MATH1013 Calculus I

Introduction to Functions[1]

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Derivatives (Chapter 3)

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[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Instantaneous Velocity

- We are now ready to give a proper meaning of instantaneous velocity that Newton could not make clear. Clearly we can now define, in the language of limit, that the instantaneous velocity or simply the velocity of the object at time t to be

$$v(t) = \lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t}.$$

- We say that $v(t)$ is the instantaneous rate of change of distance at t or simply the rate of change of S at t if the interval of time is clear.

Example revisited

- **Example** Show that the (instantaneous) velocity of the particle in the first example at the beginning of this course is a function of t and find $v(2)$. Recall that $S(t) = 20 + 4t^2$. By Definition, the velocity of the particle is given by the following limit:

$$\begin{aligned} v(t) &= \lim_{\Delta t \to 0} 4 \frac{S(t+\Delta t) - S(t)}{\Delta t} \\ &= 4 \lim_{\Delta t \to 0} \frac{(t+\Delta t)^2 - t^2}{\Delta t} \\ &= 4 \lim_{\Delta t \to 0} \frac{2t\Delta t + \Delta t^2}{\Delta t} \\ &= 4 \lim_{\Delta t \to 0} (2t + \Delta t) = 8t. \end{aligned}$$

- Hence the velocity is a function of t given by $v(t) = 8t$. In particular, the velocity at $t = 2$ is $v(2) = 8(2) = 16$, as expected.

Rate of change

- We may consider rate of change of a given function $f(x)$ not necessarily refereed to time, distance and velocity.
- **Definition** Let $f(x)$ be a function of x, then f is differentiable at x if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

  exists.
- The limit is called the derivative of f at x or the rate of change of f with respect to x, and it is denoted by $f'(x)$.
- Other notations are

$$\frac{df(x)}{dx} \quad \text{or} \quad \left.\frac{df}{dx}\right|_x \quad \text{or} \quad \frac{df}{dx}$$

- If $y = f(x)$, we also write $\frac{dy}{dx}$. We treat this notation as an operator instead of quotient of infinitesimal quantities.
- However, we shall see later that they are interchangeable.

Rate of change

- **Remarks** The definition has assumed a priori the existence of the limit $f'(x)$. However, not every function has a derivative at every point. We will give a counter-example below.
- **Example** Find the derivative of $f(x) = 3x^3 - 2x$. Evaluate $f'(5)$ and $f'(10)$. Consider

$$\begin{aligned}\frac{\Delta f}{\Delta x}(x) &= \frac{f(x+\Delta x) - f(x)}{\Delta x}\\ &= \frac{3(x+\Delta x)^3 - 2(x+\Delta x) - \left(3x^3 - 2x\right)}{\Delta x}\\ &= \frac{(9x^2-2)\Delta x + 9x(\Delta x)^2 + 9(\Delta x)^3}{\Delta x}\end{aligned}$$

  Thus

$$f'(x) = \lim_{\Delta x \to 0}\left[(9x^2 - 2) + 9x(\Delta x) + 9(\Delta x)^2\right] = 9x^2 - 2.$$

  Thus we have $f'(5) = 223$, and $f'(10) = 898$.
- We observe the larger the x the larger the $f'(x) = 9x^2 - 2$ is.

Rate of change

- **Example** It is projected that t years from now, the population of a certain suburban community will be

$$P(t) = 20 - \frac{6}{t+1}$$

thousand. Find the (instantaneous) rate of change of P at time $t = 5$ and $t = 10$.

$$\frac{\Delta P}{\Delta t}(t) = \frac{P(t+\Delta t) - P(t)}{\Delta t}$$
$$= \frac{1}{\Delta t}\left[\left(20 - \frac{6}{(t+\Delta t)+1}\right) - \left(20 - \frac{6}{t+1}\right)\right]$$
$$= \frac{6}{\Delta t}\left[\frac{\Delta t}{(t+1)[(t+\Delta t)+1]}\right] = \frac{6}{(t+1)[(t+\Delta t)+1]}.$$

Thus we have, after taking the limit $\Delta t \to 0$, $P'(t) = \frac{6}{(t+1)^2}$. $P'(5) = 1/6$, and $P'(10) = 6/101^2$. We deduce that the rate of change $P' \to 0$ as $t \to \infty$.

Exercises

- **Exercises** Find the derivatives of the following functions:
- $f(x) = 5x + 1,$ (5)
- $f(x) = -2x + 6,$ (-2)
- $f(x) = ax + b, \quad (a \text{ and } b \text{ are constants})$ (a)
- $f(x) = c, \ (c \text{ a constant})$ (0)
- $f(x) = (3x - 2)^2,$ $\Big(6(3x - 2)\Big)$
- $f(x) = x^2 + 2x,$ $(2x + 2)$
- $f(x) = ax^2 + bx + c,$ $(2ax + b)$
- $f(x) = 1/x,$ $(-1/x^2)$
- $f(x) = 1/x^2,$ $(-2/x^3)$
- $f(x) = 1/x^3,$ $(-3/x^4)$
- $f(x) = x + \dfrac{1}{x}.$ $(1 - 1/x^2)$
- $f(x) = \sqrt{x}.$ $\left(\dfrac{1}{2}x^{-1/2}\right)$

Differentiation rules

- **Theorem** Let c be a constant and that a function f is differentiable at x. Then

$$\frac{d(c\,f)}{dx} = c\,\frac{df}{dx} \qquad \text{or} \qquad (c\,f)'(x) = c\,f'(x).$$

