MATH1013 Calculus I

Introduction to Functions[1]

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Functions (Chapter 1, to be completed)

[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Definition of functions

- **Definition** A function is a rule f that assigns to each x in a set D a unique value denoted $f(x)$. $\mathbb{C}$.
- **Definitions** The set D is called the domain of the function f, and the set of values of $f(x)$ assumes, as x varies over the domain, is called the range of the function $f(x)$.
- $$x \longmapsto f(x), \qquad \text{or} \qquad y = f(x),$$
- One can think of this as a model of

  one input $\rightarrow$ **one output**

- Important point: for **each** x in D, one can find (there exists) **one** value $f(x)$ (or y) that corresponds to it.
- However, depending on the f under consideration, one could have **two or more** x that correspond to the same $f(x)$.
- This strange looking idea was created to describe dynamical

Examples of functions

- (p. 1) $x \mapsto x^2 - 2x$ or $f(x) = x^2 - 2x$.
- (p. 2) Identifying functions
- (p. 2) Domain and range
  1. $y = f(x) = x^2 + 1$, $[-3, 3] \times [-1, 5]$
  2. $y = g(t) = \sqrt{4 - t^2}$, $[-3, 3] \times [-1, 3]$
  3. $y = h(u) = \dfrac{1}{u - 1}$, $[-3, 5] \times [-4, 4]$
  4. (Ex.1.1, Q 16) $F(w) = \sqrt[4]{2 - w}$, $[-3, 2] \times [0, 2]$
  5. (Ex.1.1, Q 14) $g(y) = \dfrac{y + 1}{(y + 2)(y - 3)}$, $[-4, 6] \times [-3, 3]$
- (p.3: In context) At time $t = 0$ a stone is thrown vertically upward from the ground at a speed of $30m/s$. Its height above the ground in meters is approximated by the function $h = f(t) = 30t - 5t^2$, where t is in seconds. Find the domain and range of this function as they apply to this particular problem.

Different types of functions

- $y = f(x) = x + 1$. For each x there corresponds to one and only one y.
- $y = x^3$. For each x there corresponds to one and only one y
- Where $f(x_1) = f(x_2)$ implies $x_1 = x_2$, or equivalently $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, we say the function f is injective or one-one. So the above two examples are injective functions.
- (Eg revisited) $f(x) = x^2 - 2x$ is not injective, as two different x can correspond to the same $f(x_1) = y = f(x_2)$
- (Non-function) $y^2 = 1 - x^2$. Since for each x input, there always correspond to two outputs of $f(x) = \pm\sqrt{1 - x^2}$ within the domain of f.

Function indicative figures

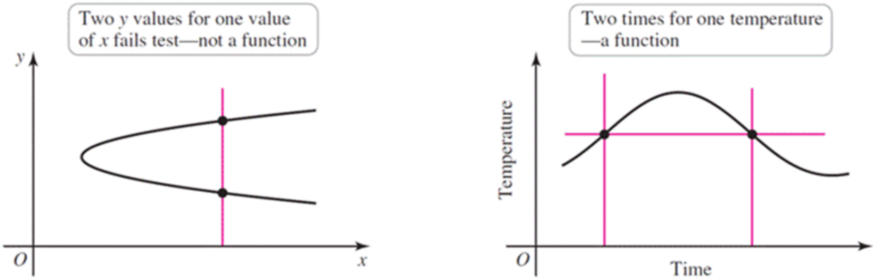

Figure: 1.2 (source textbook)

A quick test

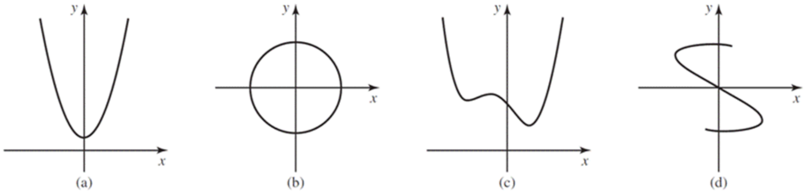

Figure: 1.3 (source textbook)

