MATH1013 Calculus I

Introduction to Functions[1]

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Derivatives IV (Chapter 4)

[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Extreme Value theorem

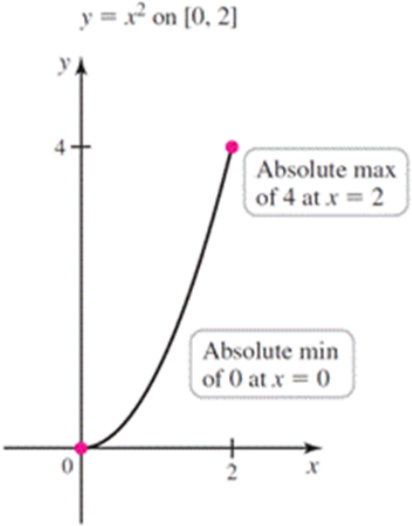
- **Theorem** A function $f(x)$ continuous on a closed interval $[a, b]$ attains its absolute maximum/minimum on $[a, b]$. That is, there exist c, d in $[a, b]$ such that

$$f(x) \geq f(c), \qquad f(x) \leq f(d) \qquad \text{for all } x \text{ in } [a, b].$$

- This result looks very trivial is in fact a deep result in elementary mathematical analysis. It is proved vigorously in chapter 5 (Theorem 5.3) of my supplementary notes on Mathematical Analysis course found in my web site of this course.
- What we will do in the following sides is to show the Extreme Value theorem does not hold when any one of the hypotheses fails to hold.

Extreme value theorem example I

$f(x) = x^2$. It requires boundedness and closedness of the interval $[a, b]$ assumption.



Extreme value theorem example II

It requires continuity $f(x)$ assumption.

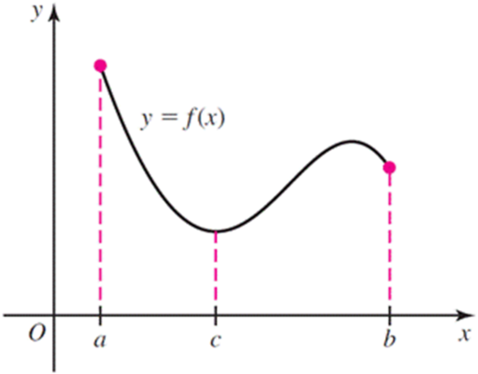


Figure: (Publisher Figure 4.4 (a))

How extreme value theorem can fail I

$f(x) = x^2$. Dropping boundedness of interval $[a, b]$ assumption.

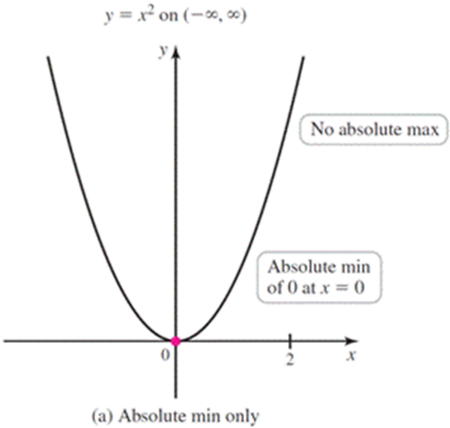
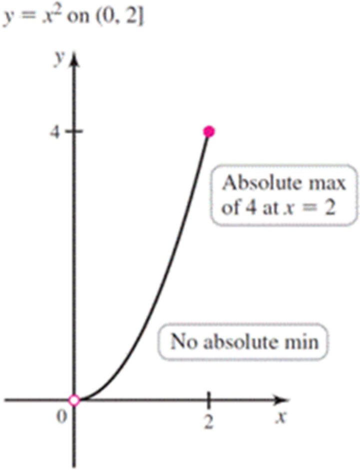


Figure: (Publisher Figure 4.2 (a))

How extreme value theorem can fail II

$f(x) = x^2$. Dropping closed interval $[a, b]$ assumption.

