MATH1013 Calculus I

Introduction to Functions[1]

Edmund Y. M. Chiang

Department of Mathematics
Hong Kong University of Science & Technology

March 6, 2013

Limits (Chapter 2) (Revised up to p. 78)

[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

The Dawn

- The need to invetigate dynamical problems in the 18th century verses static problems in the past was strongly related to the cultural and economics developments at that time.
- It was Galilei Galileo (1664-1643), called "the father of sciences" who headed the Scientific Revolution in the 17th century advocating beliefs should be built upon "experiments and mathematics" and that "Philosophy is written in this grand book, the universe ... It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures;...." (Wiki)
- He showed that the velocity of a falling body only depends on its mass and has nothing to do with its shape and size
- He invented telescope and use it to discover the four largest satellites of the planet Juipter, etc

Galilei Galileo

Figure: (Portrait in crayon by Leoni (source Wiki)

How one can describe the world?

- Newton's success is built upon Galileo's philosophy and on Kepler's experimental laws.
- How to describe an object that moves around and to know it at every instant?
- Suppose a particle that has no velocity at $t = 0$ and its velocity is 15 when $t = 10$ second. So there must be a moment or instant when the velocity of the particle is 10, say. However, this statement is very naive.
- In order to measure a change of velocity there must be an interval of time, no matter how short, to compute the velocity.
- For convenience sake, Newton invented virtual distance and virtual time to measure virtual velocity. That is,

$$\textbf{virtual velocity} = \frac{\textbf{virtual distance}}{\textbf{virtual time}}$$

or just instantaneous velocity.

A dynamical problem (p. 54)

A rock is launched vertically upward from the ground witha speed of 96 ft/s. Neglecting air resistance, a well-known formula from physics states that the position of the rock after t seconds is given by

$$s(t) = -16t^2 + 96\,t.$$

The position s is measured in feet with $s = 0$ corresponding to the ground. Find the average velocity of the rock between $t = 1$ and $t = 3$, $t = 1$ and $t = 2$.

A dynamical problem (figure 2.1)

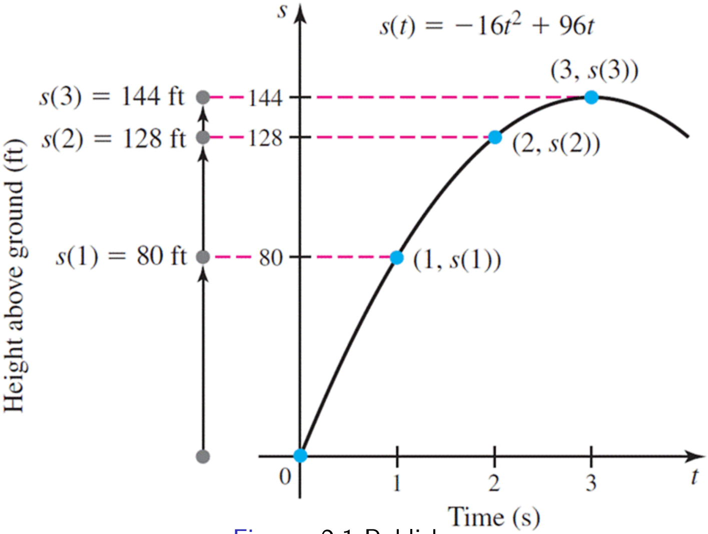


Figure: 2.1 Publisher

A dynamical problem (figure 2.2a)

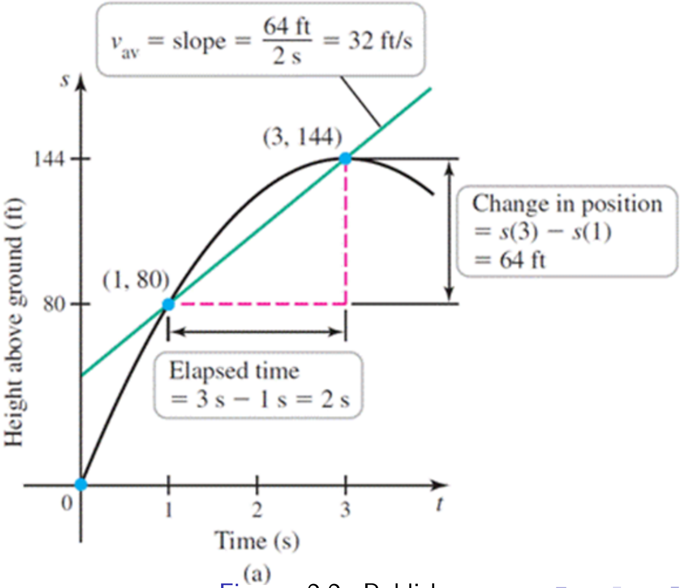
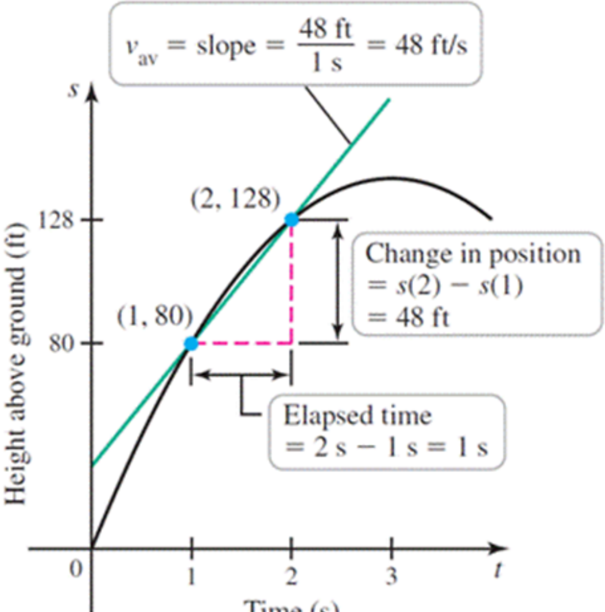
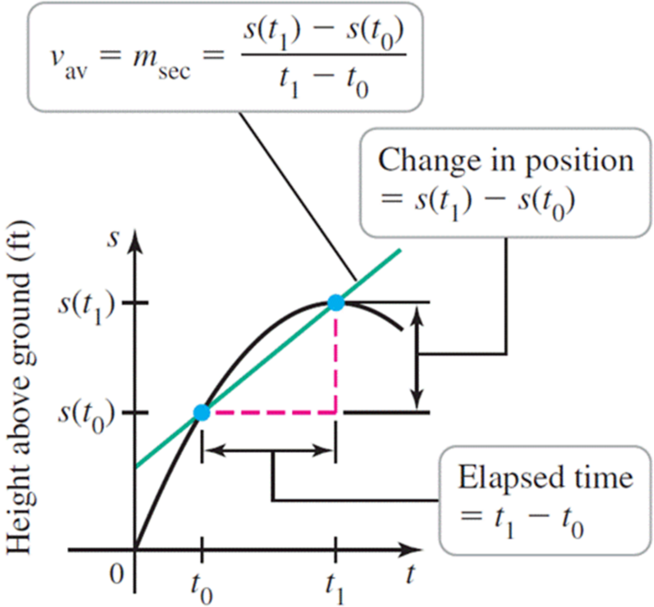


