

# MATH1013 Calculus I

## Introduction to Functions<sup>1</sup>

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**Functions (Chapter 1, to be completed)**

Definition of functions

Composition

Polynomial and Rational Functions

Transformations

Logarithm

## Definition of functions

- **Definition** A **function** is a **rule**  $f$  that assigns to each  $x$  in a set  $D$  a **unique** value denoted  $f(x)$ .  $\mathbb{C}$ .
- **Definitions** The set  $D$  is called the **domain** of the function  $f$ , and the set of values of  $f(x)$  assumes, as  $x$  varies over the domain, is called the **range** of the function  $f(x)$ .

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$$x \mapsto f(x), \quad \text{or} \quad y = f(x),$$

- One can think of this as a model of

**one input**  $\rightarrow$  **one output**

- Important point: for **each**  $x$  in  $D$ , one can find (there exists) **one** value  $f(x)$  (or  $y$ ) that corresponds to it.
- However, depending on the  $f$  under consideration, one could have **two or more**  $x$  that correspond to the **same**  $f(x)$ .
- This strange looking idea was created to describe dynamical

## Examples of functions

- (p. 1)  $x \mapsto x^2 - 2x$  or  $f(x) = x^2 - 2x$ .
- (p. 2) Identifying functions
- (p. 2) Domain and range
  1.  $y = f(x) = x^2 + 1$ ,  $[-3, 3] \times [-1, 5]$
  2.  $y = g(t) = \sqrt{4 - t^2}$ ,  $[-3, 3] \times [-1, 3]$
  3.  $y = h(u) = \frac{1}{u - 1}$ ,  $[-3, 5] \times [-4, 4]$
  4. (Ex.1.1, Q 16)  $F(w) = \sqrt[4]{2 - w}$ ,  $[-3, 2] \times [0, 2]$
  5. (Ex.1.1, Q 14)  $g(y) = \frac{y + 1}{(y + 2)(y - 3)}$ ,  $[-4, 6] \times [-3, 3]$
- (p.3: In context) At time  $t = 0$  a stone is thrown vertically upward from the ground at a speed of  $30\text{m/s}$ . Its height above the ground in meters is approximated by the function  $h = f(t) = 30t - 5t^2$ , where  $t$  is in seconds. Find the domain and range of this function as they apply to this particular problem.

## Different types of functions

- $y = f(x) = x + 1$ . For each  $x$  there corresponds to one and only one  $y$ .
- $y = x^3$ . For each  $x$  there corresponds to **one and only one  $y$**
- Where  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ , or equivalently  $x_1 \neq x_2$  implies  $f(x_1) \neq f(x_2)$ , we say the function  $f$  is **injective** or **one-one**. So the above two examples are injective functions.
- (Eg revisited)  $f(x) = x^2 - 2x$  is not injective, as two different  $x$  can correspond to the same  $f(x_1) = y = f(x_2)$
- (Non-function)  $y^2 = 1 - x^2$ . Since for each  $x$  input, there always correspond to two outputs of  $f(x) = \pm\sqrt{1 - x^2}$  within the **domain of  $f$** .

# Function indicative figures

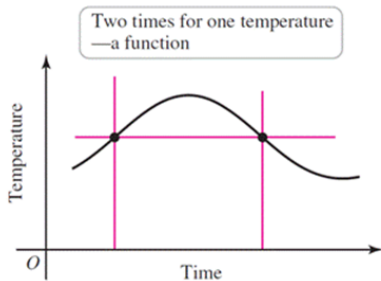
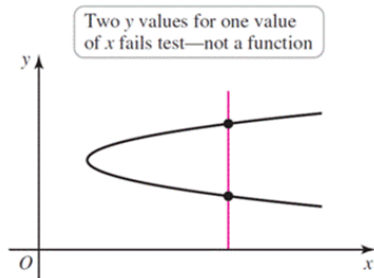


Figure: 1.2 (source textbook)

# A quick test

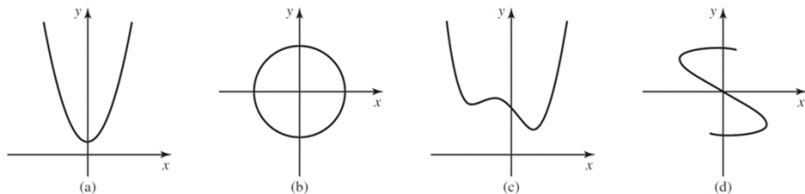


Figure: 1.3 (source textbook)