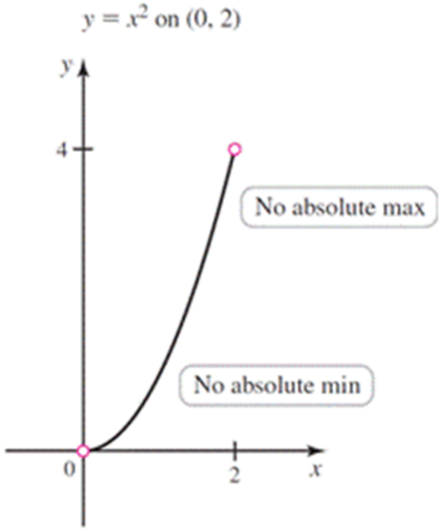


(c) Absolute max only

Figure: (Publisher Figure 4.2 (c))

How extreme value theorem can fail III

$f(x) = x^2$. Dropping closed interval $[a, b]$ assumption.



(d) No max or min

How extreme value theorem can fail IV

Dropping continuity $f(x)$ assumption.

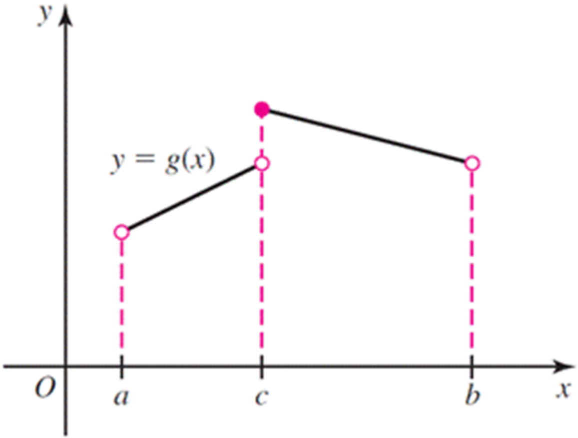


Figure: (Publisher Figure 4.4 (b))

Rolle's theorem

- **Theorem** Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.
- **Proof** Case I: If $f(x)$ attains its maximum and minimum at the end points of $[a, b]$, then $f(a) = f(b)$. That is, $f(x)$ is a constant on (a, b). So $f'(c) = 0$ for all c in (a, b).
- Case II: If at least one of the extreme points doesn't coincide with the end points a, b, then this extreme point, c say must lie in (a, b). But then we know $f'(c) = 0$.

How Rolle's theorem can fail I

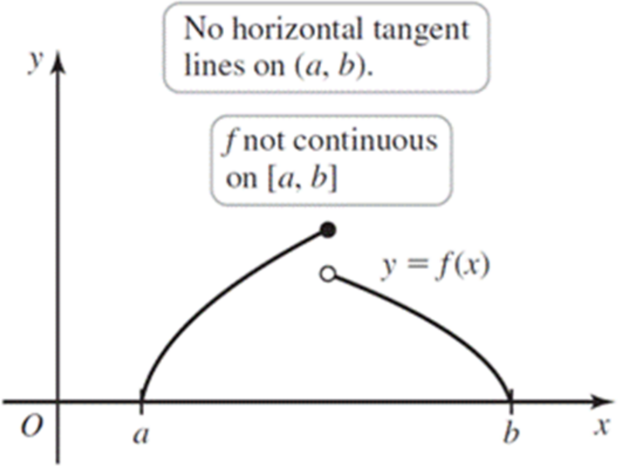


Figure: (Publisher Figure 4.66 (a))