Domain and Range figure I

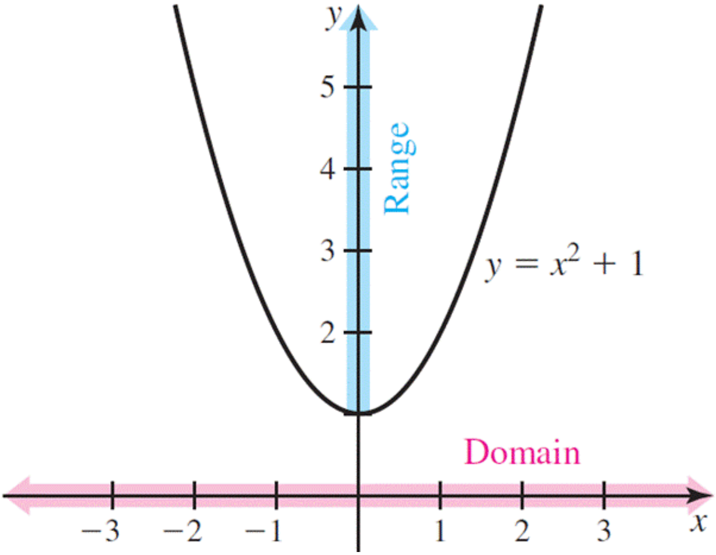

Figure: 1.4 (source: textbook)

Domain and Range figure II

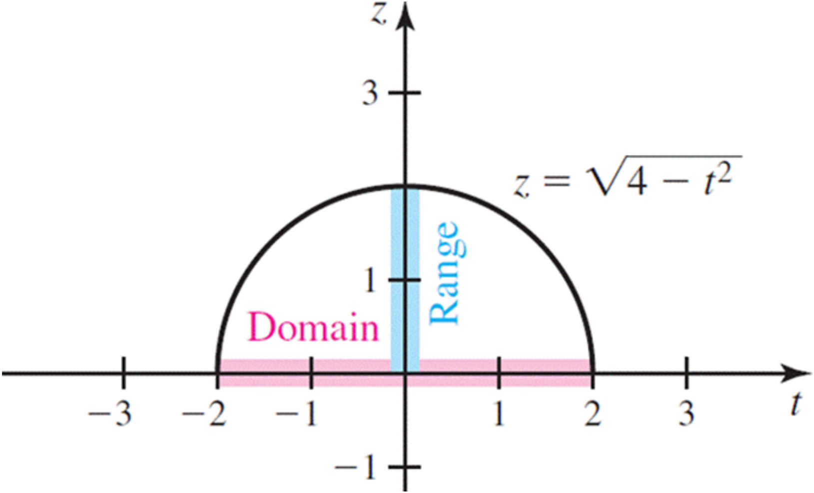

Figure: 1.5 (source: textbook)

Domain and Range figure III

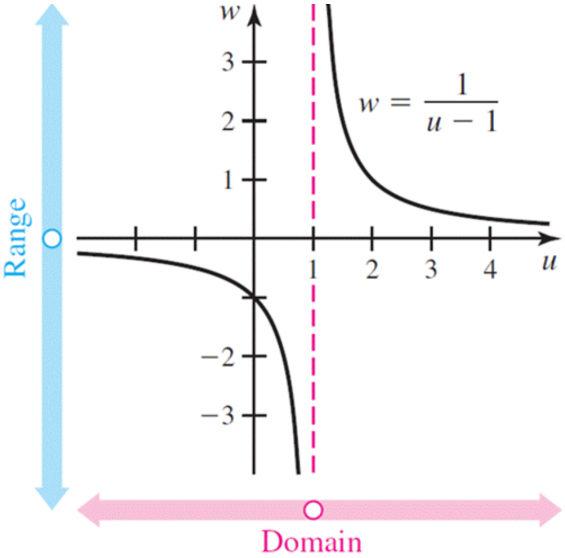

Figure: 1.6 (source: textbook)