## Domain and Range figure 1

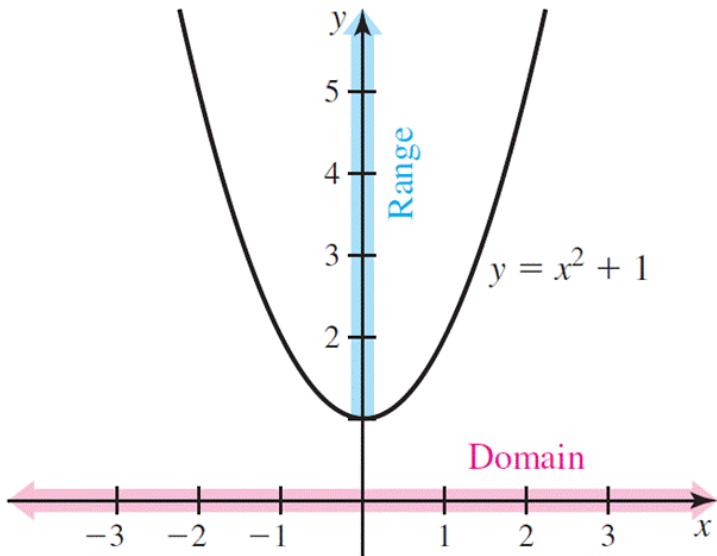


Figure: 1.4 (source: textbook)



## Domain and Range figure II

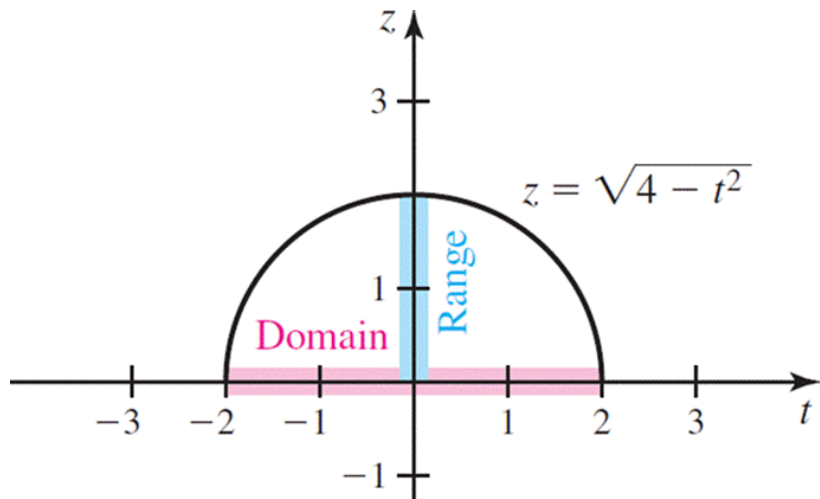


Figure: 1.5 (source: textbook)

## Domain and Range figure III

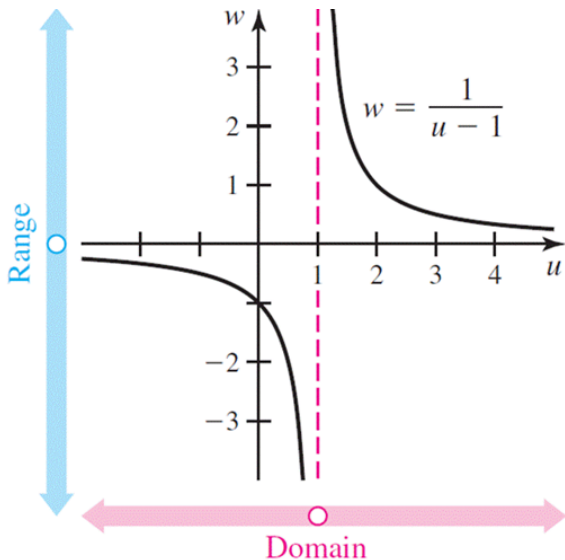


Figure: 1.6 (source: textbook)

## Composition

- **Definition** Given two functions  $f$  and  $g$ , their **composition**  $f \circ g$  is defined, by

$$(f \circ g)(x) = f(u) = f(g(x))$$

for each  $x$  in the domain of  $f \circ g$ . Let  $u = g(x)$  and  $y = f(u)$ , then  $f \circ g$  is understood as

$$y = (f \circ g)(x) = f(g(x)) = f(u), \quad u = g(x),$$

- as shown in

$$x \mapsto u = g(x) \mapsto y = f(u)$$

- with  $g$  takes the **domain of  $g$**  (range) into (part of) **domain of  $f$** , and  $f$  maps that into (part of) the **range of  $f$** . The two together thus forms a new function  $f \circ g$ .