How Rolle's theorem can fail II

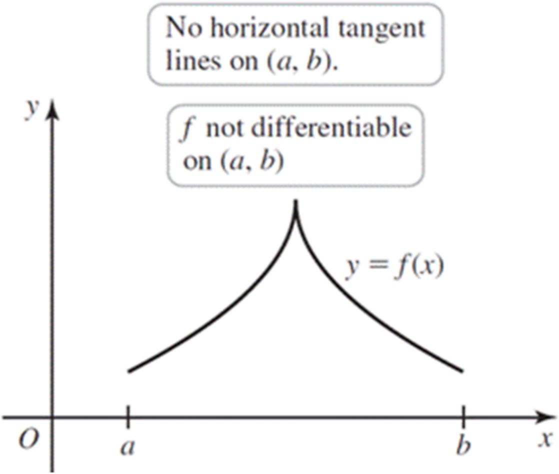
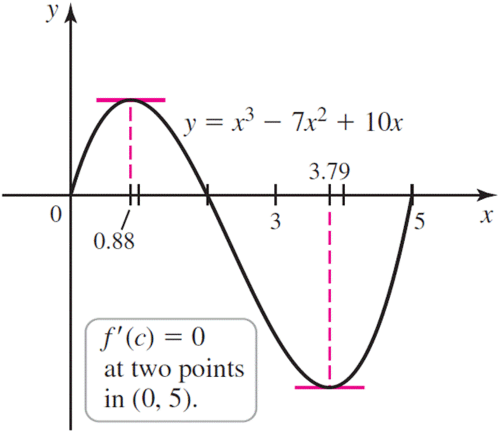


Figure: (Publisher Figure 4.66 (b))

Rolle's theorem example

$f(x) = x^3 - 7x^2 + 10x$ has critical points that satisfy the Rolle's theorem at

$$x = \frac{7 \pm \sqrt{19}}{3}, \text{ or } x \approx 0.88 \text{ and } x \approx 3.79.$$



Mean Value theorem

Theorem If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is a c in $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

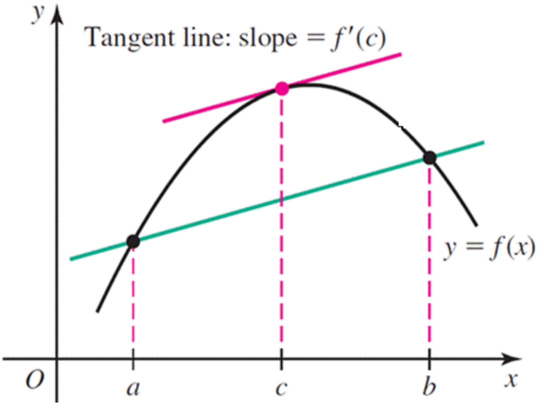


Figure: (Publisher Figure 4.68)

Proof of Mean Value theorem

The idea is to transform the problem into the setting of the Rolle theorem.
The gradient of the straight line equation $\ell(x)$ that connects the pair of points $(a, f(a))$ and $(b, f(b))$ is given by:

$$y = \ell(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

where $\ell(a) = f(a)$ and $\ell(b) = f(b)$. So the function $F(x) = f(x) - \ell(x)$ satisfies $F(a) = F(b) = 0$. Since $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b), so the Rolle theorem implies that there is a c in (a, b) so that $0 = F'(c) = f'(c) - \ell'(c)$. That is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof of Mean Value theorem II

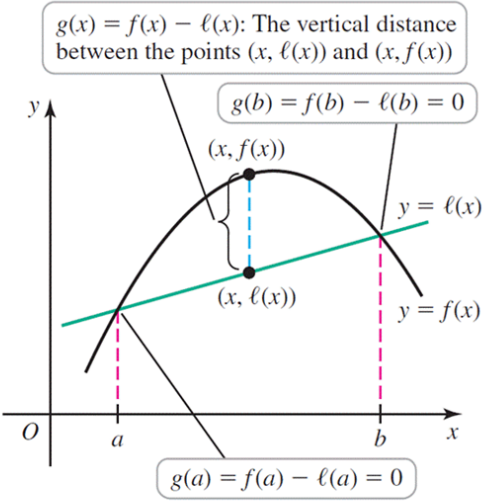


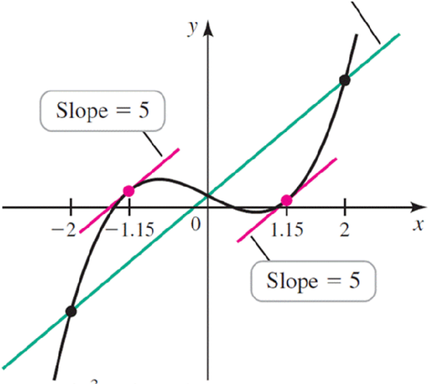
Figure: (Publisher Figure 4.69)