Composition

- **Definition** Given two functions f and g, their composition $f \circ g$ is defined, by

$$(f \circ g)(x) = f(u) = f(g(x))$$

for each x in the domain of $f \circ g$. Let $u = g(x)$ and $y = f(u)$, then $f \circ g$ is understood as

$$y = (f \circ g)(x) = f(g(x)) = f(u), \qquad u = g(x),$$

- as shown in

$$x \longmapsto u = g(x) \longmapsto y = f(u)$$

- with g takes the domain of g (range) into (part of) domain of f, and f maps that into (part of) the range of f. The two together thus forms a new function $f \circ g$.

Diagram of composition

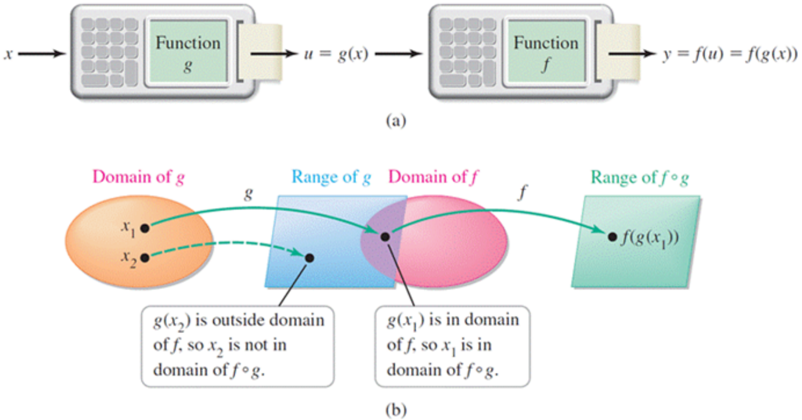

Figure: 1.8 (source: textbook)

Examples of composition

- (p. 4) Let $f(x) = 3x^2 - x$ and $g(x) = 1/x$.
  1. $f \circ g$
  2. $g \circ f$
  3. domain and ranges
- (p. 4) Recognising composition
  1. $h(x) = \sqrt{9x - x^2}$
  2. $h(x) = \dfrac{2}{(x^2 - 1)^3}$
- (p. 5) Given $f(x) = \sqrt[3]{x}$ and $g(x) = x^2 - x - 6$.
  1. $f \circ g$
  2. $g \circ f$

Composite functions Eg from table

- (p. 10, Ex 56)

x	-1	0	1	2	3	4
$f(x)$	3	1	0	-1	-3	-1
$g(x)$	-1	0	2	3	4	5
$h(x)$	0	-1	0	3	0	4

- Compute
  - (a) $h(g(0))$,
  - (b) $g(f(4))$,
  - (c) $h(h(0))$,
  - (d) $g(h(f(4)))$,
  - (e) $f(f(f(1)))$,
  - (f) $h(h(h(0)))$,
  - (g) $f(g(h(2)))$

Examples of composition: Gradients

- (p. 6) Given $f(x) = 3x^2 - x$. Find $\dfrac{f(x+h) - f(x)}{h}$
- The above quantity is called the gradient (or *slope* in less formal language) of f between the two points x and $f(x+h)$. Alternatively, it is the average rate of change of f between x and $x + h$. It is an important quantity as we enter Chapters two and three.
- (p. 6) let $I = \frac{P}{4\pi r^2}$ measures sound intensity in watts per square meter (W/m^2), at a point r meters from a sound source with acoustic power $P = 100\,W$. Find the gradient of the secant line through (i) the points $(r_1,\ I(r_1))$ and $(r_2,\ I(r_2))$ $(-\frac{P(r_1+r_2)}{4\pi r_1^2 r_2^2}\ W/m^2)$, and (ii) the points $(10,\ I(10))$ and $(15,\ I(15))$ $(-1/(36\pi)\ W/m^2)$.