Proof Consider the following limit

$$\lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} = \lim_{\Delta x \to 0} c\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$
$$= c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Differentiation rules

- **Theorem** Let $f(x)$ and $g(x)$ be differentiable at x. Then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{df}{dx} + \frac{dg}{dx}.$$

Proof We have

$$\begin{aligned}
&\frac{d}{dx}(f(x)+g(x)) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-(f(x)+g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)+(g(x+\Delta x)-g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&= \frac{df}{dx} + \frac{dg}{dx}.
\end{aligned}$$

Differentiation of x^n

- **Theorem** Let n be a real number. Then

$$\frac{dx^n}{dx} = nx^{n-1}.$$

Proof We could only provide a proof of the above statement for n to be a natural number. Recall the binomial theorem

$$(a+b)^n = a^n + na^{n-1}b + B_2a^{n-2}b^2 + \cdots + B_{n-1}ab^{n-1} + b^n,$$

where $B_2, \ldots, B_{n-1}$ are certain constants. Thus putting $a = x$ and $b = \Delta x$, we have

$$\begin{aligned}
\frac{dx^n}{dx} &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left(x^n + nx^{n-1}\Delta x + B_2x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n\right) - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \cdots + B_{n-1}x(\Delta x)^{n-1} + (\Delta x)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(nx^{n-1} + \cdots + B_{n-1}x(\Delta x)^{n-1} + (\Delta x)^{n-1}\right) = nx^{n-1}.
\end{aligned}$$

Differentiation exercises

Exercises Applying the above theorems to find the derivatives of the following functions.

- Repeat the questions in the previous exercises, you should obtain the same answer,
- x^{-4}, $x^{8/5}$, $x^{4/5}$, $\frac{3}{2x^2}$ $\quad$ $(-4x^{-5}, \frac{8}{5}x^{3/5}, \frac{4}{5}x^{-1/5}, 3x^{-3})$
- $f(x) = 3x^5 - 4x^3 + 9x - 6$, $\quad$ $(15x^4 - 12x^2 + 9)$
- $f(x) = x^9 - 5x^8 + x + 12$, $\quad$ $(9x^8 - 40x^7 + 1)$
- $f(x) = \frac{1}{4}x^8 - \frac{1}{2}x^6 - x + 2$, $\quad$ $\left(\frac{1}{2}x^7 - 3x^5 - 1\right)$
- $f(x) = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{\sqrt{x}}$, $\quad$ $\left(-\frac{1}{x^2} - \frac{2}{x^3} + \frac{1}{2\sqrt[3]{x^2}}\right)$
- $f(x) = \sqrt{x^3} + \frac{1}{\sqrt{x^3}}$, $\quad$ $\left(\frac{3}{2}\sqrt{x} - \frac{3}{2}\frac{1}{\sqrt{x^5}}\right)$
- $f(x) = -\frac{x^2}{15} + \frac{3}{x} + x^{1/3} + \frac{1}{2x^4}$, $\left(-\frac{2}{15}x - \frac{3}{x^2} + \frac{1}{3}x^{-2/3} - 2x^{-5}\right)$

Differentiation exercises (cont.)

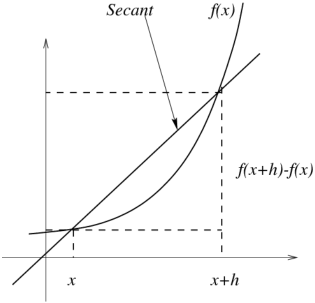
- **Exercises**
- $f(x) = x^3(x^2 - 3\sqrt{x} + 2)$, $\left(5x^4 - \frac{21}{2}x^{5/2} + 6x^2\right)$
- $f(x) = \frac{x^5 - 4x^2}{x^3}$. $\left(2x + \frac{4}{x^2}\right)$
- **Exercises** Find the rate of changes of the following functions at the specified points.
- $f(x) = 2(3x - 1)$; $x = 2, 4$; $(6, 6)$
- $f(x) = x^2$; $x = 1, 3, 5$; $(1, 6, 10)$
- $f(x) = \sqrt{x}$; $x = 1, 3, 5$; $(1/2, 1/2\sqrt{3}, 1/\sqrt{5})$.

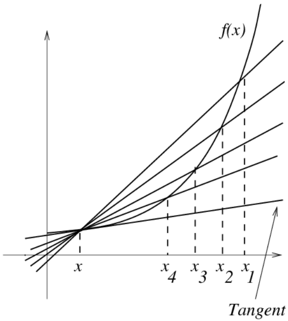
Graphical Interpretation

- Draw a straight line called secant passing through the pair of points $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$. Then the

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

gives the gradient (slope) of the above secant. See the diagram.





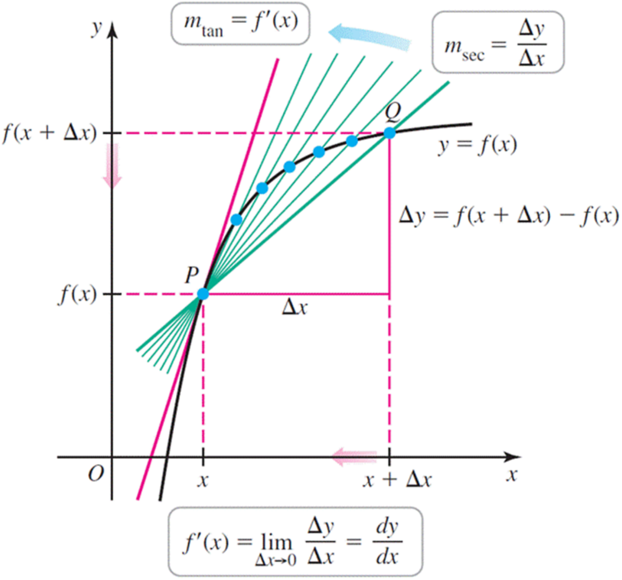
Graphical Interpretation (cont.)

