dfrac{6x^3-1}{2x-1}}$

$$\frac{dy}{dx} = y\left[\frac{1}{2}\left(\frac{18x^2}{6x^3-1} - \frac{2}{2x-1}\right)\right]$$

- $y = \log_{10} x$

$$\frac{dy}{dx} = \frac{1}{x \ln 10}$$

Exercises

Differentiate the following functions

(i) $f(x) = \ln x^3$ $(3/x)$

(ii) $f(x) = \ln(2x)$ $(1/x)$

(iii) $f(x) = x^2 \ln x$ $(x(1 + 2\ln x))$

(iv) $f(x) = \ln \sqrt{x}$ $\left(\frac{1}{2}(1 + \ln x)\right)$

(v) $f(x) = \dfrac{\ln x}{x}$ $\left(\frac{1}{x^2}(1 - \ln x)\right)$

(vi) $f(x) = \ln\left(\dfrac{x+1}{x-1}\right)$ $(2/(1 - x^2))$

(vii) $f(x) = \dfrac{(x+2)^5}{\sqrt[6]{3x-5}}$ $\left(\frac{(x+2)^5}{\sqrt[6]{3x-5}}\left(\frac{5}{x+2} - \frac{1}{2(3x-5)}\right)\right)$

(viii) $f(x) = \sqrt{\dfrac{2x+1}{1-3x}}$, $\left(\frac{1}{2}\frac{(3-4x)}{(2x+1)^{1/2}(1-3x)^{3/2}}\right)$

(ix)
$f(x) = (x+1)^3(6-x)^2\sqrt[3]{2x+1}$ $\left(\left(\frac{3}{x+1} - \frac{2}{6-x} + \frac{2}{3(2x+1)}\right)f(x)\right)$

Different bases

- **Theorem** Let a be any positive real number. Then

$$\frac{d}{dx}\log_a x = \frac{1}{\ln a}\frac{1}{x}.$$

Similarly, if u is a function of x, then

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}.$$

- **Example** Find the derivative of $y = \log_3(x^2+1)$

$$\frac{dy}{dx} = \frac{2x}{(x^2+1)\ln 3}$$

Implicit Differentiation

We have learned to find derivatives of functions in the form $y = f(x)$, i.e., y can be expressed as a function of x only. However, this is not always the case:

$$xe^y + ye^x = y.$$

if there is a change of x by Δx then there must be a corresponding change in y by a certain amount Δy say, in order the keep the equality. So how can we find $\frac{dy}{dx}$? We illustrate the method called *implicit differentiation* by the following example.

Example

Example Find the rate of change of y with respect to x if $x^2 + y^2 = 5$. Find also the gradient of the circle at $x = 1$. Applying chain rule to the both sides of $x^2 + y^2 = 5$ yields

$$0 = \frac{d}{dx}5 = \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 2x + 2y\frac{dy}{dx}.$$

Thus

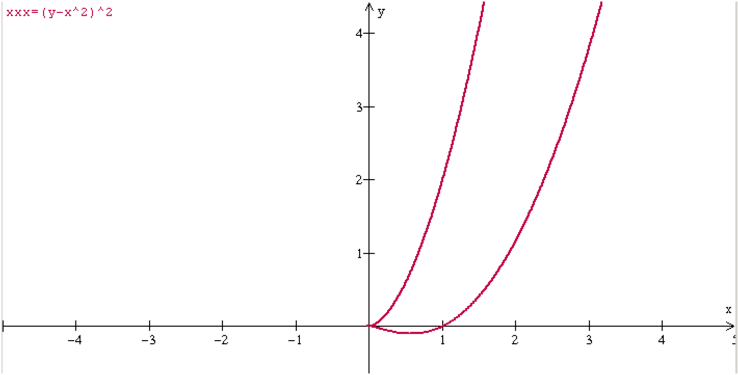
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Since $y(0)^2 = 5 - 1 = 4$, so $y(0) = \pm 2$. Hence