Figure: 2.2a Publisher

A dynamical problem (figure 2.2b)



A dynamical problem (figure 2.3)



A dynamical problem (table 2.1)

Time interval	Average velocity
[1, 2]	48 ft/s
[1, 1.5]	56 ft/s
[1, 1.1]	62.4 ft/s
[1, 1.01]	63.84 ft/s
[1, 1.001]	63.984 ft/s
[1, 1.0001]	63.9984 ft/s

A dynamical problem (figure 2.4)

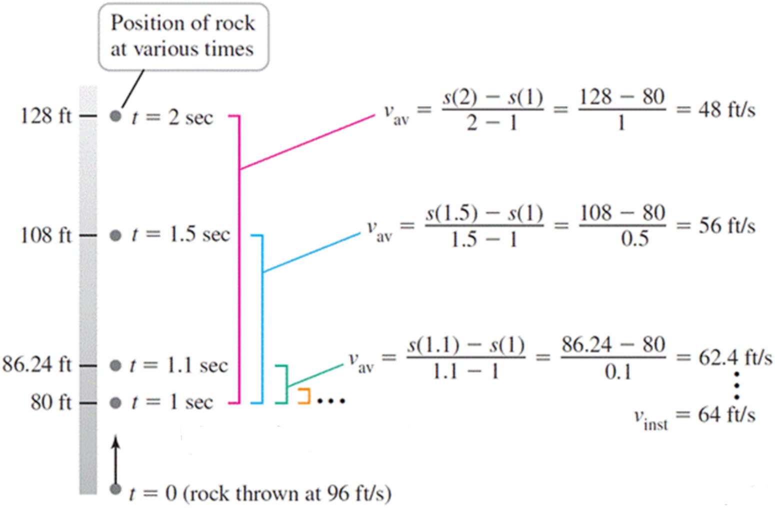


Figure: 2.4 Publisher

A dynamical problem (figure 2.5)

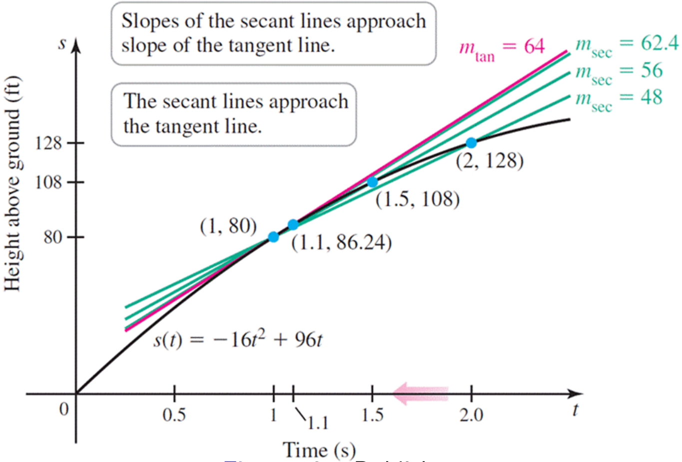


Figure: 2.5 Publisher

Newton's trouble

- Suppose an object moves according to the rule $S(t) = 20 + 4t^2$ where S measures the distance of the object from the initial position t seconds later.
- We now compute instantaneous velocity of the object at time t: let dt and dS be the virtual time and virtual distance respectively. Then the change of virtual distance is given by $dS = S(t + dt) - S(t)$. So the virtual velocity is

$$\frac{dS}{dt} = \frac{S(t+dt) - S(t)}{dt} = \frac{4(t+dt)^2 - 4t^2}{dt} = 8\,t + 4\,dt.$$

- Newton then delete the last dt:

$$\frac{dS}{dt} = 8\,t + 4\,\not{d}\not{t} = 8\,t.$$

- So do we have $dt = 0$? If so, then one would have $\frac{dS}{dt} = \frac{0}{0}$. That was the question that Newton could not answer satisfactorily during his life time.

Sir Issac Newton

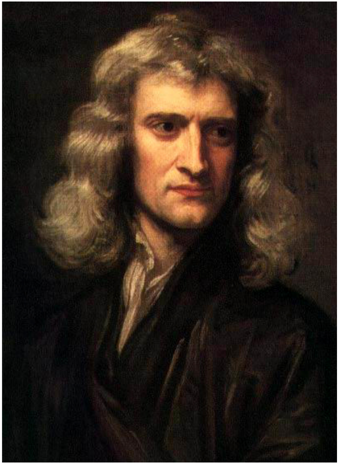


Figure: (1689 by Sir Godfrey Kneller (Newton Institute))

Newton's thought

- So he simply considers that is a virtual distance dS traveled by the object in a virtual time dt. He considers both to be infinitesimal small quantities.
- So do we have $dt = 0$? If so, then one would have $\frac{dS}{dt} = \frac{0}{0}$. That was the question that Newton could not answer satisfactorily during his life time.
- To put the question differently, is an infinitesimal quantity equal to zero? If dt is infinitely small then it would have to be less than any positive quantity, and we conclude it must be equal to zero. For suppose $dt \neq 0$ then $dt > 0$. Hence $dt = r > 0$ is an actual positive quantity. But then we could find $r/2 < dt$, contradicting the fact that dt is smaller then any positive quantity. Hence $dt = 0$.
- Newton was actually attacked by many people, and among them was the Bishop Berkeley. But he method of calculation of instantaneous velocity has been used by other since then.

Finding a remedy

- Let's get close but not when $dt = 0$. Find the average velocity of the object between
  - $t = 2$ and $t = 2.5$
    $$\frac{S(2.5) - S(2)}{2.5 - 2} = \frac{\left(20 + 4(2.5)^2\right) - \left(20 + 4(2)^2\right)}{2.5 - 2} = 18;$$
  - $t = 2$ and $t = 2.1$
    $$\frac{S(2.1) - S(2)}{2.1 - 2} = \frac{\left(20 + 4(2.1)^2\right) - \left(20 + 4(2)^2\right)}{2.1 - 2} = 16.4$$
  - $t = 2$ and $t = 2.01$
    $$\frac{S(2.01) - S(2)}{2.01 - 2} = \frac{\left(20 + 4(2.01)^2\right) - \left(20 + 4(2)^2\right)}{2.01 - 2} = 16.04$$
  - $t = 2$ and $t = 2.001$
    $$\frac{S(2.001) - S(2)}{2.001 - 2} = \frac{\left(20 + 4(2.001)^2\right) - \left(20 + 4(2)^2\right)}{2.001 - 2} = 16.004$$
  - $t = 2$ and $t = 2.0001$
    $$\frac{S(2.0001) - S(2)}{2.0001 - 2} = \frac{\left(20 + 4(2.0001)^2\right) - \left(20 + 4(2)^2\right)}{2.0001 - 2} = 16.0004$$