Representing Functions

- Polynomial functions

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the $a_{n-1}, \ldots a_1, a_0$ are some constants.

- Rational functions

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0},$$

where the $a_n, \ldots a_1, a_0, b_m, \ldots b_1, b_0$ are constants

- Piecewise function (p. 15)

$$f(x) = \begin{cases} x & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ -\frac{1}{2}x + 5 & \text{if } x > 2. \end{cases}$$

Transformations

- Given a function $y = f(x)$ and its graph.
- Then the function

$$y = f(x - b) + d,$$

where $b > 0$, d are constants, has the same shape as $f(x)$ but which is shifted to the right by b units and shifted up/down by d units.
- Then the function

$$y = f(x + b) + d$$

where $b > 0$, d are constants, has the same shape as $f(x)$ but which is shifted to the left by b units and shifted up/down by d units.
- The graph of the function

$$y = c\,f(a\,x)$$

has horizontal magnification factor a, and vertical magnification factor b.

Transformation figure I

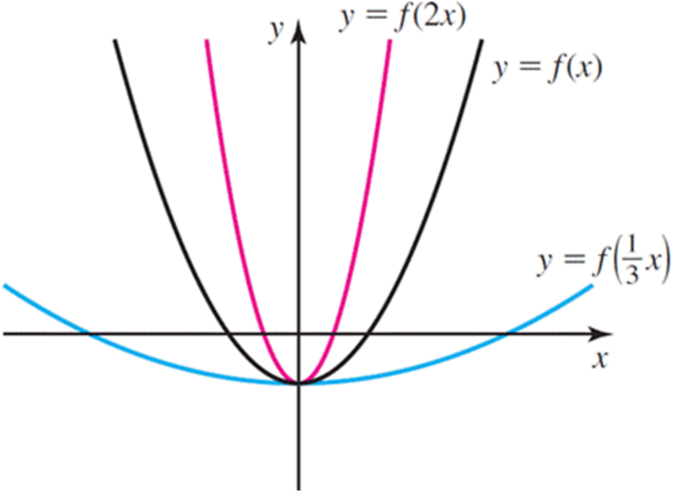

Figure: 1.38 (source: textbook)

Transformation figure II

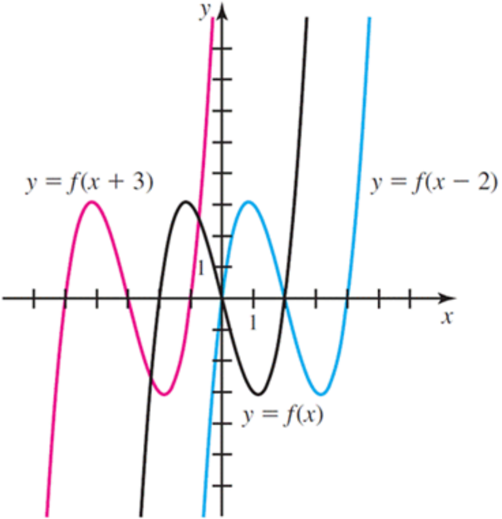

Figure: 1.35 (source: textbook)

Examples of Transformation

- (p. 20, E.g. 8) Sketch the graph of $y = (x-2)^2 - 3$
- (p. 15, E.g. 4b) Recall the absolute value function is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

- (p. 20, E.g. 9) Sketch $|2x+1|$.
- (p. 21, Ex. 10) Sketch the graph of $y = 4(x+3)^2 + 6$
- (p. 23, Ex. 44) Sketch the graph according to Exercise 44:

$$f(x) = \begin{cases} |x| - 1 & \text{if } |x| \geq 2; \\ |x| & \text{if } |x| < 1. \end{cases}$$

(a) $y = -f(x)$, (b) $y = f(x+2)$, (c) $y = f(x-2)$, (d) $y = f(2x)$, (e) $y = f(x-1) + 2$, (f) $y = 2f(x)$.