$$\frac{dy}{dx} = \pm\frac{1}{2}.$$

More examples I

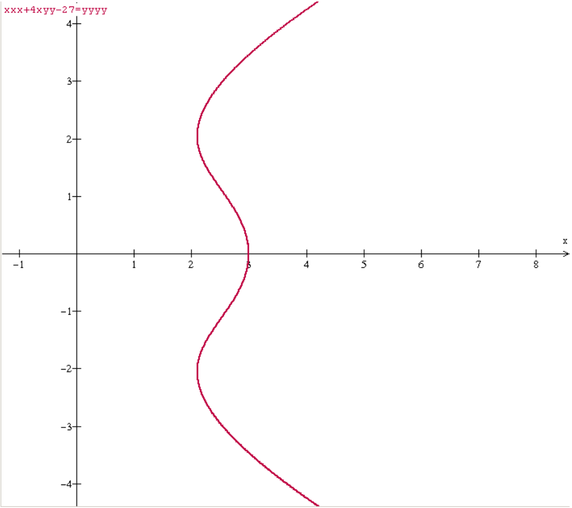
Example Find the equations of tangents to $x^3 = (y - x^2)^2$ when $x = 1$.



$\left(Ans.\ \frac{dy}{dx} = \frac{3x^2}{2(y-x^2)} + x^2;\ y = \frac{1}{2}x - \frac{1}{2},\ y = \frac{7}{2}x - \frac{3}{2}.\right)$

More examples II

Example Find $\frac{dy}{dx}$ if $x^3 + 4xy^2 - 27 = y^4$. $\qquad$ ($Ans.\ \ y' = \frac{3x^2+4y^2}{4y^3-8xy}$)



Exponential functions

- **Theorem** Let x be any real number. Then

$$\frac{d}{dx}e^x = e^x.$$

- Writing $y = f(x) = e^x$. Recall that

$$\log y = \log e^x = x$$

  Differentiating both sides implicitly yields

$$\frac{d \log y}{dx} = \frac{1}{y}\frac{dy}{dx} = \frac{dx}{dx} = 1.$$

  That is,

$$\frac{dy}{dx} = y = e^x.$$

- $\left.\dfrac{de^x}{dx}\right|_{x=0} = 1.$

Different bases

- **Theorem** If u is a function of x then

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

Furthermore, if a is a positive real number and let u be a function of x, then

$$\frac{d}{dx}a^x = a^x \ln a, \qquad \frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}.$$

- Let $y = a^x$. Then $x = \log_a y$. Differentiating both sides implicitly yields

$$1 = \frac{dx}{dx} = \frac{d \log_a y}{dx} = \frac{y'}{(\log a)\, y}.$$

So $y' = (\log a)\, a^x$.

Examples

- $y = e^{\sqrt{x}+x^3}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} 3x^2 e^{\sqrt{x}+x^3}$$

- $y = xe^{\ln x + x}$

$$\frac{dy}{dx} = (1 + \frac{2}{x}) xe^{x+\ln x}$$

- $y = e^{\frac{x}{x+1}}$

$$\frac{dy}{dx} = \frac{1}{(x+1)^2} e^{\frac{x}{x+1}}$$

- $y = a^{x^2+x}$

$$\frac{dy}{dx} = (\ln a)\, a^{x^2+x} (2x+1)$$

Further examples

- $y = x^x$

$$\frac{dy}{dx} = (\ln x + 1)\, x^x$$

- $y = (1 + e^x)^{\ln x}$

$$\frac{dy}{dx} = y[\frac{\ln(1+e^x)}{x} + \frac{xe^x}{1+e^x}]$$

- **Example** $xe^y + ye^x = y$

Exercises

Differentiate the following functions.

(i) $f(x) = e^{x^2+2x-1}$, $\left(2(x+1)e^{x^2+2x-1}\right)$

(ii) $f(x) = e^{1/x}$, $\left(-1/x^2\ e^{1/x}\right)$

(iii) $f(x) = 30 + 10e^{-0.05x}$, $\left(-0.5e^{-0.05x}\right)$

(iv) $f(x) = x^2e^x$, $\left((2x+1)e^x\right)$

(v) $f(x) = (x^2+3x+5)e^{6x}$, $\left((6x^2+20x+33)e^{6x}\right)$

(vi) $f(x) = xe^{-x^2}$, $\left(e^{-x^2}(1-2x^2)\right)$

(vi) $f(x) = e^{\sqrt{3x}}$, $\left(\frac{\sqrt{3}}{\sqrt{x}}e^{\sqrt{3x}}\right)$

(vii) $f(x) = e^{-1/2x}$, $\left(\frac{1}{2x^2\ e^{-1/2x}}\right)$

(viii) $f(x) = e^x \ln x$, $\left(e^x(\ln x + 1/x)\right)$

(ix) $f(x) = e^{-3x}\sqrt{2x-5}/(6-5x)^4$ $\left((-3+\frac{1}{2x-5}+\frac{20}{6-5x})f(x)\right)$