Re-assessing the problem

- Let us begin with the above example about the movement of the object P. Since we are interested to know the magnitude of the average velocity of P near 2, so let us rewrite the expression in the following form:

$$g(x) = \frac{S(2+x) - S(2)}{x}.$$

- This is a function g depends on the variable x, which can be made as close to 16 as we wish by chooesing t close to 2.
- That is, $g(x)$ approaches the value 16 as x approaches 0. On the other hand, we **cannot** put $x = 0$ in the function $g(x)$, since both the numerator $S(2+x) - S(x)$ and the denominator x would be zero.
- We say that the function g has limit equal to 16 as x approaches 0 abbreviated as

$$\lim_{x \to 0} g(x) = 16.$$

Limit definition

- Note that the above statement is merely an abbreviation for the statement: *The function g can get as close to 16 as possible if we let x approach* 0 *as close as we wish.*
- It is important to note that we are not allowed to put $x = 0$ above
- **Definition** *Let a and l be two real numbers. If the value of the funciton $f(x)$ approaches l as close as we wish as x approaches a, then we say the limit of f is equal to l as x tends to a. The statement is denoted by*

$$\lim_{x \to a} f(x) = l.$$

Alternatively, we may also write

$$f(x) \to l \quad as \quad x \to a.$$

Examples

- Find $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$.
- Note that we can not substitute $x = 2$ in the expression. For then both the numerator and denominator will be zero. Consider

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)}{x - 2}$$
$$= \lim_{x \to 2} (x^2 + 2x + 4) = 12.$$

- The above is an abbreviation of the expression:

$$\frac{x^3 - 8}{x - 2} = \frac{(x-2)(x^2 + 2x + 4)}{x - 2} = x^2 + 2x + 4$$

tends to the value 12 as x tends to 2.

- or more briefly

$$\frac{x^3 - 8}{x - 2} = x^2 + 2x + 4 \longrightarrow 12, \qquad \text{as } x \longrightarrow 2.$$

Illegal step

If

$$f(x) = \frac{x^3 - 8}{x - 2},$$

then it is absolutely forbidden to write

$$\lim_{x \to 2} f(x) = \frac{x^3 - 8}{x - 2} = f(2)$$

since the function f is simply undefined at $x = 2$.

Figure 2.7

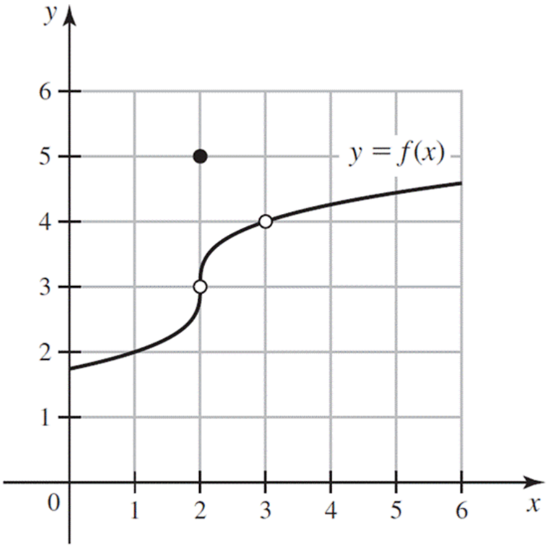


Figure: 2.7 (Publisher)

Figure 2.8

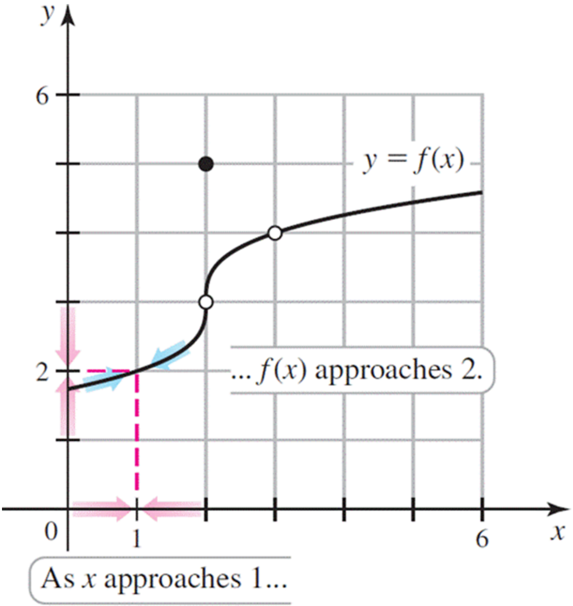


Figure 2.9

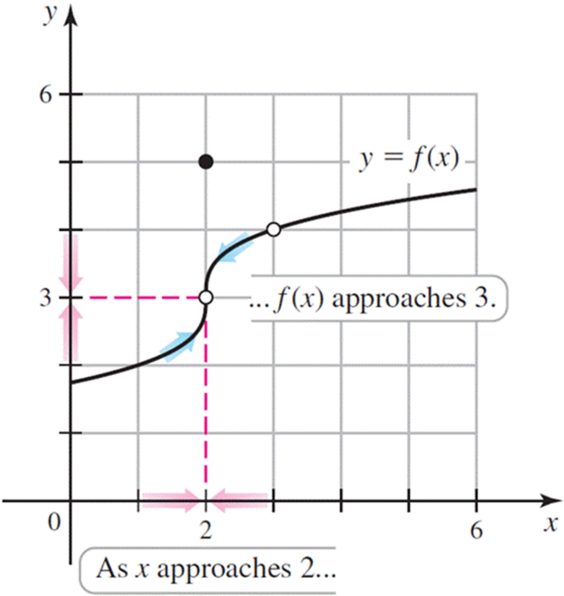
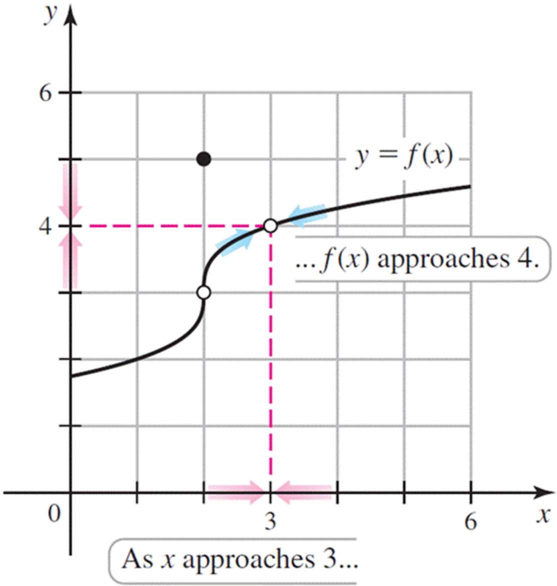


Figure 2.10



Examples

- **Exercises** Find $\lim_{x \to 4} \frac{x^{3/2} - 8}{x^{1/2} - 2}$ (12)
- $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$, (4)
- $\lim_{x \to 4} \frac{\sqrt{x} - 2}{4 - x}$, $(-1/4)$
- $\lim_{h \to 0} \frac{(2 + h)^4 - 16}{h}$, (32)
- The above examples could be misleading. There could be situations that no easy simplification when finding limit as in the above examples. We will show in the next chapter that

$$\frac{e^x - 1}{x} \mapsto 1, \qquad x \mapsto 0.$$

Artificial examples

- **Remark** We remark that the above definition does not mention whether we could substitute $x = a$ in $f(x)$. In fact, $f(a)$ may or may not be meaningful. This is slightly different from the physical problem about the object P where we were not allowed to put $x = 0$ in $g(x)$.
- The following examples do not have the kind of physical context about having $0/0$ problem that we encountered earlier. They are simply created to illustrate what one should interpret the limit definition properly, even though they seems to be trivial:
- **Example** Let $f(x) = 4x^2 + 20$. Then
  1. $\lim_{x\to 1}(4x^2 + 20) = 24$,
  2. $\lim_{x\to -1}(4x^2 + 20) = 24$,
  3. $\lim_{x\to 3}(4x^2 + 20) = 56$.