# Diagram of composition

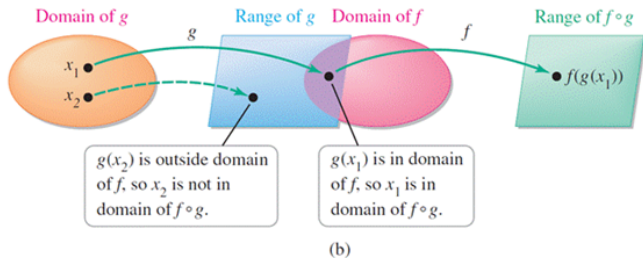
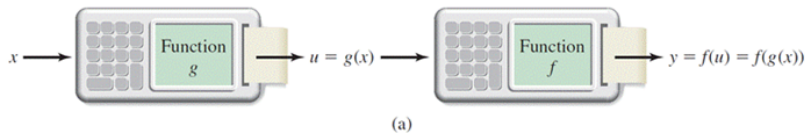


Figure: 1.8 (source: textbook)

## Examples of composition

- (p. 4) Let  $f(x) = 3x^2 - x$  and  $g(x) = 1/x$ .
  1.  $f \circ g$
  2.  $g \circ f$
  3. domain and ranges
- (p. 4) Recognising composition
  1.  $h(x) = \sqrt{9x - x^2}$
  2.  $h(x) = \frac{2}{(x^2 - 1)^3}$
- (p. 5) Given  $f(x) = \sqrt[3]{x}$  and  $g(x) = x^2 - x - 6$ .
  1.  $f \circ g$
  2.  $g \circ f$

## Composite functions Eg from table

- |  |        |    |    |   |    |    |    |
|--|--------|----|----|---|----|----|----|
|  | $x$    | -1 | 0  | 1 | 2  | 3  | 4  |
|  | $f(x)$ | 3  | 1  | 0 | -1 | -3 | -1 |
|  | $g(x)$ | -1 | 0  | 2 | 3  | 4  | 5  |
|  | $h(x)$ | 0  | -1 | 0 | 3  | 0  | 4  |
- (p. 10, Ex 56)
  - Compute
    - (a)  $h(g(0))$ ,
    - (b)  $g(f(4))$ ,
    - (c)  $h(h(0))$ ,
    - (d)  $g(h(f(4)))$ ,
    - (e)  $f(f(f(1)))$ ,
    - (f)  $h(h(h(0)))$ ,
    - (g)  $f(g(h(2)))$

## Examples of composition: Gradients

- (p. 6) Given  $f(x) = 3x^2 - x$ . Find  $\frac{f(x+h) - f(x)}{h}$
- The above quantity is called the **gradient** (or *slope* in less formal language) of  $f$  between the two points  $x$  and  $f(x+h)$ . Alternatively, it is the **average rate of change** of  $f$  between  $x$  and  $x+h$ . It is an important quantity as we enter Chapters two and three.
- (p. 6) let  $I = \frac{P}{4\pi r^2}$  measures sound intensity in watts per square meter ( $W/m^2$ ), at a point  $r$  meters from a sound source with acoustic power  $P = 100 W$ . Find the gradient of the secant line through (i) the points  $(r_1, I(r_1))$  and  $(r_2, I(r_2))$  ( $-\frac{P(r_1+r_2)}{4\pi r_1^2 r_2^2} W/m^2$ ), and (ii) the points  $(10, I(10))$  and  $(15, I(15))$  ( $-1/(36\pi) W/m^2$ ).

# Representing Functions

- **Polynomial** functions

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the  $a_{n-1}, \dots, a_1, a_0$  are some constants.

- **Rational** functions

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0},$$

where the  $a_n, \dots, a_1, a_0, b_m, \dots, b_1, b_0$  are constants

- **Piecewise** function (p. 15)

$$f(x) = \begin{cases} x & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ -\frac{1}{2}x + 5 & \text{if } x > 2. \end{cases}$$



## Transformations

- Given a function  $y = f(x)$  and its graph.
- Then the function

$$y = f(x - b) + d,$$

where  $b > 0$ ,  $d$  are constants, has the **same shape** as  $f(x)$  but which is shifted to the **right** by  $b$  units and shifted up/down by  $d$  units.

- Then the function

$$y = f(x + b) + d$$

where  $b > 0$ ,  $d$  are constants, has the **same shape** as  $f(x)$  but which is shifted to the **left** by  $b$  units and **shifted up/down** by  $d$  units.

- The graph of the function

$$y = c f(ax)$$

has **horizontal** magnification factor  $a$ , and **vertical** magnification factor  $b$ .