Function made up by two pieces

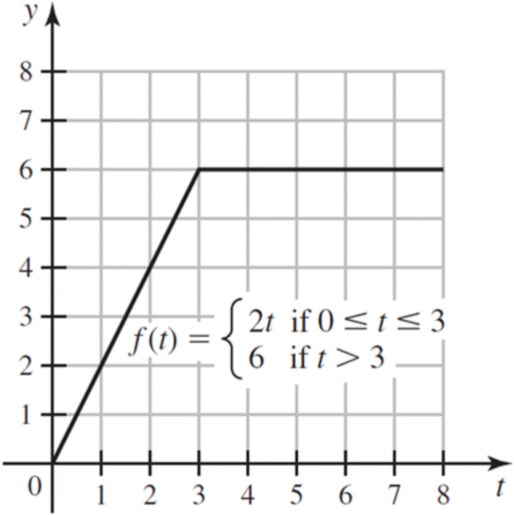

Figure: 1.30 (source: textbook)

Function made up by three pieces

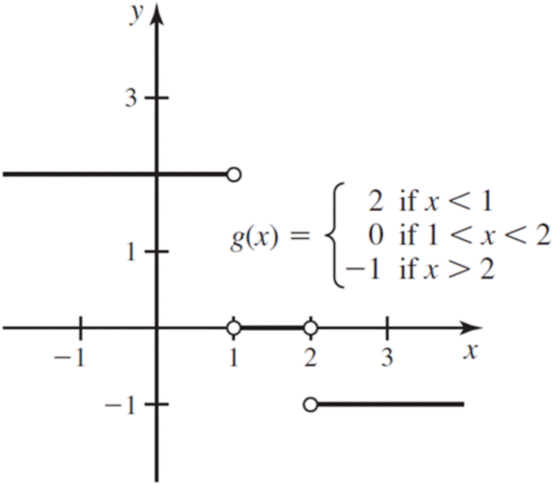

Figure: 1.29 (source: textbook)

Exponential Functions

- The function
$$y = f(x) = 2^x$$
is called binary exponential function with base 2.
- Let $b \neq 1$ be an arbitrary positive number. Then function
- The function
$$y = f(x) = b^x$$
is called exponential function with base b.
- The exponential function grows very fast without bound if $b > 1$ and tends to zero fast.
- Amongst all exponential functions, the natural exponential function
$$y = f(x) = e^x$$
where $2 < e \approx 2.71828 < 3$ is amongst the most important.
- The simpler one is the one with common base 10
$$y = f(x) = 10^x$$

Logarithm

- $b^x \times b^y = b^{x+y}, \qquad b^x \div b^y = b^{x-y}$
- $(b^x)^y = b^{x \cdot y}$.
- These observations, about turning multiplication or division into addition or subtraction of the exponents respectively, may seem elementary, but it actually becomes so important in all kinds of applications, some quite unexpected.
- The application of these rules were known since Babylonians' time in 2000-1600 BC. But it was the Scottish John Napier (1550-1617) who wrote the book entitled *Mirifici Logarithmorum Canonis Descriptio* (Description of the Wonderful Rule of Logarithms) in 1614 that popularized their use (source from Wiki)
- First called artificial number, then "logarithm" meaning from Latin "proportional-arithmetic
- Applications: http://en.wikipedia.org/wiki/Logarithm

John Napier

Figure: (1550-1617) Source from Wiki

Tables published by Cambridge

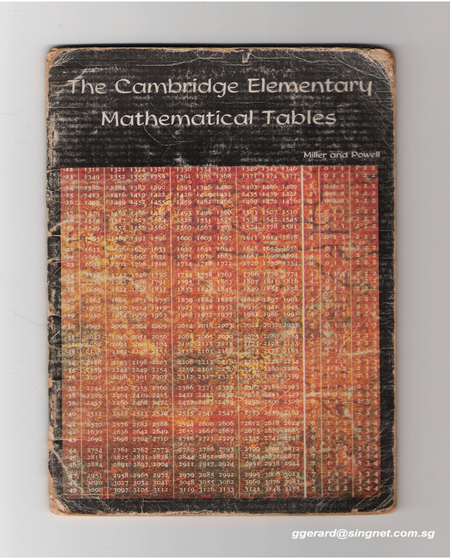

Figure: (Source from Wiki)