Simple exercises

- **Example** Let $f(x) = \sqrt{x^2+3}$. Then
  1. $\lim_{x\to 1} \sqrt{x^2+3} = \sqrt{1^2+3} = \sqrt{4} = 2$;
  2. $\lim_{x\to -1} \sqrt{x^2+3} = \sqrt{(-1)^2+3} = \sqrt{4} = 2$.
- Let $g(x) = \dfrac{1}{x-2}$. Then
  - $\displaystyle\lim_{x\to 3} \frac{1}{x-2} = \frac{1}{3-2} = 1$;
  - $\displaystyle\lim_{x\to -1} \frac{1}{x-2} = \frac{1}{-1-2} = -1/3$,
  - **Exercises**
    - $\lim_{x\to 2} 5 =$ (5)
    - $\lim_{x\to 1}(x^3-1) =$ (0)
    - $\lim_{x\to -1}(x^3-1) =$ (-2)
    - $\lim_{x\to -1}(ax^3-1) =$ $(-a-1)$
    - $\lim_{x\to 0} \sqrt{\frac{4x+2}{2}} =$ (1)
    - $\lim_{x\to 1}\left(\frac{1}{x}+\frac{1}{x+1}\right) =$ (3/2)
    - $\lim_{x\to 2}(2+x)^5-1 =$ (4^5-1)
    - $\lim_{x\to 3}(x^2-3x+2) =$ (2)
    - $\lim_{x\to -1}\left(\frac{1}{2x-5}\right) =$ $(-1/7)$

More examples

- **Example** Consider

$$f(x) = \begin{cases} 2 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

We see that $\lim_{x\to a} f(x) = 2$ whenever $a \neq 1$. This is different from the value of f at 1. So $\lim_{x\to 1} f(x) = 2 \neq 1 = f(1)$.

- **Example** Consider the function

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1,\ 2; \\ 3 & \text{if } x = 1; \\ 1 & \text{if } x = 2. \end{cases}$$

Thus $x = a$ and other than $a = 1,\ 2$, then $f(x)$ approaches the value $a+1$ as x approaches a. In fact $f(a) = a+1$. Although when $a = 1,\ 2$, we still have $\lim_{x\to a} f(x) = a+1$, it is not equal to the values of $f(1) = 3$ and $f(2) = 1$. Thus there are two "jumps" on the graph of f.

More examples

- **Example** Suppose

$$f(x) = \begin{cases} x^2 - 3 & \text{if } x < 2; \\ \dfrac{1}{x-1} & \text{if } x \geq 2. \end{cases}$$

For any $a < 2$, $f(x)$ approaches $a^2 - 3$ as x approaches a, and for any $b > 2$, $f(x)$ approaches $1/(b-1)$ as x approaches b. When $x = 2$, $x^2 - 3$ approaches 1 as x approaches 2 on the left, and $1/(x-1)$ approaches 1 as x approaches 2 on the right. Hence we conclude that f approaches 1 as x approaches 2, i.e., The limit $\lim_{x\to 2} f(x) = 1$ exists.

Right limit

Let a and l be two real numbers. If the values of the function $f(x)$ approaches l as close as we wish as x approaches a from the right then we say the right limit of f is equal to l as x tends to a from above. The statement is denoted by

$$\lim_{x \to a+} f(x) = l.$$

We may also write

$$f(x) \to l \quad \text{as} \quad x \to a+\,.$$

We have a similar definition for left limit, denoted by $\lim_{x \to a-} f(x) = l$ or $f(x) \to l \quad \text{as} \quad x \to a-$. We note again that both definitions do not say anything about f at the point $x = a$.

Left and Right limits

- It is not difficult to see that $\lim_{x \to a} f(x) = l$ exists if and only if both

$$\lim_{x \to a+} f(x) = l = \lim_{x \to a-} f(x).$$

- The previous example shown three slides before clearly illustrates this statement
- **Example** Show $|x|$ has limit at all points on the real line.
- **Example** (p. 68) Let

$$f(x) = \frac{|x|}{x}, \qquad x \neq 0.$$

  - Does $\lim_{x \to a} f(x)$ exist, where $a = 0$ or $a \neq 0$?
  - Sketch a graph of $f(x)$.

Example

- Let
$$f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x+1 & \text{if } x \geq 1. \end{cases}$$

- Since f remains at 1 for all $x < 1$, f approaches 1 when x tends to 1 on the left. So
$$\lim_{x \to 1-} f(x) = 1.$$

- Note that
$$f(1) = 2 \neq \lim_{x \to 1-} f(x).$$

- On the other hand,
$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x + 1 = 2.$$

  And we have $f(1) = 2 = \lim_{x \to 1+} f(x)$.

Exercises

- Let

$$f(x) = \begin{cases} 3x - 1, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ 2x + 5, & \text{if } x > 0. \end{cases}$$

- Evaluate
  1. $\lim_{x \to 2} f(x)$,
  2. $\lim_{x \to -3} f(x)$,
  3. $\lim_{x \to 0+} f(x)$,
  4. $\lim_{x \to 0-} f(x)$,
  5. $\lim_{x \to 0} f(x)$.
  6. $\left(\text{Answers } (1)\ 9, (2)\ -10, (3)\ 5, (4)\ -1, (5) \text{ does not exist}\right)$

An example that has no limit

- Recall that the earlier example $f(x) = \begin{cases} 2 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$ has no limit at $x = 1$ which is a discontinuity of f. But we could still correct f to be continuous at $x = 1$ again by re-defining $f(1) = 2$.
- Consider the example on page 64:

$$f(x) = \cos \frac{1}{x}$$

on the interval $(0, 1]$. It is not defined at $x = 0$. We see that even a small change in x near zero would result in a large change of $\frac{1}{x}$.So there would be an unlimited number of oscillations between the values $\{\pm 1\}$ throughout $(0, a]$. Hence no correction of value of $f(x)$ would make $f(x)$ continuous at $x = 0$ again.