- We choose $\Delta x_1, \Delta x_2, \Delta x_3, \ldots$ with magnitudes decreasing to zero. Then we have a sequence of secants passing through the points $\big((x, f(x)), \ (f(x + \Delta x_i)\big)$.
- The corresponding gradients of the secants are given by

$$m_i = \frac{f(x + \Delta x_i) - f(x)}{\Delta x_i}.$$

- Suppose we already know that f has a derivative at x , we conclude that the sequence of gradients $\{m_i\}$ must tend to $f'(x)$ as Δx_i tends to zero.
- The point $(x + \Delta x, f(x + \Delta x))$ is getting closer and closer to $(x, f(x))$, as $\Delta x \to 0$, they *eventually coincide* to become a single point.
- It follows that the corresponding secants are *tending* to a straight line with only *one* point of contact with f at x. See the diagram from last slide. This line is called the *tangent* to f *at* x.

Publisher's Figure 3.9



Application

- **Example** Suppose a manufacturer's profit of the sale of certain commodity is given by the function

$$P(x) = 400(15 - x)(x - 2),$$

where x is the price at which the commodity is sold. Find the selling price that would maximize the profit.

- The profit function is a quadratic function of x,

$$P(x) = -400x^2 + 6800x - 12000.$$

Since the coefficient of x^2 is negative, we deduce immediately that P must have a maximum, where $P'(x) = 0$ must hold. So we only need to look for points at which $P'(x) = 0$. Thus we solve $0 = P'(x) = -800x + 6800$. Thus there is only one point $x = 6800/800 = 8.5$ at which $P' = 0$. So P must attain its maximum at $x = 8.5$.

Last example's sketch

It is instructive to plot the curve of P and P' on the same coordinate axis. See the diagram on the left below.

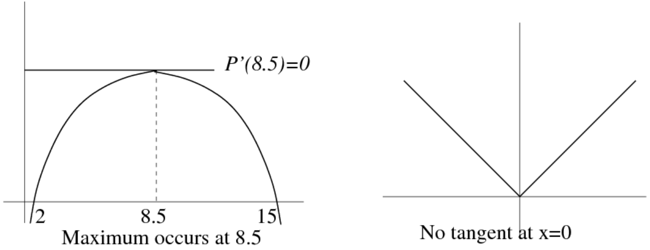


Figure: (Left: Profit function; Right: $|x|$ has no tangent at 0)

An example has no tangent

- **Example** The $|x|$ is not differentiable at 0.
- Consider

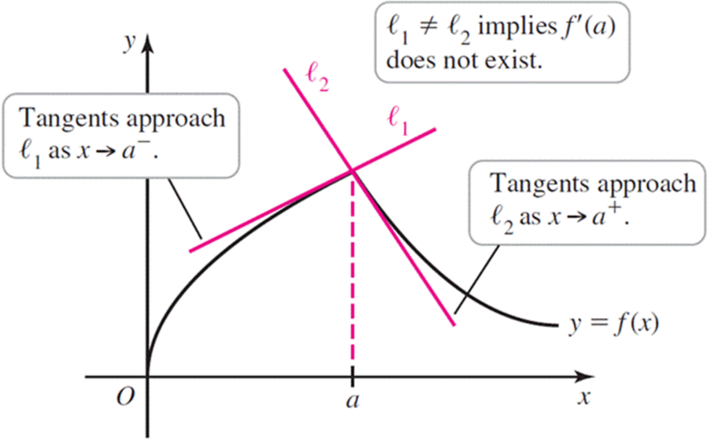
$$\lim_{\Delta x\to 0+}\frac{|0+\Delta x|-|0|}{\Delta x}=\lim_{\Delta x\to 0+}\frac{|\Delta x|}{\Delta x}=\lim_{\Delta x\to 0+}\frac{\Delta x}{\Delta x}=1.$$

- On the other hand, we have, according to the definition of $|x|$ that

$$\lim_{\Delta x\to 0-}\frac{|0+\Delta x|-|0|}{\Delta x}=\lim_{\Delta x\to 0-}\frac{|\Delta x|}{\Delta x}=\lim_{\Delta x\to 0-}\frac{-\Delta x}{\Delta x}=-1.$$

- So the left and right limits are not the same, and we conclude that $|x|$ does not have a derivative at 0 (however, it has derivatives at all other points). It is important to understand the corresponding situation on its graph drawn on last slide (right figure).