## Transformation figure I

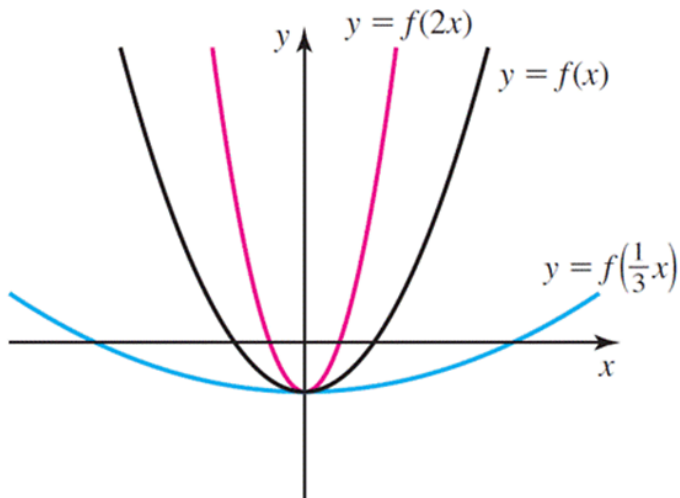


Figure: 1.38 (source: textbook)

## Transformation figure II

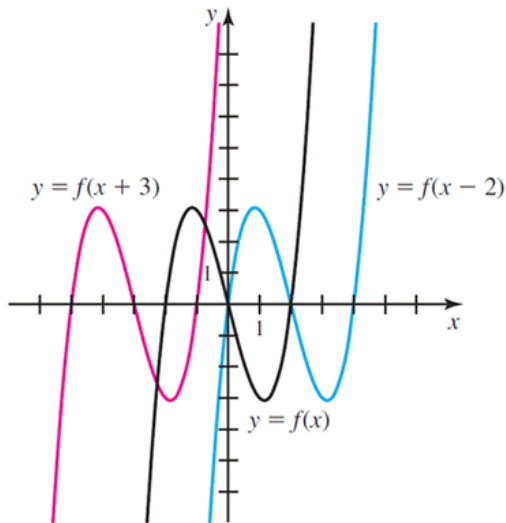


Figure: 1.35 (source: textbook)

## Examples of Transformation

- (p. 20, E.g. 8) Sketch the graph of  $y = (x - 2)^2 - 3$
- (p. 15, E.g. 4b) Recall the **absolute value** function is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

- (p. 20, E.g. 9) Sketch  $|2x + 1|$ .
- (p. 21, Ex. 10) Sketch the graph of  $y = 4(x + 3)^2 + 6$
- (p. 23, Ex. 44) Sketch the graph according to Exercise 44:

$$f(x) = \begin{cases} |x| - 1 & \text{if } |x| \geq 2; \\ |x| & \text{if } |x| < 1. \end{cases}$$

(a)  $y = -f(x)$ , (b)  $y = f(x + 2)$ , (c)  $y = f(x - 2)$ , (d)  
 $y = f(2x)$ , (e)  $y = f(x - 1) + 2$ , (f)  $y = 2f(x)$ .

## Function made up by two pieces

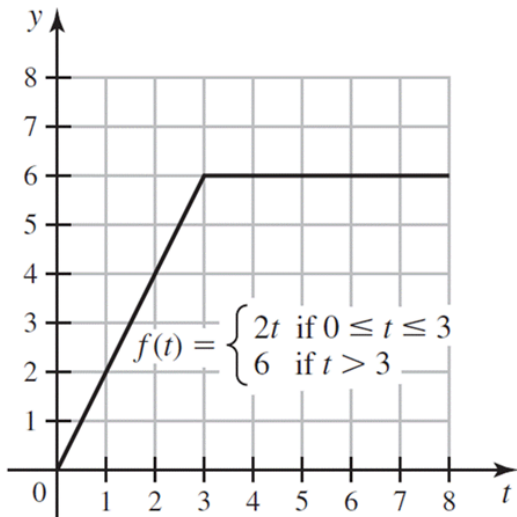


Figure: 1.30 (source: textbook)

## Function made up by three pieces

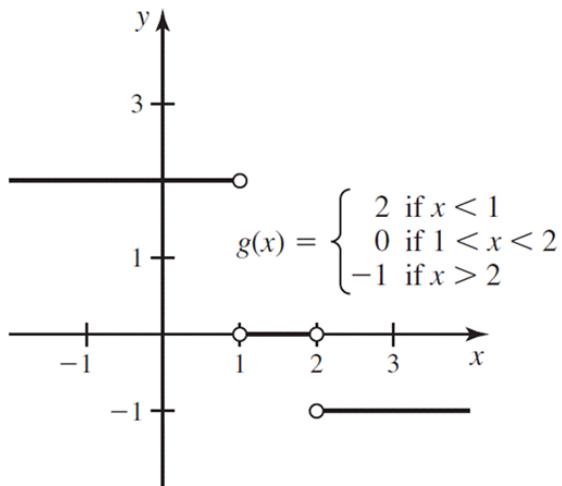


Figure: 1.29 (source: textbook)

# Exponential Functions

- The function

$$y = f(x) = 2^x$$

is called **binary exponential function with base 2**.

- Let  $b \neq 1$  be an arbitrary positive number. Then function
- The function

$$y = f(x) = b^x$$

is called **exponential function with base  $b$** .