Example of Mean Value theorem

Determine the points in $[-2, 2]$ for $f(x) = 2x^3 - 3x + 1$ that are guaranteed to exist by the Mean Value theorem.
There are point(s) c in $[-2, 2]$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = 5.$$

But $f'(x) = 6x^2 - 3 = 5$ so that $x^2 = 4/3$ or $x = \pm 2/\sqrt{3}$.



Consequences of Mean Value theorem

- **Theorem** If $f'(x) = 0$ at all points of an interval I, then f is a constant on I.
- **Theorem** If two functions have the property that $f'(x) = g'(x)$ on I, then $f(x) = g(x) + k$ holds for all x in I for some constant k.
- **Theorem** Suppose $f(x)$ is continuous on an interval I and differentiable at all interior points of I, then
    - if $f'(x) > 0$ at all interior points of I, then f is increasing on I;
    - if $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

Application I (p. 286)

In meteorology, let $T(z)$ be the temperature of of the atmosphere at the altitude z. Then

$$\text{lapse rate } = \gamma = -\frac{dT(z)}{dz}$$

that is, it is the rate of decrease in the temperature of the atmosphere at the altitude z. It is known from experience that if the lapse rate is more than 7 degrees/km, then the chance of thunderstorm/tornado is large.

Example Suppose $T(2.9) = 7.6$ and $T(5.6) = -14.3$. What will a meteorologist deduce from this data?

We assume that the temperature T function is differentiable. Then

$$T'(c) = \frac{T(5.6) - T(2.9)}{5.6 - 2.9} = \frac{-14.3 - 7.6}{5.6 - 2.9} = -8.1,$$

which is an average rate of change. The mean value theorem guarantees that there's at least one c lies between 2.9 and 5.6 such that $T(c) = -8.1$. The chance of thunderstorm/tornado is large.

Application II (p. 286)

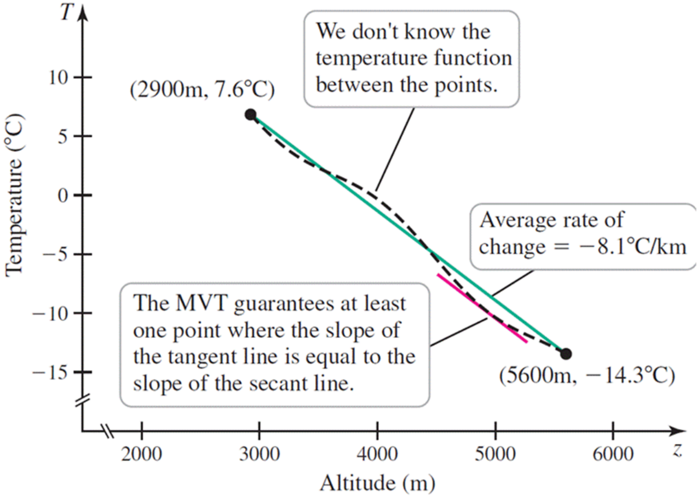


Figure: (Publisher Figure 4.70)

$0/0$ form

- **Theorem (p.290)** *Suppose f and g are differentiable on (a, b) and c lies in (a, b) such that $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ and $g'(x) \neq 0$ $(x \neq c)$. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the last limit exists, including $\pm\infty$.