Publisher's No tangent figure 3.16



Finding extremal exercises

Determine any max/min for the following $f(x)$:

- $f(x) = x^2 + 2x + 1$, $(x = 0$, min.)
- $f(x) = 3x^2 + 5x + 1$, $(x = \frac{-5}{6}$, min.)
- $f(x) = -\frac{1}{2}x^2 + 3x - 1$, $(x = 3$, max.)
- $f(x) = ax^2 + bx + c$, $(x = -b/2a$, min. if $a > 0$; max. if $a < 0)$
- $f(x) = 3x^3 - 2x - 1$, $(x = 1/\sqrt{3}$, max.; $x = -1/\sqrt{3}$, min.)
- $f(x) = x^2$, $(x = 0$, min.)
- $f(x) = x^3$. $(x = 0$, non)

Finding tangents

- **Example** Find the equation of tangent of $f(x) = x^3$ at the point where $x = -1/2$.
- Since $f' = 3x^2$. The gradient of $f(x)$ at $x = -1/2$ is $f'(-1/2) = 3(-1/2)^2 = 3/4$. We suppose that the equation of the tangent line at $x = -1/2$ is $y = ax + b$. Then it must pass through the point $(-1/2, -1/8)$. Hence $a = +3/4$ and

$$-1/8 = 3/4(-1/2) + b.$$

Thus $b = 1/4$. The tangent equation is $4y = 3x + 1$.

Finding tangents exercises

- **Exercises** Let $y = -3x^2 + 2x$. Find (i) the equation of tangent to y at $x = -1$ and $x = 2$. Find (ii) the $x-$coordinate so that y has a maximum.
  (Ans. (i) The tangent at $x = -1$ is $y = 8x + 3$, and the tangent at $x = 2$ is $y = -10x + 12$. (ii) The maximum occurs at $x = -1/3$.)
- Find the equation of the tangent of the following functions at the specified points:
  - $f(x) = x^2 + 2x + 1;\ x = 1/2,$ $(4y = 12x + 3)$
  - $f(x) = -1/2x^2 + 3x - 1;\ x = 1,$ $(2y = 4x - 1)$
  - $f(x) = 2x^3 - x;\ x = -1,$ $(y = 5x + 4)$
  - $f(x) = 2\sqrt{x+1};\ x = 2.$ $(y = \frac{1}{\sqrt{2}}x + \sqrt{2} + 1)$

First Order Approximation

- When f is differentiable at x, the quotient

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is very close to $f'(x)$, provided that Δx is taken to be small. See the diagram. Hence we have

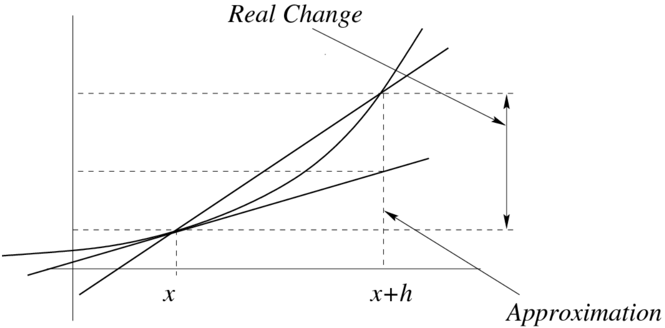
$$\frac{\Delta f}{\Delta x} \approx \frac{dy}{dx} = f'(x).$$

In other words, we have

$$\Delta f = f(x+\Delta x) - f(x) \approx \frac{dy}{dx}\Delta x = f'(x)\Delta x,$$

i.e., the change in f due to a small change Δx at x can be approximated by $f'(x)\Delta x$.

$$\Delta f \approx f'(x)\Delta x,$$



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- In particular, when $\Delta x = 1$, we have

$$\Delta f \approx f'(x),$$

- That is, the $f'(x)$ approximates the change of $f(x)$ when x is increased by one unit.

First Order Approximation

- **Example** Suppose $y = f(x) = x^2$. Find an approximate change of $f(x)$ when x is increased from 2 to 2.5.
- We have $f'(x) = 2x$. And $f'(2) = 2(2) = 4$. Hence

$$\Delta y = y(2 + 0.5) - y(2) \approx y'(2)\,(0.5) = 4\,(0.5) = 2.$$

  Note that the real change of y can be computed directly by $y(2.5) - y(2) = 2.25$. The approximation will become more accurate if we involve changes much smaller than 0.5.
- **Exercise** Repeat the above example when x is increased from 2 to 2.005. How accurate is it?
- **Exercise** Without using the calculus find an approximate value of $3.98^{1/2}$.

Economics example

- **Example** Suppose the total cost in dollars if manufacturing q units of a certain commodity is

$$c(q) = 3q^2 + 5q + 10.$$

If the current level of production is 40 units, estimate the change of cost if 40.5 units are produced.

- By the first order approximation,

$$c(40 + 0.5) - c(40) \approx c'(40)\,(0.5).$$

- But $c'(q) = 6q + 5$ so that $c'(40) = 245$. We deduce that the change in cost to be

$$c(40.5) - c(40) \approx 245 \times 0.5 = 122.5$$

Application: Marginal Analysis

- Economists are interested in the change $f(x+1) - f(x)$. Suppose that f is differentiable at x, then the first order approximation gives

$$\Delta f = f(x+1) - f(x) \approx f'(x)\,\Delta x = f'(x),$$

  i.e., the $f'(x)$ approximates the change in f, when x is increased by 1 (assuming f is well-behaved). The above is called *marginal analysis* by economists.
- Let $c(q)$ and $r(q) = qp(q)$ to denote total cost function and total revenue function respectively (the $p(q)$ is the price function). Then

$$c'(q) = \frac{dc}{dq}, \qquad \text{and} \qquad r'(q) = \frac{dr}{dq}$$

  are called the *marginal cost* and *marginal revenue* respectively.

Marginal Analysis example

- Thus

  $\Delta c = c(q+1)-c(q) \approx c'(q)$ cost of prod'ing $(q+1)^{\text{th}}$unit,

  and

  $\Delta r = r(q+1)-r(q) \approx r'(q)$ revenue of prod'ing $(q+1)^{\text{th}}$unit.