- The exponential function grows very **fast without bound** if  $b > 1$  and **tends** to zero fast.
- Amongst all exponential functions, the **natural exponential function**

$$y = f(x) = e^x$$

where  $2 < e \approx 2.71828 < 3$  is amongst the most important.

- The simpler one is the one with **common base 10**

$$y = f(x) = 10^x$$

# Logarithm

- $b^x \times b^y = b^{x+y}$ ,  $b^x \div b^y = b^{x-y}$
- $(b^x)^y = b^{x \cdot y}$ .
- These observations, about turning **multiplication or division** into **addition or subtraction** of the **exponents** respectively, may seem elementary, but it actually becomes so important in all kinds of applications, some quite unexpected.
- The application of these rules were known since Babylonians' time in 2000-1600 BC. But it was the Scottish **John Napier** (1550-1617) who wrote the book entitled *Mirifici Logarithmorum Canonis Descriptio* (Description of the Wonderful Rule of Logarithms) in 1614 that popularized their use (source from Wiki)
- First called **artificial number**, then "logarithm" meaning from Latin "proportional-arithmetical"
- Applications: <http://en.wikipedia.org/wiki/Logarithm>



# John Napier



Figure: (1550-1617) Source from Wiki



# A page in Log tables

COMMON LOGARITHMS  $\log_{10} x$

x	0	1	2	3	4	5	6	7	8	9	$\Delta_m$	1	2	3	4	5	6	7	8	9		
10	0000	0043	0086	0128	0170	0212						41	4	2	1	3	2	5	2	4	1	3
11	0414	0457	0491	0531	0571	0612	0653	0694	0734	0774		40	4	0	1	2	1	0	2	0	1	3
12	0792	0834	0874	0913	0951	0987	1025	1062	1099	1135		39	4	0	1	1	0	0	0	0	0	2
13	1179	1219	1258	1296	1333	1369	1404	1439	1473	1507		38	4	0	1	0	0	0	0	0	0	1
14	1549	1588	1626	1663	1699	1734	1769	1803	1837	1870		37	3	0	0	0	0	0	0	0	0	0
15	1904	1938	1971	2003	2035	2066	2096	2126	2155	2184		36	3	0	0	0	0	0	0	0	0	0
16	2212	2239	2265	2291	2316	2340	2364	2387	2410	2432		35	3	0	0	0	0	0	0	0	0	0
17	2454	2475	2496	2516	2535	2554	2572	2589	2606	2622		34	3	0	0	0	0	0	0	0	0	0
18	2639	2654	2669	2683	2696	2709	2721	2733	2744	2755		33	3	0	0	0	0	0	0	0	0	0
19	2766	2776	2786	2795	2804	2812	2820	2828	2835	2842		32	3	0	0	0	0	0	0	0	0	0
20	2849	2856	2862	2868	2874	2879	2884	2889	2894	2898		31	3	0	0	0	0	0	0	0	0	0
21	2902	2906	2910	2914	2918	2921	2924	2927	2930	2933		30	3	0	0	0	0	0	0	0	0	0
22	2936	2938	2940	2942	2944	2946	2947	2948	2949	2950		29	3	0	0	0	0	0	0	0	0	0
23	2951	2952	2953	2954	2955	2955	2956	2956	2957	2957		28	3	0	0	0	0	0	0	0	0	0
24	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		27	3	0	0	0	0	0	0	0	0	0
25	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		26	3	0	0	0	0	0	0	0	0	0
26	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		25	3	0	0	0	0	0	0	0	0	0
27	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		24	3	0	0	0	0	0	0	0	0	0
28	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		23	3	0	0	0	0	0	0	0	0	0
29	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		22	3	0	0	0	0	0	0	0	0	0
30	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		21	3	0	0	0	0	0	0	0	0	0
31	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		20	3	0	0	0	0	0	0	0	0	0
32	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		19	3	0	0	0	0	0	0	0	0	0
33	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		18	3	0	0	0	0	0	0	0	0	0
34	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		17	3	0	0	0	0	0	0	0	0	0
35	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		16	3	0	0	0	0	0	0	0	0	0
36	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		15	3	0	0	0	0	0	0	0	0	0
37	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		14	3	0	0	0	0	0	0	0	0	0
38	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		13	3	0	0	0	0	0	0	0	0	0
39	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		12	3	0	0	0	0	0	0	0	0	0
40	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		11	3	0	0	0	0	0	0	0	0	0
41	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		10	3	0	0	0	0	0	0	0	0	0
42	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		9	3	0	0	0	0	0	0	0	0	0
43	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		8	3	0	0	0	0	0	0	0	0	0
44	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		7	3	0	0	0	0	0	0	0	0	0
45	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		6	3	0	0	0	0	0	0	0	0	0
46	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		5	3	0	0	0	0	0	0	0	0	0
47	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		4	3	0	0	0	0	0	0	0	0	0
48	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		3	3	0	0	0	0	0	0	0	0	0
49	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		2	3	0	0	0	0	0	0	0	0	0
50	2958	2958	2958	2958	2958	2958	2958	2958	2958	2958		1	3	0	0	0	0	0	0	0	0	0