(x) $f(x) = 2^{x^2}$, $\left(\frac{2x}{\ln 2}2^{x^2}\right)$

(xi) $f(x) = x^{1-x}$, $\left((\frac{1}{x}-1-\ln x)x^{1-x}\right)$

General principle (not required)

Consider the equation

$$F(x, y(x)) = 0$$

where $F(X, Y)$ is polynomial in X and Y. If this $F = 0$ for all X and Y, then the X and Y are related to each other. i.e., , when X changes, there must be a corresponding change in Y (or vice verse). So $Y = G(X)$ for some function G whose explicit form is (generally) unknown. However, it is possible that one finds its derivative $\frac{dY}{dX}$. This follows from multi-variable calculus when treating $Z = F(X, Y)$ as a function of two (independent) variables X, Y. Then the

$$dZ = \frac{\partial F}{\partial X}\, dX + \frac{\partial F}{\partial Y}\, dY.$$

But $dZ = 0$ and $Y = Y(X)$ so that we can apply the Chain rule to obtain

$$\frac{dY}{dX} = -\frac{\frac{\partial F}{\partial X}}{\frac{\partial F}{\partial Y}}$$

.

Inverse Sine function

(pp. 209-210) Here is another application of implicit differentiation.
Consider $y = \arcsin x$ on the domain $[-1, 1]$ has range $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. On the other hand $\sin y = x$. Differentiating this equation on both sides yields

$$1 = \frac{dx}{dx} = \frac{d}{dx} \sin y = \cos y \, \frac{dy}{dx}.$$

Notice that we have $1 - x^2 = \cos^2 y$. So

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

on $(-1, 1)$. Note that we have chosen the positive branch of $\pm\sqrt{1 - x^2}$ since $\cos y \geq 0$ on $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
Note that $y' \to +\infty$ as $x \to \pm 1$.

Figure 3.57 (Publisher)

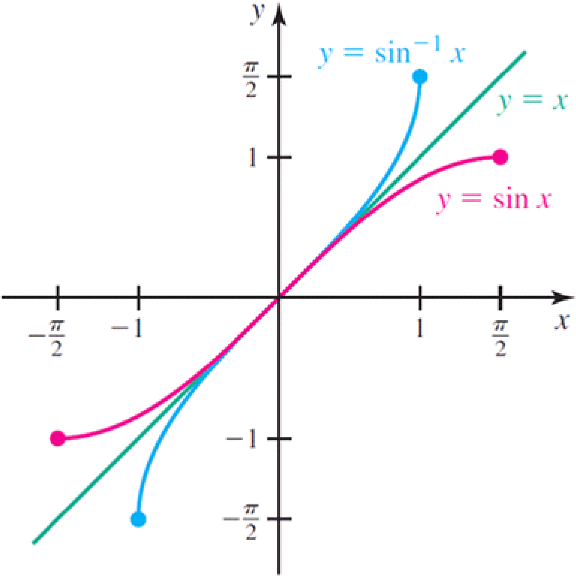
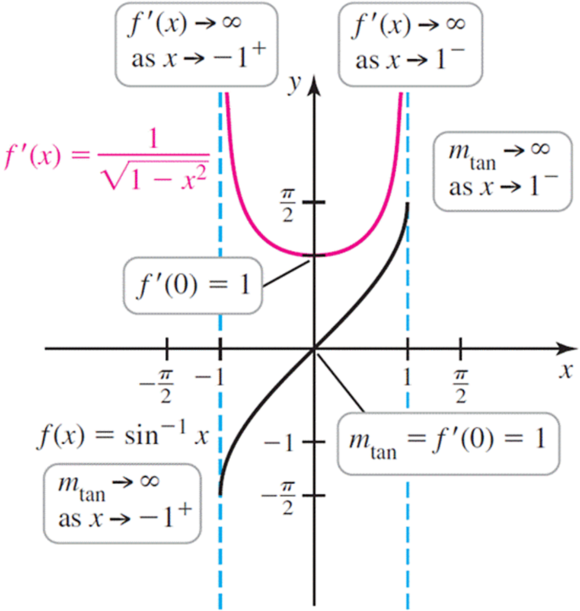


Figure 3.58 (Publisher)