- **Example** Let the cost function of certain commodity be govened by $c(q) = \frac{1}{8}q^2 + 3q + 98$. Approximate the cost for producing the 9th unit.

  The marginal cost is

  $$c'(q) = \frac{1}{4}q + 3.$$

  Thus the approximate cost of producing the 9th unit is

  $$c(8+1) - c(8) \approx c'(8)(1) = \frac{1}{4}(8) + 3 = 5.$$

  The real change can be computed by $c(9) - c(8) = 5.13$.

Marginal Analysis example

- **Example** Suppose the cost function $c(q)$ is as in the last Example, and that $p(q) = \frac{1}{3}(75 - q)$ is the selling price of the commodity.
  - Estimate the revenue of the commodity for the 9th unit.
  - Estimate of the profit of selling of the 9th unit.
- We note that the revenue function is given by $r(q) = qp(q)$. So the profit of producing the 9th unit is

$$r(9) - r(8) \approx r'(8) = 25 - \frac{2}{3}q = 19\frac{2}{3} \quad \text{at } q = 8.$$

Since the profit is given by $P(9) - P(8)$ and $P(q) = r(q) - c(q)$ is the profit function. Then

$$P(9) - P(8) \approx P'(8) = 44/3.$$

Marginal Analysis exercises

Using the marginal cost and revenue to estimate

- the cost of producing the fourth unit;
- the revenue from the sale of the fourth unit;
- the profit of the selling of the fourth unit, for the following functions for the following cost and price functions
  - $c(q) = \frac{1}{5}q^2 + 4q + 57; p(q) = \frac{1}{4}(36 - q);$ $(c'(3) = 5.20,\ r'(3) = 7.20)$
  - $c(q) = \frac{1}{3}q^2 + 2q + 39; p(q) = -q^2 + 4q + 10;$ $(c'(3) = 4,\ r'(3) = 7)$
  - $c(q) = \frac{1}{4}q^2 + 43; p(q) = \frac{3 + 2q}{1 + q}.$ $(c'(3) = 1.5,\ r'(3) = 33/16)$

More Rules for Differentiation

- **Theorem** Suppose f is differentiable at a then f must also be continuous at a.
- When a function is differentiable at x, i.e., $f'(x)$ exists, it means that the curve of f has a tangent at x. For it is not difficult to see that f must be *nice* there. That is, f is continuous at x.
- **Proof** We need to show $\lim_{x\to a} f(x) = f(a)$. Consider

$$\begin{aligned}\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)\\ &= f'(a)\cdot 0 = 0.\end{aligned}$$

- So $\lim_{x\to a} f(x) = f(a)$.

Product Rule

- **Theorem** Let $f(x)$ and $g(x)$ both be differentiable at x. Then
$$\frac{d(fg)}{dx} = g(x)\frac{df(x)}{dx} + f(x)\frac{dg(x)}{dx}.$$

- **Proof**
$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(f(x+\Delta x)-f(x))g(x+\Delta x)+f(x)(g(x+\Delta x)-g(x))}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(f(x+\Delta x)-f(x))g(x+\Delta)}{\Delta x}$$
$$+ \lim_{\Delta x\to 0}\frac{f(x)(g(x+\Delta x)-g(x))}{\Delta x}$$
$$= f'(a)\cdot \lim_{\Delta x\to 0} g(x+\Delta x) + \lim_{\Delta x\to 0} f(x)\cdot g'(x)$$
$$= f'(x)\cdot g(x) + f(x)\cdot g'(x).$$

Product Rule examples

- **Example** Let $F(z) = (x+1)(x^2+1)$. Find F'.
- Writing $f(x) = x+1$, $g(x) = x^2+1$. We have, by the product rule,

$$\begin{aligned}
F'(z) &= (x^2+1)\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}(x^2+1) \\
&= (x^2+1)(1) + (x+1)(2x) \\
&= x^2 + 1 + 2x^2 + 2x \\
&= 3x^2 + 2x + 1.
\end{aligned}$$

- **Example** Find $(x^5)'$.
- Writing $x^5 = x^2 \cdot x^3$. Then

$$\frac{d}{dx}(x^5) = (x^3)(x^2)' + (x^2)(x^3)' = 2(x^3)(x) + 3(x^2)(x^2) = 5x^5.$$

Product Rule examples II

- **Example** $F(x) = (5x^4 + 2x^3 + 1)(2x^2 + x)$. Find $F^(x)$. Then

$$F'(x) = (5x^4 + 2x^3 + 1)'(2x^2 + x) + (5x^4 + 2x^3 + 1)(2x^2 + x)'$$
$$= (20x^3 + 6x^2)(2x^2 + x) + (5x^4 + 2x^3 + 1)(4x + 1).$$

- **Example** Let $F(x) = \frac{1}{x}(x^{-1} + 2x^2)$.

$$\begin{aligned} F'(x) &= (1/x)'(x^{-1} + 2x^2) + (1/x)(x^{-1} + 2x^2)' \\ &= (-1/x^2)(x^{-1} + 2x^2) + (1/x)(-x^{-2} + 4x) \\ &= -\frac{2}{x^3} - 2. \end{aligned}$$

- **Exercises** Find the derivatives of the following functions
  - $f(x) = (3x - 1)(2x + 1)$,
  - $f(x) = (x - 4)(1 - 3x)$,
  - $f(x) = 173(12 - x^2)(3x - 2)$,
  - $f(u) = 12(3u + 1)(1 - 5u)$,
  - $f(x) = -2(5x^2 - 4x^3 + 5)$.