A page in Log tables

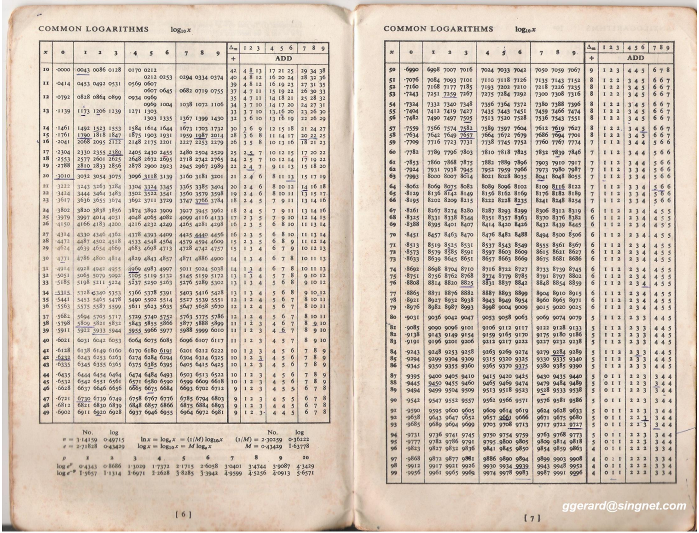

Figure: (Source from Wiki)

Notation of Logarithm

- Let us recall that after agreeing to a base b, we make use of

$$b^x \times b^y = b^{x+y}, \quad b^x \div b^y = b^{x-y}.$$

- To simplify the writing, we need notation that shows, in the case of multiplication, only the x, y and $x + y$ and to de-emphasis the base b.
- Without loss of generality, we may assume to multiply two positive numbers X, Y. We first need to turn them into exponents of b. That is, suppose we can find positive numbers x, y such that

$$b^x = X, \qquad b^y = Y.$$

- This works if there is a unique x and y that correspond to X, Y respectively. We want to work with x and y only.
- Denote $x = \log_b X$ and $y = \log_b Y$

Rules of Logarithm

- **Definition** Given any $X > 0$, we define the exponent x for which $b^x = X$ to be the logarithm of X with respect to the base b. The x is commonly denoted by $x = \log_b X$. That is,

$$X = b^x = b^{\log_b X}. \qquad (\text{so } \log_b 1 = 0)$$

- We write $X\,Y = X\,Y$ in terms of the base b in two ways

$$b^{\log_b X} \times b^{\log_b Y} = b^{\log_b XY}.$$

But $b^x \times b^y = b^{x+y}$. So

$$\log_b X + \log_b Y = \log_b XY$$

- Then one can deduce from $b^x \div b^y = b^{x-y}$ the relation

$$\log_b X - \log_b Y = \log_b \frac{X}{Y}$$

Rules of Logarithm II

- By the definition of logarithm (that is $X = b^x = b^{\log_b X}$),

$$a^m = b^{\log_b(a^m)} = b^{\overbrace{(\log_b a + \cdots + \log_b a)}^{m}} = b^{m \log_b a},$$

so

$$\log_b a^m = m \log_b a$$

holds when m is a positive integer. If $m = -n$ where n is a positive integer, the logarithm is still valid:

- $$a^m = a^{-n} = \frac{1}{a^n} = \frac{1}{b^{\log_b(a^n)}} = \frac{1}{b^{n \log_b a}} = b^{-n \log_b a} = b^{m \log_b a}.$$

Rules of Logarithm III

- How do these different logarithms for the same number X relate to each other?
- Suppose X is a given number, and

$$b^{\log_b X} = X = c^{\log_c X}$$

  are the two different logarithms with respect to base b and c respectively.
- Taking $\log_b$ on both sides yields

$$(log_b X)\underbrace{(\log_b b)}_{=1} = (\log_c X)(\log_b c)$$

  That is, $\log_b X / \log_c X = \log_b c$.
- Similarly, Taking $\log_c$ on both sides yields

$$(log_b X)(\log_c b) = (\log_c X)\underbrace{(\log_c c)}_{=1}$$

  That is, $\log_b X / \log_c X = 1/\log_c b$.
- So $\log_b c = 1/\log_c b$.