Inverse function examples

- (p. 210)

$$\begin{aligned}\frac{d}{dx}\arcsin(x^2-1) &= \frac{d}{du}\arcsin u \cdot \frac{du}{dx} \\ &= \frac{1}{\sqrt{1-u^2}} \cdot \frac{d}{dx}(x^2-1) \\ &= \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot 2x = \frac{2x}{\sqrt{2x^2-x^4}} \\ &= \frac{2x}{|x|\sqrt{2-x^2}}\end{aligned}$$

- (p. 210)

$$\begin{aligned}\frac{d}{dx}\cos(\arcsin x) &= -\sin(\arcsin x)\frac{d}{dx}(\arcsin x) \\ &= -\frac{x}{\sqrt{1-x^2}}\end{aligned}$$

Inverse tangent

- (p. 211) Inverse tangent $y = \arctan x$ has domain $(-\infty, \infty)$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$.
  Differentiating $x = \tan y$ on both sides w.r.t. x yields,

$$1 = \frac{dx}{dx} = \frac{d}{dx} \tan y = \frac{d}{dy} \tan y \, \frac{dy}{dx} = \sec^2 y \cdot \frac{dy}{dx}.$$

  Thus $\frac{dy}{dx} = \frac{1}{\sec^2 y}$. Note that $1 + x^2 = \sec^2 y$. So

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

  Thus we see that $y' \to 0$ as $x \to \pm\infty$.

Figure 3.59 (Publisher)

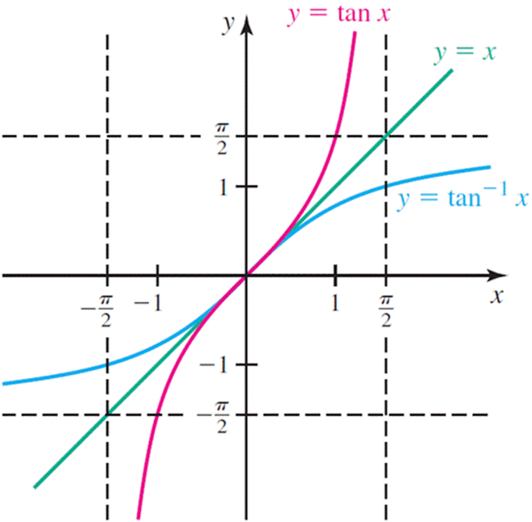
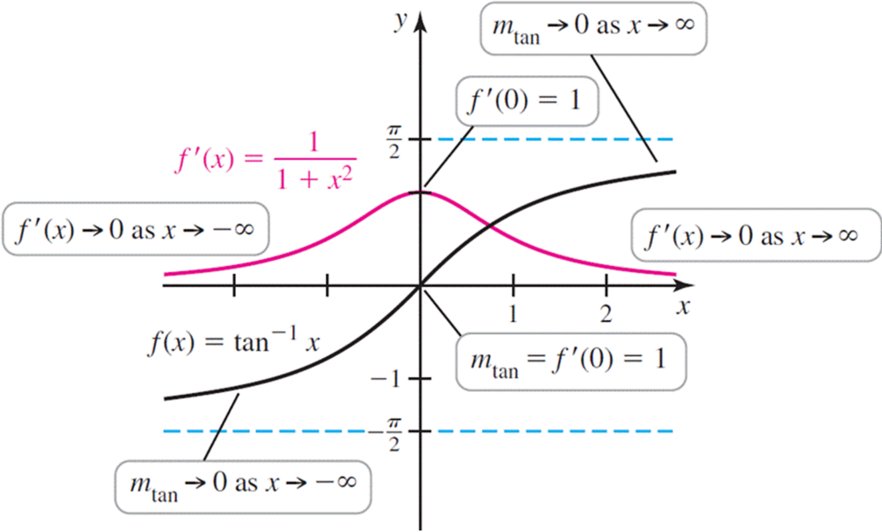


Figure 3.60 (Publisher)



Inverse secant

Since secant maps $[0, \pi/2]$ to $[1, +\infty)$ and $[\pi/2, \pi]$ to $(-\infty, -1]$, so its inverse $y = \sec^{-1} x$ and hence $\sec y = x$ is well defined on $[1, +\infty)$ and $(-\infty, -1]$ or on $|x| \geq 1$.