Quotient Rule

- **Theorem** Let $f(x)$ and $g(x)$ both be differentiable at x and that $g'(x) \neq 0$. Then

$$\frac{d}{dx}\left(\frac{f}{g}(x)\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

- **Proof**

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right) = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x+h)\,g(x)}$$

$$= \lim_{h\to 0}\frac{\left(f(x+h) - f(x)\right)g(x) - \left(g(x+h) - g(x)\right)f(x)}{h\,g(x+h)\,g(x)}$$

$$= \lim_{h\to 0}\frac{\left(f(x+h) - f(x)\right)g(x)}{h\,g(x+h)\,g(x)} - \lim_{h\to 0}\frac{\left(g(x+h) - g(x)\right)f(x)}{h\,g(x+h)\,g(x)}$$

$$= \frac{f'(x)}{g(x)} - \frac{g'(x)\,f(x)}{g(x)^2}.$$

Quotient Rule examples I

- **Example** Let $F(x) = 1/(ax+b)$. Find $F'(x)$.
  Writing $f(x) = 1$ and $g(x) = ax+b$. Then

$$F'(x) = \frac{(ax+b)(1)' - (1)(ax+b)'}{(ax+b)^2} = \frac{0-a}{(ax+b)^2} = \frac{-a}{(ax+b)^2}.$$

- **Example** Let $F(x) = \dfrac{x}{3x^2+1}$. Find $F'(x)$.
  Writing $f(x) = x$ and $g(x) = 3x^2+1$. Then

$$F'(x) = \frac{(3x^2+1)(x)' - (x)(3x^2+1)'}{(3x^2+1)^2} = \frac{(3x^2+1) - x(6x)}{(3x^2+1)^2} = \frac{-3x^2+1}{(3x^2+1)^2}.$$

Quotient Rule example II

- **Example** Let $F(x) = \dfrac{1}{x + \frac{1}{x+1}}$. Find $F'(x)$. Since

$$F(x) = \frac{1}{\frac{x(x+1)+1}{x+1}} = \frac{x+1}{x(x+1)+1} = \frac{x+1}{x^2+x+1}.$$

So

$$\begin{aligned} F'(x) &= \frac{(x^2+x+1)(x+1)' - (x+1)(x^2+x+1)'}{(x^2+x+1)^2} \\ &= \frac{(x^2+x+1) - (x+1)(2x+1)}{(x^2+x+1)^2} \\ &= \frac{(x^2+x+1) - (2x^2+3x+1)}{(x^2+x+1)^2} \\ &= \frac{-x^2-2x}{(x^2+x+1)^2}. \end{aligned}$$

Quotient Rule examples III

- **Example** Let $f(x) = \dfrac{4x^4 - 2x^2}{3x^2 - 1}$. Find $f'(x)$.

- Find the equation of the tangent line to the curve $y = \dfrac{x}{2x+3}$ at the coordinate $x = -1$.

(Ans. $y = 3x + 2$)

Quotient Rule exercises

Differentiate the following functions:

- $f(x) = \dfrac{1}{2x+1}$, $\left(-2/(2x+1)^2\right)$
- $f(x) = 1/(x+1)$, $\left(-1/(x+1)^2\right)$
- $f(x) = 1/(x+1)^2$, $\left(-2/(x+1)^3\right)$
- $f(x) = 1/(x+1)^3$, $\left(-3/(x+1)^4\right)$
- $f(x) = \dfrac{x+1}{x-1}$, $\left(-2/(x-1)^2\right)$
- $f(x) = \dfrac{4x-1}{5x+4}$, $\left(21/(5x+4)^2\right)$
- $f(x) = \dfrac{t^2+1}{1-2t^2}$, $\left(6t/(1-2t^2)^2\right)$
- $f(x) = \dfrac{x^2-3x+2}{2x^2+5x-1}$, $\left(\frac{11x^2-10x-7}{(2x^2+5x-1)^2}\right)$
- $f(x) = \dfrac{(2x-1)(x+3)}{x+1}$, $\left(\frac{2(x^2+2x+4)}{(x+1)^2}\right)$

How does composition change?

- Suppose that $y = g(u)$ and $u = f(x)$. i.e., is y is a function of u and u is a function of x.
- When we compose to get $y = g(f(x))$, which is now a function of x, written as $y = h(x)$.
- If x is changed to $x + \Delta x$, there's a corresponding change in u,

$$u + \Delta u = f(x + \Delta x).$$

- As a result, it will induce a further change in y. Hence

$$y + \Delta y = g(u + \Delta u).$$

Example

- **Example** Let $y = u^3 + 1$ and $u = 2x - 4$. Find the increases of y and u due to an increase of x from x to $x + \Delta x$ where Δx is a small increment of x. When x is increased to $x + \Delta x$ the change in u is

$$\begin{aligned} u(x + \Delta x) - u(x) &= [2(x + \Delta x) - 4] - (2x - 4) \\ &= 2\Delta x. \end{aligned}$$

$$\begin{aligned} \Delta u &= u(x + \Delta x) - u(x) \\ &= 2\Delta x. \end{aligned}$$

$$\begin{aligned} y(x + \Delta x) - y(x) &= y(u + \Delta u) - y(u) \\ &= [(u + \Delta u)^3 + 1] - [u^3 + 1] \\ &= (u^3 + 3u^2(\Delta u) + 3u(\Delta u)^2 + 1) - (u^3 + 1) \\ &= 3u^2(\Delta u) + 3u(\Delta u)^2 \\ &= 3(2x - 4)^2(2\Delta x) + 3(2x - 4)(2\Delta x)^2 \end{aligned}$$