- The above also holds for $x \to \pm\infty$ or $x \to c\pm$.
- **Example**

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} 1} = 1.$$

Proof of "0/0" form

We assume in addition that f' and g' are also continuous over (a, b) and $g'(c) \neq 0$

$$\lim_{x\to c}\frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)} = \frac{\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}}{\lim\limits_{x\to c}\dfrac{g(x)-g(c)}{x-c}}$$

$$= \lim_{x\to c}\frac{\dfrac{f(x)-f(c)}{x-c}}{\dfrac{g(x)-g(c)}{x-c}} = \lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x\to c}\frac{f(x)}{g(x)}$$

The basic rationale behind is the use of first order approximation formula: $f(x) \approx f(a) + f'(a)(x-a) = f'(a)(x-a)$.

Examples (p. 291)

- $\lim\limits_{x\to 1}\dfrac{x^3+x^2-2x}{x-1}$
- $\lim\limits_{x\to 0}\dfrac{\sqrt{9+3x}-3}{x}$

More examples (p. 292)

The following examples require repeated applications of L'Hôpital's rule:

- $\lim_{x\to 0} \dfrac{e^x - x - 1}{x^2}$
- $\lim_{x\to 2} \dfrac{x^3 - 3x^2 + 4}{x^4 - 4x^3 + 7x^2 - 12x + 12}$

Double check in the indeterminate forms

$$\lim_{x\to 0} \frac{3x - \sin x + 1}{x + 2} = \lim_{x\to 0} \frac{3 - \cos x}{1} = \frac{3 - \cos 0}{1} = 2.$$

But in fact, the above limit is NOT in the indeterminate form

$$\lim_{x\to 0} \frac{3x - \sin x + 1}{x + 2} = \frac{0 - 0 + 1}{0 + 2} = \frac{1}{2}.$$

Geometric reason I (p. 291)

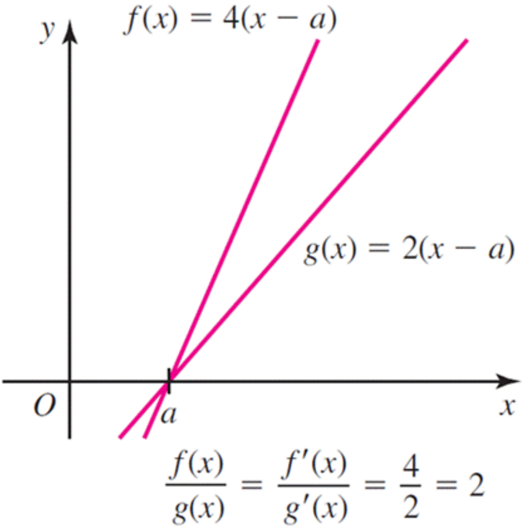
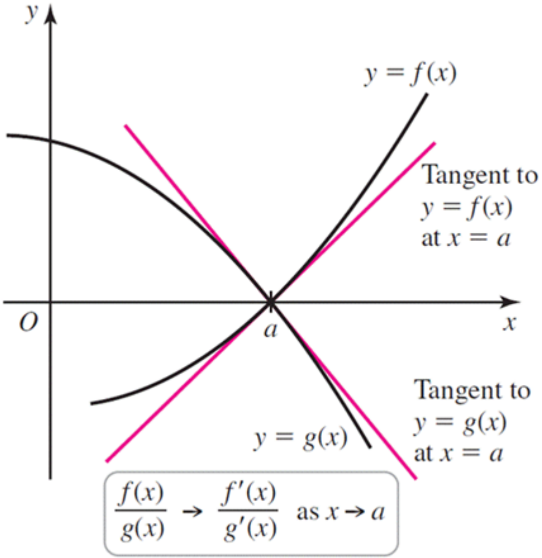


Figure: (Publisher Figure 4.72)

Geometric reason II (p. 291)



"∞/∞" form

Theorem (p.293) *Suppose f and g are differentiable on (a, b) and c lies in (a, b) such that $\lim_{x\to c} f(x) = \pm\infty = \lim_{x\to c} g(x)$ and $g'(x) \neq 0$ $(x \neq c)$. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the last limit exists, including $\pm\infty$.

Remark The above also holds for $x \to \pm\infty$ or $x \to a\pm$.