Figure 2.14

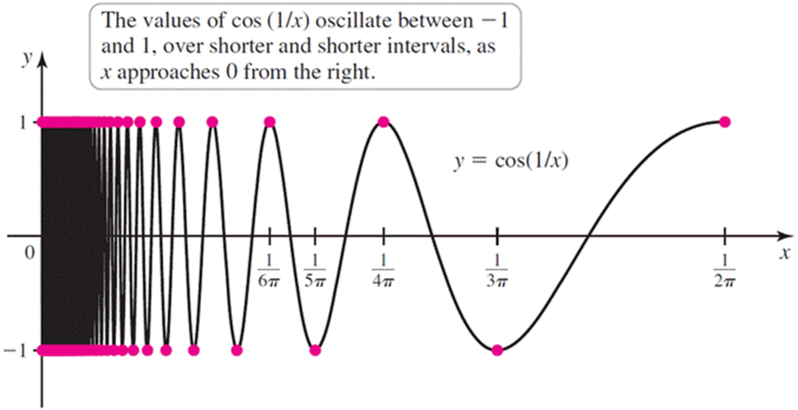


Figure: 2.14 (Publisher)

How to avoid the "infinitesmal"?

Here is the real difficulty:

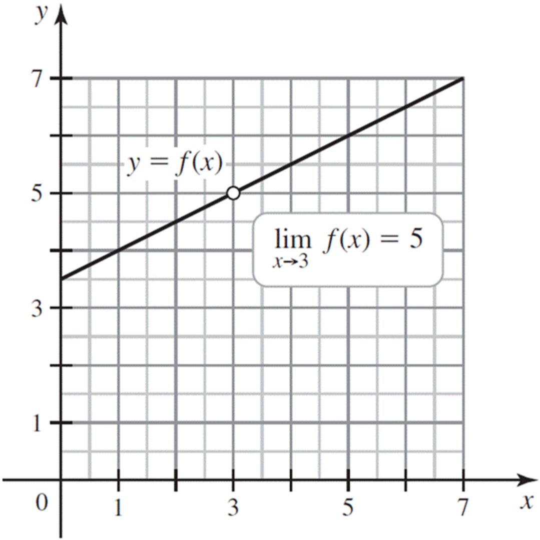
- Our thinking process and/or language usage generally does not allow us to describe infinitesmal quantities clearly
- Mathematicians have found a way to get around describing infinitesmal directly. We say that the function can get as close to a number (limit ℓ) as possble.
- But we need to pay a heavy price if we want to do so precisely. Here it is. The abbreviation $\lim_{x \to a} f(x) = \ell$ really means: Given an arbitrary $\varepsilon > 0$, one can find a $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon, \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

- Both ε and δ represent positive real numbers. Given each/any $\varepsilon > 0$ one can (always) find a $\delta > 0$ such that ... holds
- we refer to this kind of statement as $\varepsilon - \delta$ language interpretation.

A linear function example

How do we use $\delta - \varepsilon$ to describe $\lim_{x\to 3} = 5$?



$\varepsilon = 1$

How do we use $\delta - \varepsilon$ to describe $\lim_{x\to 3} = 5$?

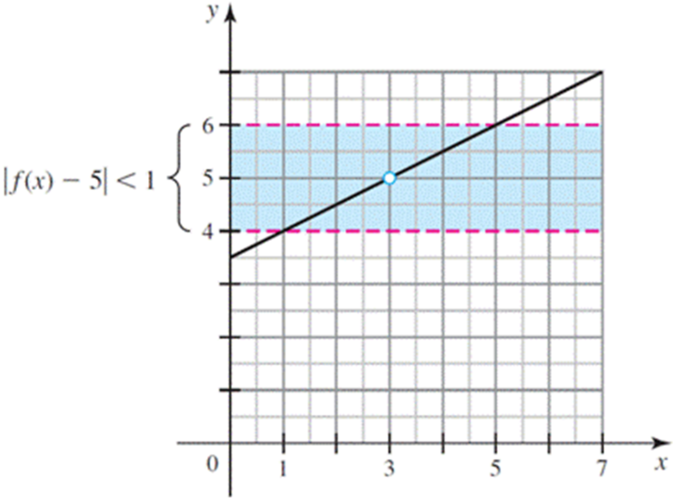
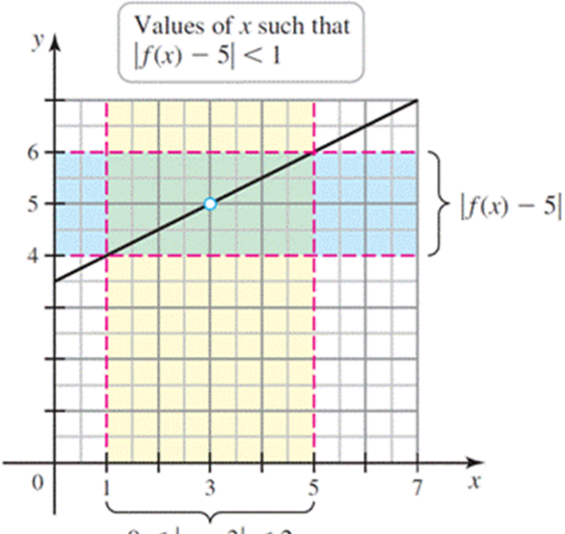


Figure: 2.57a (Publisher)

$\delta = 2$

The corresponding $\delta = 2$. That is, $0 < |x - 3| < 2$ guarantees $|f(x) - 5| < 1$.



$\varepsilon = 1/2$

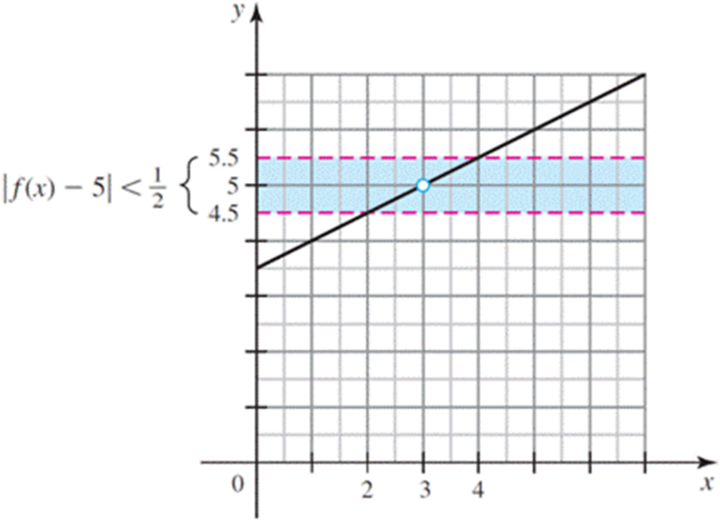
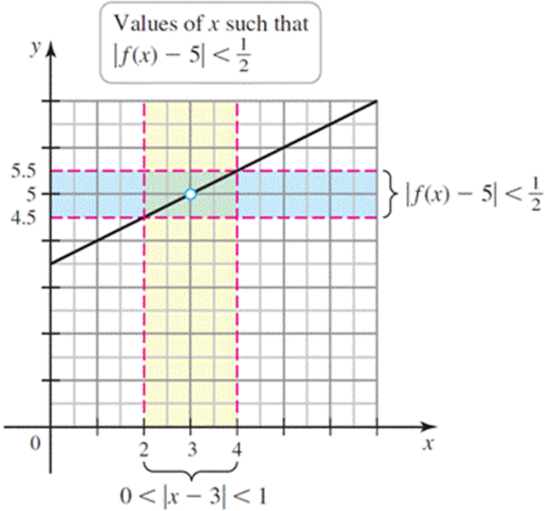


Figure: 2.58a (Publisher)

$\delta = 1$

The corresponding $\delta = 2$. That is, $0 < |x - 3| < 1$ guarantees $|f(x) - 5| < 1/2$.