Chain Rule

- **Theorem** *Let* $y = g(u)$, $u = f(x)$ *and* $b = f(a)$, $c = g(b)$. *Suppose that* g *is differentiable at* $u = b$, *and that* f *is differentiable at* $x = a$. *Then the function* $y = h(x) = g(f(x))$ *is also differentiable at* $x = a$, *and the relationship is given by*

$$\left.\frac{dh}{dx}\right|_{x=a} = \left.\frac{dg}{du}\right|_{u=b} \times \left.\frac{df}{dx}\right|_{x=a},$$

or

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=b} \times \left.\frac{du}{dx}\right|_{x=a},$$

or simply

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Chain Rule examples

- Let $y = u^3 + 1$ and $u = 2x - 4$. Find the derivative of y with respect to x by (i) chain rule, (ii) by direct differentiation.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= (3u^2)(2) = 6u^2 = 6(2x-4)^2.$$

$$y = u^3 + 1 = (2x-4)^3 + 1$$
$$= 8(x-2)^3 + 1 = 8[x^3 - 3x^2 \cdot 2 + 3x2^2 - 8] + 1$$
$$= 8x^3 - 48x^2 + 96x - 63.$$

Hence

$$\frac{dy}{dx} = 24x^2 - 96x + 96$$
$$= 24(x^2 - 4x + 4) = 24(x-2)^2.$$

We note that the two answers obtained above are the same.

Chain Rule examples

- **Example** Let $F(x) = \sqrt{7 - x^3}$. Find $F'(x)$.
  Let $u = 7 - x^3$ and $y = f(u) = \sqrt{u}$. Then $F(x) = f(u) = f(7 - x^3)$. Hence

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} &= \frac{1}{2}u^{-1/2} \cdot \frac{d}{dx}(7 - x^3) \\ &= \frac{1}{2\sqrt{u}}(-3x^2) = \frac{-3x^2}{2\sqrt{7 - x^3}}.\end{aligned}$$

- **Example** Let $y = (3x - 4)^{20}$. Find dy/dx.
  Let $u = 3x - 4$. Then $y = u^{20}$. So

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} &= \frac{d}{du}(u^{20})\frac{d}{dx}(3x - 4) \\ &= 20u^{19}(3) = 60(3x - 4)^{19}.\end{aligned}$$

Chain Rule examples

- **Example** $y = \dfrac{1}{(x+3)^3}$. Find dy/dx.

  Let $u = x + 3$. Then $y = 1/u^3 = u^{-3}$. Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}u^{-3} \cdot \frac{d}{dx}(x+3)$$
$$= -3u^{-4} \cdot (1) = \frac{-3}{u^4} = \frac{-3}{(x+3)^4}.$$

- **Example** $y = \left(\dfrac{2x-1}{1-x}\right)^3$. Find dy/dx.

  Let $u = \dfrac{2x-1}{1-x}$. Then $y = u^3$. So

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(u^3)\frac{d}{dx}\left(\frac{2x-1}{2-x}\right) = 3u^2 \cdot \frac{d}{dx}\left(\frac{2x-1}{2-x}\right)$$
$$= 3\left(\frac{2x-1}{1-x}\right)^2 \cdot \frac{2-2x+2x-1}{(1-x)^2} = 3\frac{(2x-1)^2}{(1-x)^4}.$$

Chain rule exercises

- **Exercise** Find y' given that (i) $y(x) = \sqrt{3x^2 - 4x + 1}$, (ii) $y = (2x+3)^7$.

  (Ans.(i) $\frac{3x-2}{\sqrt{3x^2-4x+1}}$, (ii) $14(2x+3)^6$.)

- **Exercises** Find y' if (i) $y = \dfrac{1}{(2x+3)^4}$, (ii) $y = \left(\dfrac{2x+1}{3x-1}\right)^5$.

  (Ans. (i) $\frac{-8}{(2x+3)^5}$, (ii) $\frac{-15}{(3x-1)^2}\left(\frac{2x+1}{3x-1}\right)^4$.)

- **Exercises** Find the equation of the tangent line to $y = (3x^2+1)^2$ at the point where $x = -1$. (Ans. $y = -48x - 32$)

- **Exercises** Find all values of x where the tangent line to $y = \dfrac{x}{(3x-2)^2}$ is horizontal. (Ans. $x = 0, -1, -1/2$)

More Chain rule exercises

Find y' of the following functions:

- $y = (3x-2)^2 + 1,$ $\quad (6(3x-2))$
- $y = \sqrt{x^2+2x-3},$ $\quad \left(\frac{(x+1)}{\sqrt{x^2+2x-3}}\right)$
- $y = \dfrac{1}{(x^2+1)^2},$ $\quad \left(\frac{-4x}{(x^2+1)^3}\right)$
- $y = \dfrac{1}{\sqrt{x^2-1}},$ $\quad \left(\frac{-x}{(x^2-9)^{3/2}}\right)$
- $y = \dfrac{1}{x^2-1},$ $\quad \left(\frac{-2x}{(x^2-1)^2}\right)$
- $y = 3(x^3-2x-5)^4 - 4(x^3-2x-5) + 5; (x=2)$ $\quad (-160)$
- $y = \sqrt{x^2-2x+6}; (x=3);$ $\quad (2/3)$
- $y = \dfrac{1}{3-1/x^2}; (x = 1/2),$ $\quad (-16)$
- $y = (2x+1)^4,$ $\quad (8(2x+1)^3)$
- $y = (x^5-4x^3-7)^8,$ $\quad (8x^2(x^5-4x^3-7)(5x^2-12))$

More Chain rule exercises (cont.)