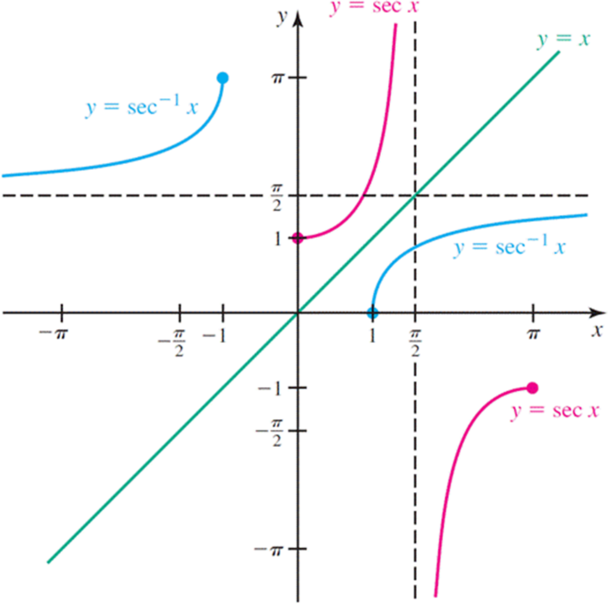
Note that $x^2 = 1 + \tan^2 y$. Differentiating $\sec y = x$ implicitly on both sides yields

$$\begin{aligned} 1 = \frac{dx}{dx} = \frac{d}{dx} \sec y &= \frac{d \sec y}{dy} \cdot \frac{dy}{dx} \\ &= \sec y \tan y \cdot \frac{dy}{dx} \end{aligned}$$

That is,

$$\begin{aligned} (\sec^{-1} x)' = \frac{dy}{dx} = \frac{1}{\sec y \tan y} &= \frac{1}{\pm x\sqrt{x^2 - 1}} \\ &= \begin{cases} \dfrac{1}{x\sqrt{x^2 - 1}}, & \text{if } x > +1; \\ -\dfrac{1}{x\sqrt{x^2 - 1}}, & \text{if } x < -1 \end{cases} \end{aligned}$$

Figure 3.61 (Publisher)



Other Inverses

Differentiating

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

on both sides yields

$$\frac{d}{dx}\cos^{-1} x = -\frac{d}{dx}\sin^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

Similarly, differentiating

$$\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}, \quad \text{and} \quad \csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

yields

$$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}, \quad \text{and} \quad \frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

respectively.

Example

(p. 213) Compute the derivative of $f(x) = x\tan^{-1} x/2$ at $x = 2\sqrt{3}$. Differentiating $f(x)$ with product and chain rules yield

$$f'(x) = \tan^{-1}\frac{x}{2} + \frac{x\cdot\frac{1}{2}}{1+x^2/4}.$$

Since $\tan\pi/3 = \sqrt{3}$, so

$$f'(2\sqrt{3}) = \tan^{-1}\sqrt{3} + \frac{\sqrt{3}}{1+(\sqrt{3})^2} = \frac{\pi}{3} + \frac{\sqrt{3}}{4}.$$

Derivative of inverse

Theorem (p. 215) Let f be differentiable and have an inverse on an interval I. If x_0 is in I such that $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

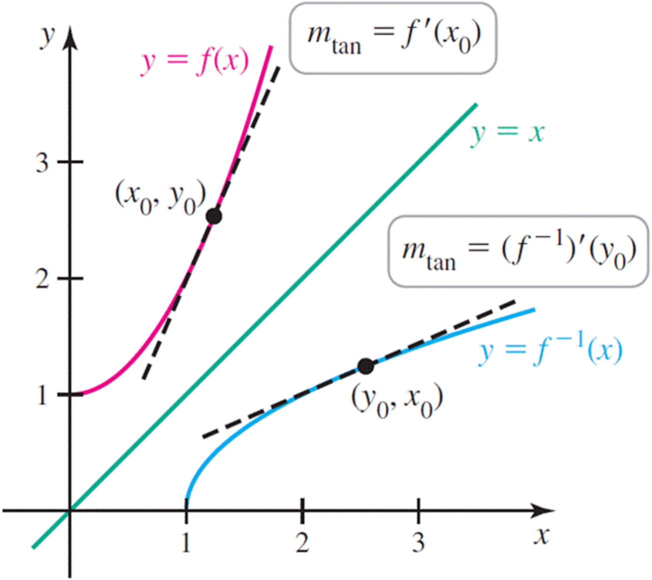
$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof Since $x_0 = f^{-1}(y_0)$ and $x = f^{-1}(y)$ so

$$\begin{aligned}(f^{-1})'(y_0) &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \\ &= \lim_{y \to y_0} \frac{x - x_0}{f(x) - f(x_0)} \\ &= \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}\end{aligned}$$

since (f^{-1}) is continuous. In particular, we note that $f'(x_0)(f^{-1})'(y_0) = 1$.