$\varepsilon = 1/8,\ \delta = 1/4$

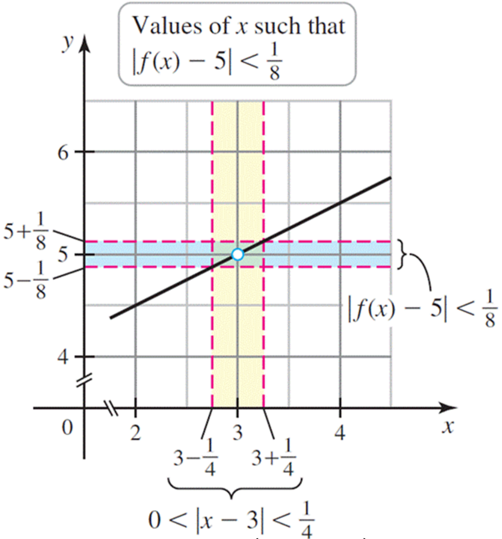


Figure: 2.59 (Publisher)

General $\varepsilon - \delta$

That is, $0 < |x - 3| < \delta$ guarantees $|f(x) - 5| < \varepsilon$.

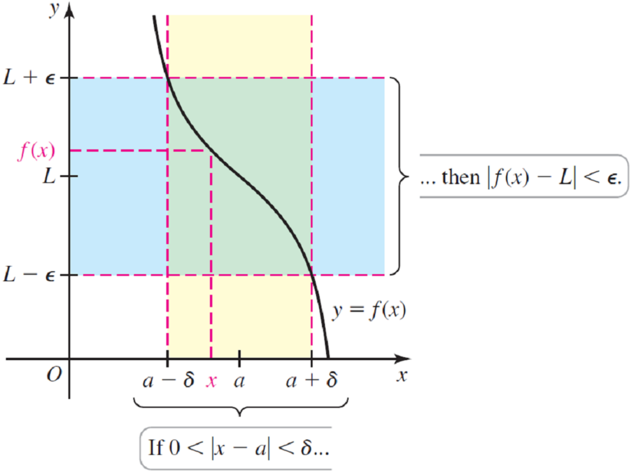


Figure: 2.60 (Publisher)

Example p. 115

$f(x) = x^3 - 6x^2 + 12x - 5$. In order to show $\lim_{x\to 2} f(x) = 3$, given $\varepsilon = 1$, find the corresponding δ.

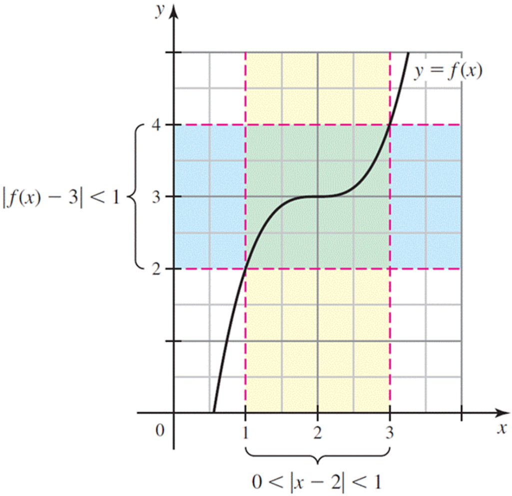


Figure: 2.61 (Publisher)

Example p. 115 (cont.)

$f(x) = x^3 - 6x^2 + 12x - 5$. In order to show $\lim_{x\to 2} f(x) = 3$, given $\varepsilon = 1$, find the corresponding δ.

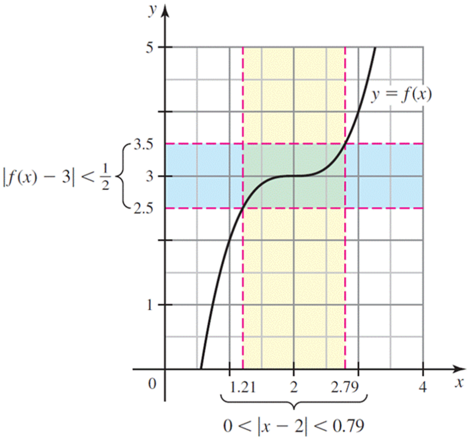


Figure: 2.62 (Publisher)

$\varepsilon - \delta$ definition example

- Let $f(x) = 2x$. Show $\lim_{x\to 0} 2x = 0$ by the $\varepsilon - \delta$ argument of limit.
- Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that

$$|2x - 0| < \varepsilon, \qquad \text{whenever } 0 < |x - 0| < \delta.$$

- Notice that $|2x - 0| = |2x| = 2|x|$. So if we impose that $0 < \delta < \varepsilon/2$ and that $|x| < \delta < \varepsilon/2$. Hence under this restriction of δ and x, we have

$$|2x - 0| = |2x| = 2|x| < 2\,\delta < 2\,\frac{\varepsilon}{2} = \varepsilon$$

Thus given the $\varepsilon > 0$, we have found $\delta > 0$ (namely $\delta < \varepsilon/2$). Since this argument works for every $\varepsilon > 0$. We conclude that $\lim_{x\to 0} 2x = 0$.

$\varepsilon - \delta$ definition example

- Let $f(x) = 3x + 1$. Show $\lim_{x\to 1} 3x + 1 = 4$ by the $\varepsilon - \delta$ argument of limit.
- Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that

$$|(3x + 1) - 4| < \varepsilon, \qquad \text{whenever } 0 < |x - 1| < \delta.$$

- Notice that $|(3x + 1) - 4| = |3x - 3| = 3|x - 1|$. So if we impose that $0 < \delta < \varepsilon/3$ and that $|x - 1| < \delta < \varepsilon/3$. Hence under this restriction of δ and x, we have

$$|(3x + 1) - 4| = |3x - 3| = 3|x - 1| < 3\,\delta < 3\,\frac{\varepsilon}{3} = \varepsilon$$

Thus given the $\varepsilon > 0$, we have found $\delta > 0$ (namely $\delta < \varepsilon/3$). Since this argument works for every $\varepsilon > 0$. We conclude that $\lim_{x\to 1} 3x + 1 = 4$.