- **Example** $\displaystyle\lim_{x\to\infty} \frac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$
- **Example** $\displaystyle\lim_{x\to\pi/2-} \frac{1 + \tan x}{\sec x}$

Double check "∞/∞" form

Example

$$\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x-5}} = \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x+2}}}{\frac{1}{2\sqrt{x-5}}} = \lim_{x\to\infty}\frac{\sqrt{x-5}}{\sqrt{x+2}}$$
$$= \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x-5}}}{\frac{1}{2\sqrt{x+2}}} = \lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x-5}}.$$

But in fact,

$$\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x-5}} = \lim_{x\to\infty}\frac{\sqrt{1+2/x}}{\sqrt{1-5/x}} = \frac{\sqrt{1+0}}{\sqrt{1-0}} = 1.$$

"$0 \cdot \infty$", "$\infty - \infty$", "1^∞", "0^∞", "∞^0"

They can be changed to $0/0$ or ∞/∞.

- "$0 \cdot \infty$": $\lim_{x\to\infty} x \sin \frac{1}{x} = 1$
- "$\infty - \infty$": $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$
- "0^0": $\lim_{x\to 0+} x^x = 1$
- 1^∞: $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$
- Cannot apply L'Hôpital's rule to 1^∞, 0^∞, ∞^0.
- Rewrite $\lim_{x\to a} f(x)^{g(x)}$ as

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}.$$

Thus it remains to compute

$$\lim_{x\to a} g(x)\ln f(x)$$

perhaps again using L'Hôpital's rule.

Ranking growth rates to ∞ I

Definition Suppose f and g are such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0, \quad \text{or} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

Then f and g have comparable growth rate if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = M, \qquad 0 < M < \infty.$$

Ranking growth rates to ∞ II

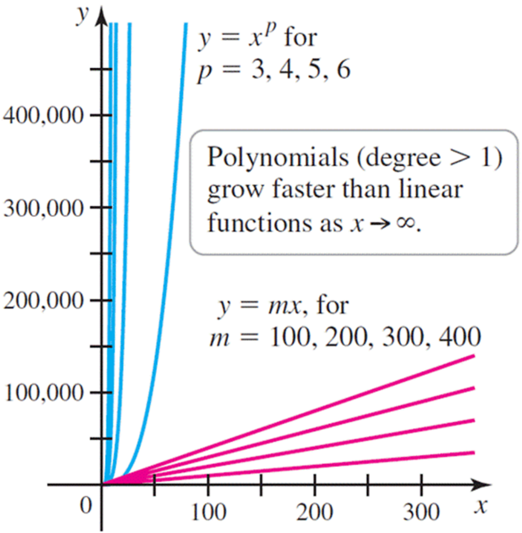


Figure: (Publisher Figure 4.76)

Ranking growth rates to ∞: III

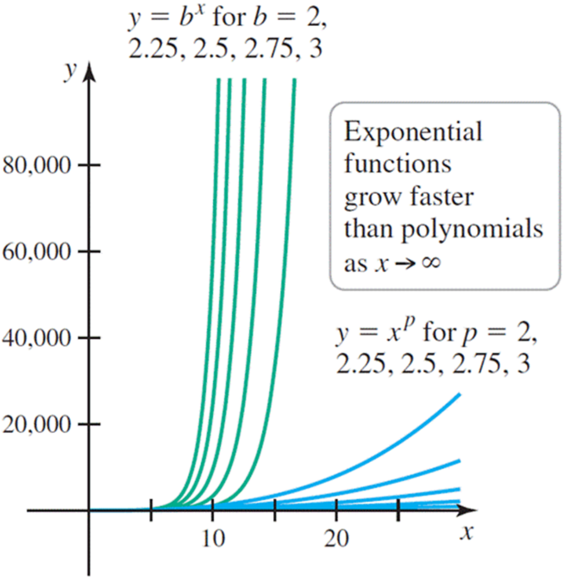


Figure: (Publisher Figure 4.77)

Ranking growth rates theorem

Theorem 4.15 (p.299) *Let $f \ll g$ to mean that g grows faster than f as $x \to \infty$. Suppose p, q, r, s and $b > 1$ are positive real numbers. Then*

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x,$$

as $x \to \infty$.