Find y' of the following functions:

- $y = \dfrac{1}{5t^2 - 6t + 2}$, $\left(\frac{-2(5t-3)}{(5t^2-6t+2)^2}\right)$
- $y = \dfrac{1}{\sqrt{4x^2 + 1}}$, $\left(\frac{-4x}{(4x^2+1)^{3/2}}\right)$
- $y = 3/(1 - x^2)^4$, $\left(\frac{24x}{(1-x^2)^5}\right)$
- $y = (x + 2)^3(2x - 1)^5$, $\left((x + 2)^2(2x - 1)(16x + 17)\right)$
- $y = \sqrt{(3x + 1)/(2x - 1)}$, $\left(- 5/2(3x + 1)^{-1/2}(2x - 1)^{-3/2}\right)$
- $y = \dfrac{(x + 1)^5}{(1 - x)^4}$. $\left(\frac{(x+1)^4(9-x)}{(1-x)^5}\right)$
- $y = \dfrac{3x + 1}{\sqrt{1 - 4x}}$, $\left(\frac{5-6x}{(1-4x)^{3/2}}\right)$

Proof of Chain rule I

Recall that $y = g(u)$, $u = f(x)$ and $b = f(a)$, $c = g(b)$. And that g is differentiable at $u = b$, and that f is differentiable at $x = a$. So $y = h(x) = g(f(x))$. Thus

$$\begin{aligned} h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} &= \frac{g(f(x)) - g(f(a))}{x - a} \\ &= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \\ &= \lim_{u \to b} \frac{g(u) - g(b)}{u - b} \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= g'(b) \cdot f'(a) \end{aligned}$$

since $f(x) \to f(a)$ as $x \to a$, so we must have $u \to b$ as $x \to a$. But this argument is incomplete since $f(x) = f(a)$ and hence $1/(u - b) =$ "1/0" would be meaningless in the limiting process.

Proof of Chain rule II

We can consider

$$G(u) = \begin{cases} \dfrac{g(u) - g(b)}{u - b}, & \text{if } u \neq b; \\ g'(b), & \text{if } u = b \end{cases}$$

which is a continuous at and around $u = b$ $(x = a)$. Thus

$$\begin{aligned} h'(a) &= \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \\ &= \lim_{\substack{u \to b \\ \text{"if meaningful"}}} \frac{g(u) - g(b)}{u - b} \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{u \to b} G(u) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= g'(b) \cdot f'(a) \end{aligned}$$

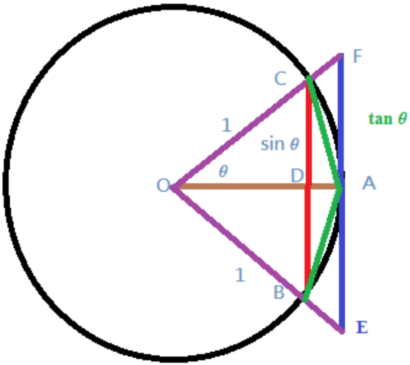
since $\lim_{u \to b} G(u) = G(b) = g'(b)$. This completes the proof.

Sine function

Area of the circular sector $OBACO$ is $\frac{1}{2}r^2\theta = \frac{1}{2}\theta$ since $r = 1$ and area $\Delta OAC = \frac{1}{2}(1)\sin\theta$. We see

$$2\,\text{area }(\Delta OAC) < \text{area (sector } OBACO) < \text{area }(\Delta OEF)$$

We deduce $\sin\theta < \theta < \tan\theta = \sin\theta/\cos\theta,$



Limit of Sine

It follows from

$$\sin\theta < \theta < \tan\theta = \frac{\sin\theta}{\cos\theta}.$$

That is,

$$\frac{\cos\theta}{\sin\theta} < \frac{1}{\theta} < \frac{1}{\sin\theta}$$

Hence

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Since $\lim_{\theta\to 0}\cos\theta = 1$, so the Squeeze theorem implies that

$$\lim_{\theta\to 0+}\frac{\sin\theta}{\theta} = 1.$$

One can show the $\lim_{\theta\to 0-}\frac{\sin\theta}{\theta} = 1$ also holds. Hence $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

Derivative of $\sin x$

We recall the sine addition formula:

$$\sin(a+b) = \sin a\,\cos b + \sin b\,\cos a$$

Then

$$\begin{aligned}
\sin'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin h}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \cos x \cdot \frac{\sin h}{h} \\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 = \cos x
\end{aligned}$$

Other trigonometric functions

- We omit the proof of $\cos' x = -\sin x$.
- When $\sin x$ reaches its maxima/minima, $0 = \sin' x = \cos x$. That is, when

$$x = \frac{(2n+1)\pi}{2}, \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

- **Exercises** Where are the maxima/minima of $\cos x$? (p. 164)

$$\left(\frac{1+\sin x}{1-\sin x}\right)', \quad \tan' x = \sec^2 x, \quad \cot' x = -\csc^2 x$$

$$\sec' x = \sec x \tan x, \quad \csc' x = -\csc x \cot x, \quad (e^x \sin x)'.$$