Inverse functions

- **Definition** Let f be a function defined on its domain D. Then a function $^{-1}$ is called an inverse of f if

$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in } D.$$

That is, $x = f^{-1}(y)$ whenever $y = f(x)$.

- **Remark 1** It follows that the domain of f^{-1} is on the range of f.
- **Remark 2** There is no guarantee that every function has an inverse.
- **Remark 3** If f has two inverse functions, then the two inverse functions must be identically the same.

Indication of inverse functions A

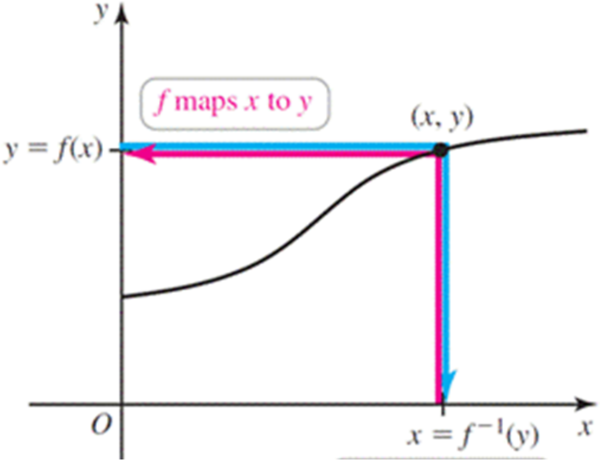

Figure: (Publisher) 1.49a

Indication of inverse functions b

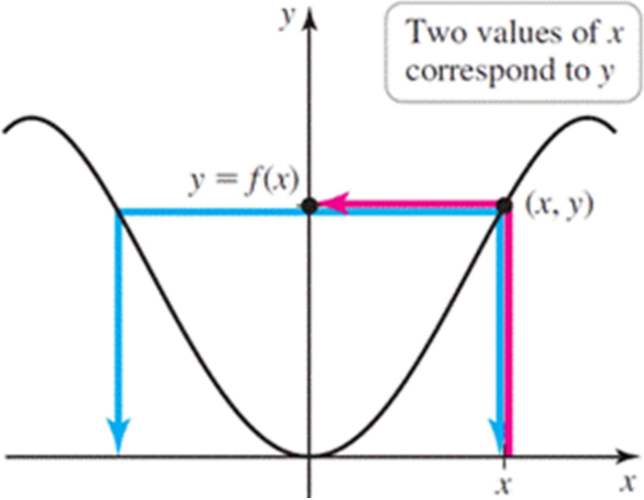

Figure: (Publisher) 1.49b

Examples of inverse function

- Let $y = f(x) = ax + b$ where a, b are constants. Then

$$x = \frac{y-b}{a}$$

is the inverse function of f. That is,

$$x = f^{-1}(y) = \frac{1}{a}(y-b), \qquad x = (f^{-1} \circ f)(x).$$

- $y = f(x) = 2x + 6$.

$$x = f^{-1}(y) = \frac{1}{2}(y-6).$$

- In fact, a criterion of a given f has an inverse is that it is an injective (one-one) mapping.
- In terms of the graph of $y = f(x)$, the graph of f is either increasing or decreasing against the $x - axis$.