Figure 3.65 (Publisher)



Example

(p. 216) Let $f(x) = \sqrt{x} + x^2 + 1$ is one-one for $x \geq 0$. Find the gradient of $y = f^{-1}(x)$ at the point $(3, 1)$.

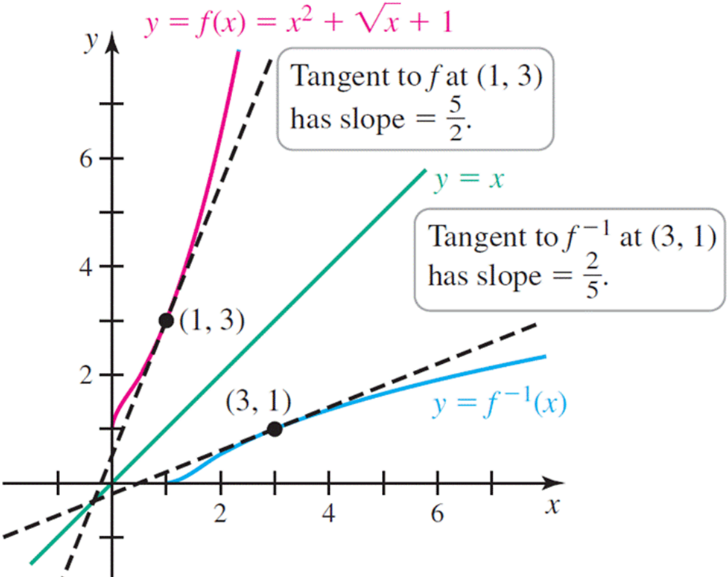
We first check that $3 = f(1) = \sqrt{1} + 1^2 + 1$. Then

$$f'(x) = \frac{1}{2\sqrt{x}} + 2x.$$

So

$$(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{2}{5}.$$

Figure 3.66 (Publisher)



Higher order derivatives

Consider $f(x) = x^3 + 2x$. Its derivative is given by $f'(x) = 3x^2 + 2$ which is again a function of x. Therefore we may ask what is *its* rate of change with respect to x? In fact

$$\lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x} = 6x.$$

We call the above limit the *second derivative* of f at x. It is denoted by $f''(x)$, or $\frac{d^2 f}{dx^2}$. This definition can easily be extended to any function. If x is replaced by time t and $y$stands for the distance travelled by an object. Then y'' is interpreted as the *acceleration* of the object. That is the rate of change of the velocity of the object.

Higher order derivatives

We can define the third order derivative of f by

$$\lim_{\Delta x \to 0} \frac{f''(x+\Delta x) - f''(x)}{\Delta x} = 6.$$

In general we defin the *kth-order derivative* of f by

$$\lim_{\Delta x \to 0} \frac{f^{(k-1)}(x+\Delta x) - f^{(k-1)}(x)}{\Delta x}.$$

We denote this by $f^{(3)}(x)$, $\frac{d^3y}{dx}$ and $f^{(n)}(x)$, $\frac{d^ny}{dx}$. Higher order derivatives have very important geometrical meaning.Thus we have $f'''(x) = 0$ for the above example since 6 is a constant function

Examples

- $\frac{d^2}{dx^2} e^{x^2} = e^{x^2}(2+4x^2)$
- $\frac{d^2}{dx^2} \frac{e^x}{x+1} = \frac{e^x(x^2+1)}{x+1)}$
- Find $\frac{d^2y}{dx^2}$ in $x^2 + ky^2 = 4$:

$$\frac{d^2y}{dx^2} = -\frac{1}{k^2}(\frac{ky^2+x^2}{y^3})$$

- Find the rate of change of y' if $y^2 = e^{x+y}$.