Proof Most of these relationships are quite clear to hold. It only remains to check $\ln^q x \ll x^p$. We first check for $p > 0$ that

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0.$$

Hence

$$\lim_{x\to\infty} \frac{\ln^q x}{x^p} = \lim_{x\to\infty} \left(\frac{\ln x}{x^{p/q}}\right)^q = \left(\lim_{x\to\infty} \frac{\ln x}{x^{p/q}}\right)^q = 0$$

by the above consideration.

Principle of Newton's method

We want to solve the equation $f(r) = 0$ where we don't have any formula for this equation. In fact, it is rare that such formula exists for a general equation.

- A tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is drawn.
- Suppose the tangent line drawn above intersects with the $x-$axis at $(x_1, 0)$. Then the x_1 is a new approximation to the real root r.
- We repeat the above procedure of constructing tangents using the sequence of points $\{x_1, x_2, x_3, \cdots\}$ obtained from the intersection of previous tangent line and the $x-$axis.

Deriving Newton's method I

- The tangent line equation to the curve $f(r) = 0$ at x_0 can be written in the form

$$y - f(x_0) = f'(x_0)(x - x_0).$$

- When the tangent line at $(x_0, f(x_0))$ intersects the $x-$axis, we have

$$0 - f(x_0) = f'(x_0)(x - x_0).$$

- Writting the new point to be $x = x_1$ and rearranging:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Deriving Newton's method I

- We can repeat this procedure as many steps as needed unless we have found the exact solution.
- Suppose we have already reached the approximation x_n with the tangent line at $(x_n, f(x_n))$. When the tangent line intersects the $x-$axis, we have

$$0 - f(x_n) = f'(x_n)(x - x_n).$$

  We label the new approximation by $x = x_{n+1}$.
- We have, after rearranging,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example (p. 304)

Approximate the roots of $f(x) = x^3 - 5x + 1$ by Newton's method

$$f'(x) = 3x^2 - 5.$$

So the Newton approximation formula becomes:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 5x_n + 1}{3x_n^2 - 5} = \frac{2x_n^3 - 1}{3x_n^2 - 5}, \quad n = 0, 1, 2, 3, \cdots$$

We try a first approximation to be $x_0 = -3$. Then

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5} = \frac{2(-3)^3 - 1}{3(-3)^2 - 5} = -2.5.$$

So

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5} = \frac{2(-2.5)^3 - 1}{3(-2.5)^2 - 5} = -2.345455.$$

Example (p. 304) II

Subsequence values of approximations are given by the table below. Note that two other initial guess values, namely $x_0 = 1$ and $x_0 = 4$ are also given:

Table 4.5

k	x_k	x_k	x_k
0	−3	1	4
1	−2.500000	−0.500000	2.953488
2	−2.345455	0.294118	2.386813
3	−2.330203	0.200215	2.166534
4	−2.330059	0.201639	2.129453
5	−2.330059	0.201640	2.128420
6	−2.330059	0.201640	2.128419
7	−2.330059	0.201640	2.128419

Remarks

- **Example** (p. 306). The problem of solving $f(x) = g(x)$ involving two functions $f(x)$ and $g(x)$ can be put in the form of $F(x) = f(x) - g(x)$. For example we can use Newton's method to solve $\cos x = x$ which has no known formula for a solution. So

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = \frac{x_n \sin x_n + \cos x_n}{\sin x_n + 1}.$$

- **Example** (p. 307) Find local max/min of $f(x) = e^{-x} \sin 2x$. That is to use Newton's method to solve

$$f'(x) = e^{-x}(2\cos 2x - \sin 2x) = 0.$$

- When $f'(x_n) = 0$ for some x_n during the Newton's method process. Then the method breaks. In fact, the approximation may converge to the real root very slowly or even diverges from the real root.