No.  $\log$   $\log x = \log_{10} x$   $\log x = \log_{10} x$   $\log x = \log_{10} x$   $\log x = \log_{10} x$   
 $x = 10^{\log x}$   $x = 10^{\log x}$   $x = 10^{\log x}$   $x = 10^{\log x}$   
 $\log x = \frac{\log x}{\log 10}$   $\log x = \frac{\log x}{\log 10}$   $\log x = \frac{\log x}{\log 10}$   $\log x = \frac{\log x}{\log 10}$

COMMON LOGARITHMS  $\log_{10} x$

x	0	1	2	3	4	5	6	7	8	9	$\Delta_m$	1	2	3	4	5	6	7	8	9
50	6900	6909	7007	7016	7044	7033	7043	7050	7059	7067		9	1	3	4	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7151		8	1	2	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7200	7208	7216	7224	7232		7	1	2	3	4	5	6	7	8
53	7240	7248	7256	7264	7272	7280	7288	7296	7304	7312		6	1	2	3	4	5	6	7	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396		5	1	2	3	4	5	6	7	8
55	7404	7412	7420	7428	7436	7444	7452	7460	7468	7476		4	1	2	3	4	5	6	7	8
56	7484	7492	7500	7508	7516	7524	7532	7540	7548	7556		3	1	2	3	4	5	6	7	8
57	7564	7572	7580	7588	7596	7604	7612	7620	7628	7636		2	1	2	3	4	5	6	7	8
58	7644	7652	7660	7668	7676	7684	7692	7700	7708	7716		1	1	2	3	4	5	6	7	8
59	7724	7732	7740	7748	7756	7764	7772	7780	7788	7796		0	1	2	3	4	5	6	7	8
60	7804	7812	7820	7828	7836	7844	7852	7860	7868	7876		0	1	2	3	4	5	6	7	8
61	7884	7892	7900	7908	7916	7924	7932	7940	7948	7956		0	1	2	3	4	5	6	7	8
62	7964	7972	7980	7988	7996	8004	8012	8020	8028	8036		0	1	2	3	4	5	6	7	8
63	8044	8052	8060	8068	8076	8084	8092	8100	8108	8116		0	1	2	3	4	5	6	7	8
64	8124	8132	8140	8148	8156	8164	8172	8180	8188	8196		0	1	2	3	4	5	6	7	8
65	8204	8212	8220	8228	8236	8244	8252	8260	8268	8276		0	1	2	3	4	5	6	7	8
66	8284	8292	8300	8308	8316	8324	8332	8340	8348	8356		0	1	2	3	4	5	6	7	8
67	8364	8372	8380	8388	8396	8404	8412	8420	8428	8436		0	1	2	3	4	5	6	7	8
68	8444	8452	8460	8468	8476	8484	8492	8												

## Notation of Logarithm

- Let us recall that after agreeing to a **base**  $b$ , we make use of

$$b^x \times b^y = b^{x+y}, \quad b^x \div b^y = b^{x-y}.$$

- To simplify the writing, we need notation that shows, in the case of multiplication, only the  $x$ ,  $y$  and  $x + y$  and to de-emphasis the **base**  $b$ .
- Without loss of generality, we may assume to multiply two positive numbers  $X$ ,  $Y$ . We first need to turn them into exponents of  $b$ . That is, suppose we can find positive numbers  $x$ ,  $y$  such that

$$b^x = X, \quad b^y = Y.$$

- This works if there is a unique  $x$  and  $y$  that correspond to  $X$ ,  $Y$  respectively. **We want to work with  $x$  and  $y$  only.**
- Denote  $x = \log_b X$  and  $y = \log_b Y$

## Rules of Logarithm

- **Definition** Given any  $X > 0$ , we define the exponent  $x$  for which  $b^x = X$  to be the **logarithm of  $X$  with respect to the base  $b$** . The  $x$  is commonly denoted by  $x = \log_b X$ . That is,

$$X = b^x = b^{\log_b X}. \quad (\text{so } \log_b 1 = 0)$$

- We write  $XY = X Y$  in terms of the base  $b$  in two ways

$$b^{\log_b X} \times b^{\log_b Y} = b^{\log_b XY}.$$

But  $b^x \times b^y = b^{x+y}$ . So

$$\log_b X + \log_b Y = \log_b XY$$

- Then one can deduce from  $b^x \div b^y = b^{x-y}$  the relation

$$\log_b X - \log_b Y = \log_b \frac{X}{Y}$$

## Rules of Logarithm II

- By the definition of logarithm (that is  $X = b^x = b^{\log_b X}$ ),

$$a^m = b^{\log_b(a^m)} = b^{\overbrace{(\log_b a + \cdots + \log_b a)}^m} = b^{m \log_b a},$$

so

$$\log_b a^m = m \log_b a$$

holds when  $m$  is a positive integer. If  $m = -n$  where  $n$  is a positive integer, the logarithm is still valid:

- 

$$a^m = a^{-n} = \frac{1}{a^n} = \frac{1}{b^{\log_b(a^n)}} = \frac{1}{b^{n \log_b a}} = b^{-n \log_b a} = b^{m \log_b a}.$$

## Rules of Logarithm III

- How do these different logarithms for the same number  $X$  relate to each other?
- Suppose  $X$  is a given number, and

$$b^{\log_b X} = X = c^{\log_c X}$$

are the two different logarithms with respect to **base  $b$**  and  **$c$**  respectively.

- Taking  $\log_b$  on both sides yields

$$(\log_b X) \underbrace{(\log_b b)}_{=1} = (\log_c X)(\log_b c)$$

That is,  $\log_b X / \log_c X = \log_b c$ .

- Similarly, Taking  $\log_c$  on both sides yields

$$(\log_b X)(\log_c b) = (\log_c X) \underbrace{(\log_c c)}_{=1}$$

That is,  $\log_b X / \log_c X = 1 / \log_c b$ .

- So  $\log_b c = 1 / \log_c b$ .

# Inverse functions

- **Definition** Let  $f$  be a function defined on its domain  $D$ . Then a **function**  $f^{-1}$  is called an **inverse** of  $f$  if

$$(f^{-1} \circ f)(x) = x, \quad \text{for all } x \text{ in } D.$$

That is,  $x = f^{-1}(y)$  whenever  $y = f(x)$ .

- **Remark 1** It follows that the domain of  $f^{-1}$  is on the **range** of  $f$ .
- **Remark 2** There is **no** guarantee that every function has an inverse.
- **Remark 3** If  $f$  has two inverse functions, then the two inverse functions must be identically the **same**.



# Indication of inverse functions A

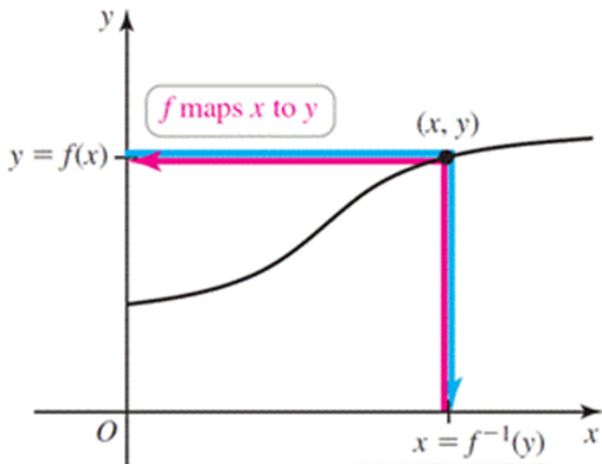


Figure: (Publisher) 1.49a

## Indication of inverse functions b

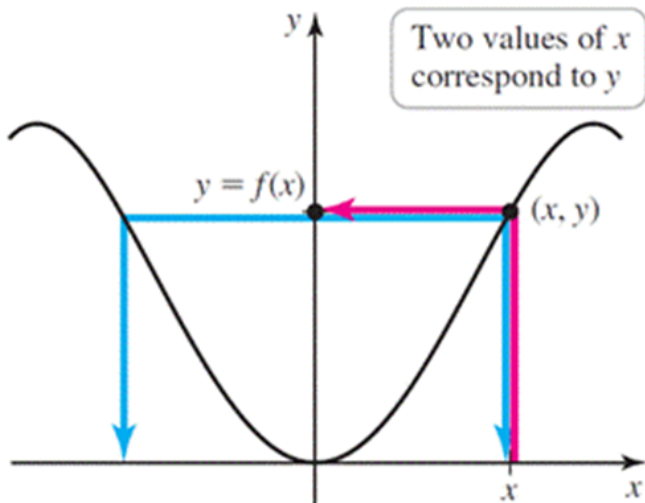


Figure: (Publisher) 1.49b

## Examples of inverse function

- Let  $y = f(x) = ax + b$  where  $a, b$  are constants. Then

$$x = \frac{y - b}{a}$$

is the inverse function of  $f$ . That is,

$$x = f^{-1}(y) = \frac{1}{a}(y - b), \quad x = (f^{-1} \circ f)(x).$$

- $y = f(x) = 2x + 6$ .

$$x = f^{-1}(y) = \frac{1}{2}(y - 6).$$

- In fact, a criterion of a given  $f$  has an inverse is that it is an **injective (one-one)** mapping.
- In terms of the **graph** of  $y = f(x)$ , the graph of  $f$  is either **increasing** or **decreasing** against the  $x$  - **axis**.