$$\frac{d^2y}{dx^2} = \frac{2y}{(2-y)^3}.$$

Applications I

- (p. 219) **Spreading oil** An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 meters/hour. How fast is the area of the patch increasing when the patch has a radius reaches 100 meters.
- Recall

$$\text{area formula of a circle } = A = \pi r^2.$$

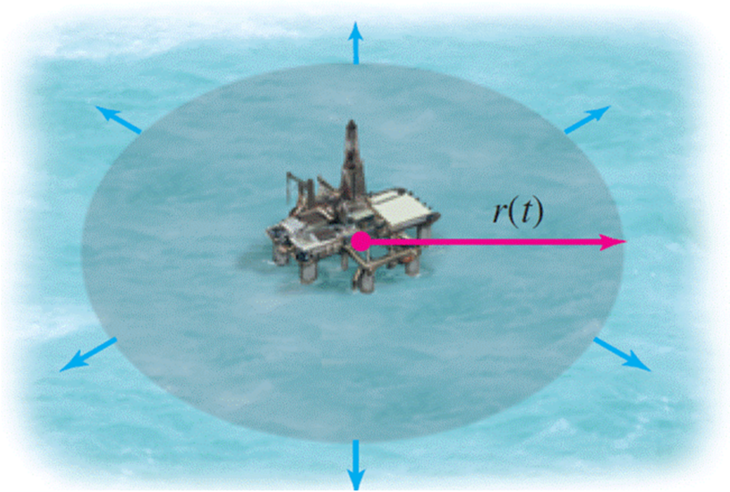
- But the radius $r(t)$ is a function of time.
- So $A(t) = \pi r(t)^2$ and so

$$\frac{d}{dt} A(t) = \frac{d}{dr} A \cdot \frac{d}{dt} r(t) = 2\pi r(t) \cdot r'(t).$$

- Since $r'(t) = 30$ is a constant rate, so when $r(t) = 100$, we have

$$A'(t)\Big|_{r=100} = 2\pi (100\text{m}) \cdot 30\text{m/hr} = 6,000\,\pi\,\text{m}^2/\text{hr} \approx 18,850\,\text{m}^2/\text{hr}.$$

Figure 3.67 (Publisher)

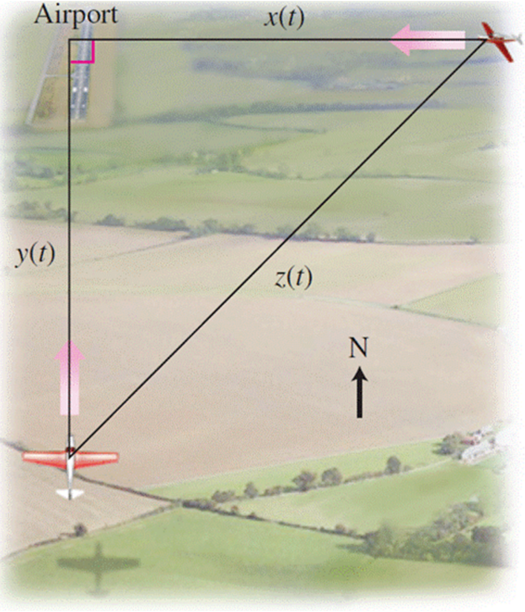


Application II

- (p. 220) **Coverging airplanes** Two small planes approach an airport, one flying due west at 120 mile/hr and the other flying due north at 150 mile/hr. Assuming they fly at the constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 miles from the airport and the northbound plane is 225 miles from the airport?
- Let $x(t)$ and $y(t)$ be the distances from the airport of the westbound plane and northbound plane respectively
- Then the Pythagora's theorem that $z^2 = x^2 + y^2$, where the $z(t)$ denotes the distance between the planes. So

$$2z(t)\frac{dz}{dt} = \frac{d}{dt}z(t)^2 = \frac{d}{dt}(x(t)^2+y(t)^2) = 2x(t)\,\frac{dx}{dt}+2y(t)\,\frac{dy}{dt}.$$

Figure 3.68 (Publisher)



Application II (cont.)