Inverse linear functions

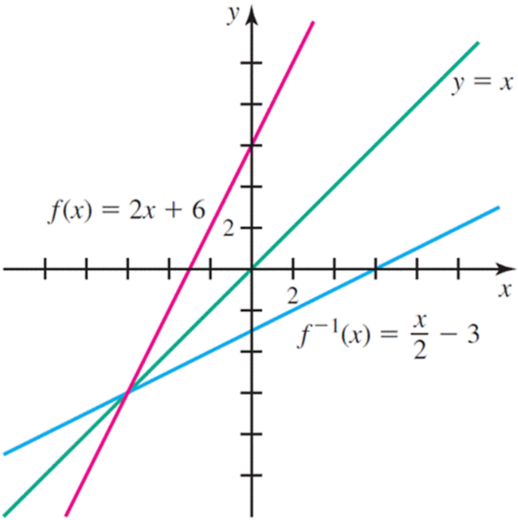

Figure: (Publisher) 1.52

Graphing of inverse functions

- (p. 31) $y = f(x) = x^2 - 1$.
- Choose inverse as $x = +\sqrt{y+1}$
- We really consider x as a function of y now.
- Convention is that we use x for the independent variable.
- Interchange the x and y

$$y = \sqrt{x+1}.$$

- A practical graphical procedure to find the graph of the inverse $y = f^{-1}(x)$ is to rotate the graph of $y = f(x)$ along the straight line $x = y$ by 180 degrees in our three dimensional space.
- Can apply this for $x = f^{-1}(y) = \frac{1}{2}(y - 6)$.

Inverse a quadratic function example

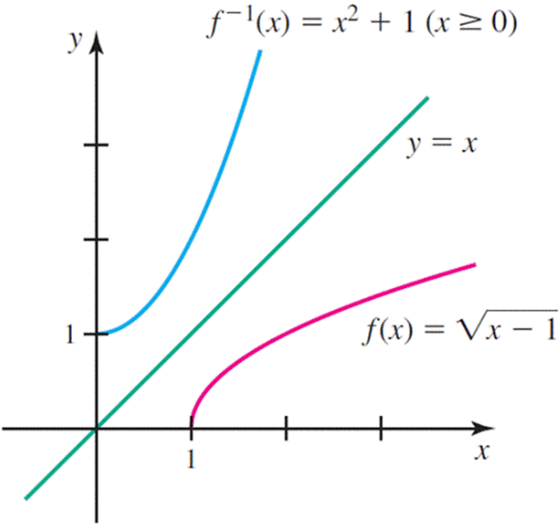

Figure: (Publisher) 1.53

Inverse of quadratic function

- Let $y = f(x) = ax^2 + bx + c$, where $a > 0$.
- We apply the method of completing the square:

$$
\begin{aligned}
y = f(x) = & ax^2 + bx + c \quad (a \neq 0) \\
& = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \\
& = a\left[x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] \\
& = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right] \\
& = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \\
& = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.
\end{aligned}
$$

Inverse of quadratic function

- That is, we have

$$x = -\frac{b}{2a} \pm \sqrt{\frac{1}{a}\Big(y + \frac{b^2 - 4ac}{4a}\Big)}.$$

Recall that the solution of x when $y = 0$ are possible only when $b^2 - 4ac \geq 0$.

- Choosing the "+" branch and switching the roles of x and y yields

$$y = f^{-1}(x) = -\frac{b}{2a} + \sqrt{\frac{1}{a}\Big(x + \frac{b^2 - 4ac}{4a}\Big)}.$$

- Explicit inverse functions are actually difficult to find.

Logarithm as inverse function

- If we view $y = f(x) = b^x$ as a given function, then its inverse is given by $y = f^{-1}(x) = \log_b x$ since we can check

$$(f^{-1} \circ f)(x) = \log_b(b^x) = x$$

by the definition of logarithm.

- In fact, even

$$(f \circ f^{-1})(x) = b^{\log_b x} = x$$

holds trivially.

- The graph of $\log_b x$ is obtained from rotating $y = b^x$ along the line $x = y$ by 180 degrees.