$\varepsilon - \delta$ definition example

- Let $f(x) = x^2$. Show $\lim_{x\to 2} x^2 = 4$ by the $\varepsilon - \delta$ argument of limit.
- Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon, \qquad \text{whenever } 0 < |x-2| < \delta.$$

- Notice that $|x^2 - 4| = |(x-2)(x+2)|$. We can control the factor $|x-2|$, but the other factor $|x+2|$ depends on x which is unlike those of previous examples. Since we are close to 2 anyway, so WLOG, we may impose $|x-2| < 1$. So $|x| - 2 \le |x-2| < 1$. So $|x| < 3$ and $|x+2| \le |x| + 3 < 5$.
- We impose $|x| < 3$ and $|x-2| < \delta < \varepsilon/5$, and whichever is smaller. i.e., $\delta < \min(1, \frac{\varepsilon}{5})$. Then we have

$$|x^2 - 4| = |(x-2)|\,|(x+2)| < 5|x-2| < 5\,\delta < 5\,\frac{\varepsilon}{5} = \varepsilon$$

Thus given any $\varepsilon > 0$, we have found a $\delta > 0$. We conclude that $\lim_{x\to 2} x^2 = 4$.

$\varepsilon - \delta$ limit exercises

Employ $\varepsilon - \delta$ arguments to prove the following limits:

- $\lim_{x\to 1} 2x - 1 = 1$;
- $\lim_{x\to -1} 2x - 1 = -3$;
- $\lim_{x\to 1} ax + b = a + b$;
- $\lim_{x\to 1} x^2 = 1$;
- $\lim_{x\to -1} x^2 = 1$;
- $\lim_{x\to 1} \frac{1}{x} = 1$;
- $\lim_{x\to 1} \frac{1}{x^2} = 1$.
- $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

Limit laws

- Suppose $\lim_{x\to a} f(x) = \ell$, $\lim_{x\to a} g(x) = m$ both exist. Let c be a constant, then the following hold:
- $$\lim_{x\to a}\Big(f(x)+g(x)\Big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = \ell + m$$
- $$\lim_{x\to a}\Big(c\,f(x)\Big) = c\lim_{x\to a} f(x) = c\ell$$
- $$\lim_{x\to a}\Big(f(x)g(x)\Big) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) = \ell m$$
- $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\ell}{m} \quad \text{provided } m \neq 0.$$

One-sided Limit laws

These properties of limit have counterparts in the left and right limits formulations. Since the formulations are exactly the same as the above results except that the number a is replaced by either $a-$ or $a+$, so we omit the details here.

The real difficulty again

- So for $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$, one needs to show, assuming that $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = s$
  Given an arbitrary $\varepsilon > 0$, one can find a $\delta > 0$ such that

$$\big|[f(x)+g(x)] - (\ell + s)\big| < \varepsilon, \qquad \text{whenever } 0 < |x-a| < \delta.$$

  with the given assumption.
- This is slightly not easy. Some other laws are more difficult to verify using this language. So this explains why one needs to state these seemingly simple laws as separate entities.

Examples

- (p. 71) Given that $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = 5$, $\lim_{x\to 2} h(x) = 8$.
- $$\begin{aligned}\lim_{x\to 2}[6f(x)g(x) + h(x)] &= 6 \lim_{x\to 2}[f(x)g(x)] + \lim_{x\to 2} h(x)] \\ &= 6 \cdot \lim_{x\to 2} f(x) \cdot \lim_{x\to 2} g(x) + \lim_{x\to 2} h(x) \\ &= 6 \cdot (4 \cdot 5) + 8 = 128.\end{aligned}$$
- $$\begin{aligned}\lim_{x\to 2} \frac{f(x) - g(x)}{h(x)} &= \frac{\lim_{x\to 2}[f(x) - g(x)]}{\lim_{x\to 2} h(x)} \\ &= \frac{\lim_{x\to 2} f(x) - \lim_{x\to 2} g(x)}{\lim_{x\to 2} h(x)} \\ &= \frac{4-5}{8} = -\frac{1}{8}.\end{aligned}$$

Examples

- By the above properties,

$$\begin{aligned}\lim_{x\to 2}(4x^2+20) &= \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 20 \\ &= 4\lim_{x\to 2} x^2 + \lim_{x\to 2} 20 \\ &= 4(4)+20 \\ &= 36.\end{aligned}$$

We note that since both $\lim_{x\to 2} x^2$ and $\lim_{x\to 2} 20$ exist, so we can apply the above properties.

Examples

- $\lim_{x\to 3} \frac{3x^2-1}{1-6x}$
  Applying the above properties give
  $$\lim_{x\to 3} \frac{3x^2-1}{1-6x} = \frac{\lim_{x\to 3}(3x^2-1)}{\lim_{x\to 3}(1-6x)} = \frac{26}{-17}.$$
  We again note both $\lim_{x\to 3}(3x^2-1)$ and $\lim_{x\to 3}(1-6x)$ exist. Hence we can apply the above result.
- $\lim_{x\to 3}(x-1)^2(x+1)$
  So
  $$\begin{aligned}\lim_{x\to 3}(x-1)^2(x+1) &= \lim_{x\to 3}(x-1)^2 \cdot \lim_{x\to 3}(x+1)\\ &= (3-1)^2\cdot(3+1)\\ &= 16\end{aligned}$$
  We could apply some of the above limit laws, this is because that both $\lim_{x\to 3}(x-1)^2$ and $\lim_{x\to 3}(x+1)$ exist.

Examples

- (p. 72)

$$\lim_{x\to 2}\frac{\sqrt{2x^3+9}+3x-1}{4x+1} = \frac{\lim_{x\to 2}(\sqrt{2x^3+9}+3x-1)}{\lim_{x\to 2}4x+1}$$

$$= \frac{\sqrt{\lim_{x\to 2}(2x^3+9)}+\lim_{x\to 2}(3x-1)}{\lim_{x\to 2}4x+1}$$

$$= \frac{\sqrt{2\cdot 2^3+9}+(3\cdot 2-1)}{4\cdot 2+1}$$

$$= \frac{\sqrt{25}+5}{9} = \frac{10}{9}.$$

Squeezed limits

- (p. 76) **Theorem** Assume that a functions f, g, f satisfy $f(x) \leq g(x) \leq h(x)$ for all x near a, except possibly at a. If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$.
- (p. 76) **E.g.** It is clear from the graph that

$$-|x| \leq \sin x \leq |x|, \qquad 0 \leq 1 - \cos x \leq |x|$$

  hold on $[-\pi/2, \pi/2]$. Since $\lim_{x\to 0} |x| = 0$, so the Squeeze theorem implies that $\lim_{x\to 0} \sin x = 0$. Similarly, $\lim_{x\to 0} \cos x = 1$.
- (p. 77) **E.g.** Show $\lim_{x\to 0} x^2 \sin \frac{1}{x} = 0$.

New situations

- **Example** Let $f(x) = 2 + \frac{1}{x^2}$ for $x > 0$.
- We want to investigate the behaviour of $f(x)$ when "x is large".
- $f(x)$ gets as close to 2 as we please by letting x "sufficiently large", i.e., $f(x)$ tends to 2 as x becomes arbitrary large and positive.
- Similarly $f(x)$ tends to 2 as x becomes arbitrary large and negative.
- On the other hand, $f(x)$ becomes arbitrary large as x approaches 0 on either sides.
- As the above description is quite long and vague, so people naturally want to find a better way to describe the situation. So they come up with the following definition.