# Inverse linear functions

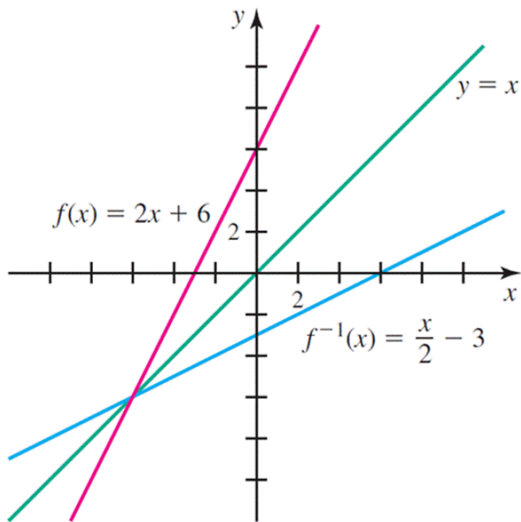


Figure: (Publisher) 1.52

## Graphing of inverse functions

- (p. 31)  $y = f(x) = x^2 - 1$ .
- Choose **inverse** as  $x = +\sqrt{y + 1}$
- We really consider  $x$  as a function of  $y$  now.
- Convention is that we use  $x$  for the independent variable.
- **Interchange** the  $x$  and  $y$

$$y = \sqrt{x + 1}.$$

- A practical graphical procedure to find the graph of the inverse  $y = f^{-1}(x)$  is to **rotate** the graph of  $y = f(x)$  along the straight line  $x = y$  by **180** degrees in our three dimensional space.
- Can apply this for  $x = f^{-1}(y) = \frac{1}{2}(y - 6)$ .

## Inverse a quadratic function example

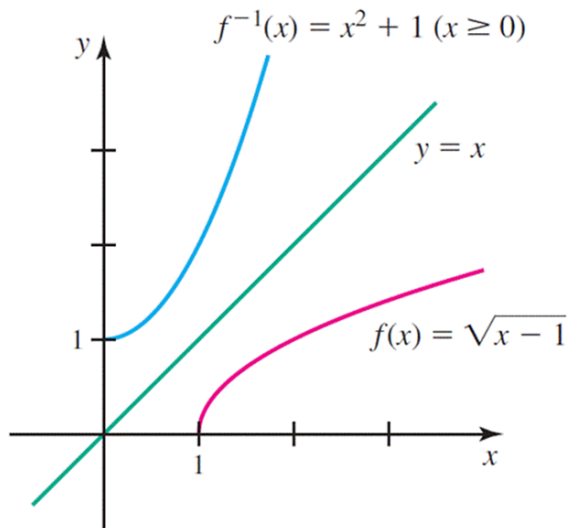


Figure: (Publisher) 1.53

## Inverse of quadratic function

- Let  $y = f(x) = ax^2 + bx + c$ , where  $a > 0$ .
- We apply the method of **completing the square**:

$$y = f(x) = ax^2 + bx + c \quad (a \neq 0)$$

$$= a \left[ x^2 + \frac{b}{a}x + \frac{c}{a} \right]$$

$$= a \left[ x^2 + 2 \left( \frac{b}{2a} \right) x + \left( \frac{b}{2a} \right)^2 + \frac{c}{a} - \left( \frac{b}{2a} \right)^2 \right]$$

$$= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right]$$

$$= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right]$$

$$= a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}.$$

## Inverse of quadratic function

- That is, we have

$$x = -\frac{b}{2a} \pm \sqrt{\frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right)}.$$

Recall that the solution of  $x$  when  $y = 0$  are possible only when  $b^2 - 4ac \geq 0$ .

- Choosing the “+” branch and switching the roles of  $x$  and  $y$  yields

$$y = f^{-1}(x) = -\frac{b}{2a} + \sqrt{\frac{1}{a}\left(x + \frac{b^2 - 4ac}{4a}\right)}.$$

- Explicit inverse functions are actually difficult to find.



## Logarithm as inverse function

- If we view  $y = f(x) = b^x$  as a given function, then its inverse is given by  $y = f^{-1}(x) = \log_b x$  since we can check

$$(f^{-1} \circ f)(x) = \log_b(b^x) = x$$

by the definition of logarithm.

- In fact, even

$$(f \circ f^{-1})(x) = b^{\log_b x} = x$$

holds trivially.

- The graph of  $\log_b x$  is obtained from rotating  $y = b^x$  along the line  $x = y$  by 180 degrees.