- (p. 220) **Coverging airplanes**
- So

$$\frac{dz}{dt} = \frac{\frac{d}{dt}(x(t)^2 + y(t)^2) = x(t)\,\frac{dx}{dt} + y(t)\,\frac{dy}{dt}}{z(t)}.$$

- when the westbound plane is 180 miles and the northbound plane is 225 miles from the airport, the distance between the planes is approximately $z(t) = \sqrt{180^2 + 225^2} \approx 288$ miles to each other.
- But both $x'(t)$ and $y'(t)$ are decreasing and hence negative, so we deduce

$$\frac{dz}{dt} = \frac{x(t)\,\frac{dx}{dt} + y(t)\,\frac{dy}{dt}}{z(t)} \approx \frac{(180)(-120) + (225)(-150)}{288}$$
$$\approx -192\text{mile/hr}.$$

Application III

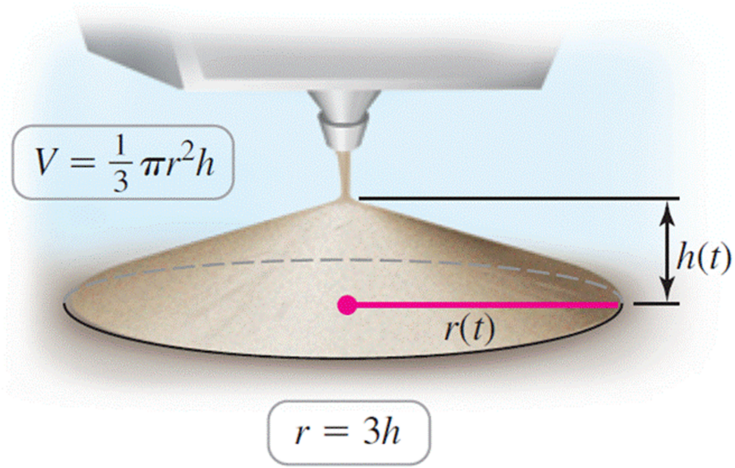
- (p. 221) **Sandile**
- Sand falls from an overhead bin, accumulating in a conical pile with a radius that is always three times it height. If the sand falls from the bin at a rate of $120\text{ft}^3/\text{min}$, how fast is the height of the sandpile changing when the pile is 10ft high?
- Recall the volume of the conical pile is given by $V = \frac{1}{3}\pi r^2 h$. Here $r = 3h$ so that $V = 3\pi h^3$.
- So when $h = 10$,

$$\left.\frac{dV}{dt}\right|_{h=10} = \left.\frac{dV}{dh}\right|_{h=10} \cdot \frac{dh}{dt}$$

- That is,

$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\left.\frac{dV}{dh}\right|_{h=10}} = \frac{120}{\left.9\pi h^2\right|_{h=10}} = \frac{120}{2\pi 10^2} \approx 0.042\text{ft}/\text{min}$$

Figure 3.69 (Publisher)



Application IV

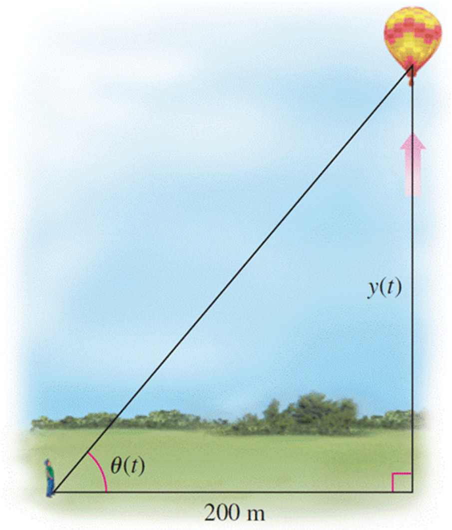
- (p. 221) **Observing launch** An observer stands 200 m from launch site of a hot-air ballon. The ballon raises vertically at a constant rate of 4 m/s. How fast is the angle of elevation of the ballon increasing 30 s after the launch?
- The elevation angle is $\tan\theta = y/200$ where $y(t)$ is the vertical height. We want $\frac{d\theta}{dt}$.
- After 30 s, $y = 30 \times 4 = 120$ m. So

$$\frac{dy}{dt} = 200\sec^2\theta\,\frac{d\theta}{dt}.$$

- That is, after 30 s

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{200}\,\frac{dy}{dt} = \frac{\cos^2\theta}{200}(4) = \frac{\cos^2\theta}{50}.$$

Figure 3.70 (Publisher)



Application IV (cont.)

- (p. 221) **Observing launch**
- It remains to compute $\cos^2\theta$.
  $\cos\theta = 200/\sqrt{120^2 + 200^2} \approx 200/233.23 \approx 0.86$. So

$$\frac{d\theta}{dt} \approx \frac{1}{50} \times (0.86)^2 = 0.015 \text{ rad/s}$$

which is slightly less than $1^\circ/\text{s}$.