Definitions

- The behaviour of the function $f(x)$ described on last slide certainly has no limit in ordinary sense. But it is still important enough to deserve a special mention.
- **Infinity** We give a meaning of the symbol $+\infty$, called **positive infinity**, that indicates a quantity described grows larger than any given positive number;
- $-\infty$ **negative infinity** that indicates a quantity described grows smaller than any given negative number.
- Both the notations "$\pm\infty$" are **NOT NUMBERS**. They are being artificially inserted on the real axis $\mathbb{R}$: So
  - $10,000 < +\infty$,
  - $10,000,000 < +\infty$,
  - $10^{10} < +\infty$
  - $\ldots\ldots$
  - $-\infty < -10,000$,
  - $-\infty < -10^{10}$.

Limits at infinity

- **Definitions** Let ℓ and a be real numbers. If f tends to ℓ as x becomes arbitrary large and positive, we say f has the limit ℓ at positive infinity, written as

$$\lim_{x\to+\infty} f(x) = \ell \qquad (f \to \ell, \quad \text{as} \quad x \to +\infty).$$

- Similarly, if f tends to ℓ as x becomes arbitrary large and negative, we say f has the limit ℓ at negative infinity, written as

$$\lim_{x\to-\infty} f(x) = \ell \qquad (f \to \ell, \quad \text{as} \quad x \to -\infty).$$

Infinity limits

- If f becomes arbitrary large and positive as x approaches a, we denote this by

$$\lim_{x\to a} f(x) = +\infty \qquad (f \to +\infty, \text{as } x \to a).$$

 and we either say f has no limit at a, or that f has the limit infinity at a.

- Similarly, if f becomes arbitrary large and negative as x approaches a, we write

$$\lim_{x\to a} f(x) = -\infty \qquad (f \to -\infty, \text{ as } x \to a).$$

- **Remark** We sometimes write ∞ for $+\infty$.
- **Remark** Both the notations "$\pm\infty$" are **NOT NUMBERS**

An example

- **Example** (revisited) For $f(x) = 2 + 1/x^2$, we clearly have:

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (2 + 1/x^2) = 2,$$

since f tends to 2 as $x \to +\infty$. Note that there is no finite value x we can find so that $f(x) = 2$.

- Similarly

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (2 + 1/x^2) = 2,$$

since f tends to 2 as $x \to -\infty$.

- Finally

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (2 + 1/x^2) = +\infty,$$

since f becomes arbitrary large as $x \to 0$.

Infinite limit examples

- (p. 82) $\lim_{x \to 1} \dfrac{x}{(x^2-1)^2}$;
- (p. 82) $\lim_{x \to -1} \dfrac{x}{(x^2-1)^2}$
- (p. 83) $\lim_{x \to 1} \dfrac{x-2}{(x-1)^2(x-3)}$
- (p. 83) $\lim_{x \to 3\pm} \dfrac{x-2}{(x-1)^2(x-3)}$
- (p. 84) $\lim_{x \to 4+} \dfrac{-x^3+5x^2-6x}{-x^3-4x^2}$
- The vertical lines where the curves that represent the above functions that become infinite that we encounter above are called vertical asymptotes of the function $f(x)$.

Properties of Limits at Infinity

Suppose $\lim_{x\to\infty} f(x) = \ell$, $\lim_{x\to\infty} g(x) = m$ both exist. Let c be a constant, then the following hold:

- $$\lim_{x\to\infty}\Big(f(x)+g(x)\Big) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = \ell + m,$$
- $\lim_{x\to\infty}\Big(c\,f(x)\Big) = c\,\lim_{x\to\infty} f(x) = c\,\ell,$
- $\lim_{x\to\infty}\Big(f(x)g(x)\Big) = \lim_{x\to\infty} f(x)\lim_{x\to\infty} g(x) = \ell\, m,$
- $\lim_{x\to\infty}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \dfrac{\ell}{m}$ provided $m \neq 0$.
- We note that the above rules do not apply when one or both of $\lim_{x\to\infty} f(x)$, and $\lim_{x\to\infty} g(x)$ are infinite. Note, however, that ∞ can be replaced by $-\infty$.

Examples of Limit at infinity

- **Example** (revisit) Let $f(x) = 2 + 1/x^2$. Then

$$\lim_{x\to\infty}(2+1/x^2) = \lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x^2} = 2+0 = 2.$$

since both $\lim_{x\to\infty} 2$ and $\lim_{x\to\infty} 1/x^2$ exist.

- **Example** (revisit) Let $f(x) = \dfrac{x^2+2x}{x^3+4}$. Then

$$\lim_{x\to\infty}\frac{x^2+2x}{x^3+4} = \lim_{x\to\infty}\frac{x^2(1+2/x)}{x^3(1+4/x^3)} = \lim_{x\to\infty}\frac{1+2/x}{x(1+4/x^3)}$$
$$= \lim_{x\to\infty}\frac{1}{x}\cdot\frac{\lim(1+2/x)}{\lim(1+4/x^3)} = 0\cdot\frac{1}{1} = 0,$$

since $\lim_{x\to\infty} 1/x$, $\lim_{x\to\infty}(1+2/x)$, $\lim_{x\to\infty}(1+4/x^3)$ exist.

More examples

- **Example** $\lim_{x\to\infty} \dfrac{2x^2+3x}{6x^2-x}$.

$$\lim_{x\to\pm\infty}\left(\frac{2x^2+3x}{6x^2-x}\right) = \lim_{x\to\pm\infty}\frac{x^2(2+3/x)}{x^2(6-1/x)}$$
$$= \frac{\lim(2+3/x)}{\lim(6-1/x)} = \frac{2+0}{6-0} = 1/3.$$

- **Example** $\lim_{x\to\infty} \dfrac{3x^3+3x}{4x^3-x^2}$.

Horizontal asymptotes

- **Definition** If $f(x) \to \ell$ or $f(x) - \ell \to 0$ as $x \to \infty$, then we say $y = \ell$ is a horizontal asymptote of $f(x)$ as $x \to \infty$.
- **Example** (revisited) Since $\lim_{x\to\infty} \left(\frac{2x^2+3x}{6x^2-x}\right) - \frac{1}{3} = 0$, so $y = \frac{1}{3}$ is a horizontal asymptote of $f(x)$ as $x \to \infty$.
- **Example** (p. 90) Since $\lim_{x\to\infty} \left(5 + \frac{\sin x}{\sqrt{x}}\right) - 5 = 0$, so $y = 5$ is a horizontal asymptote of the function as $x \to \infty$.
- **Example** (p. 90) Consider $\lim_{x\to\pm\infty} \frac{x}{2x^2 - x + 3}$.

  Since

  $$x/(2x^2 - x + 3) \to 0 \qquad \text{as } x \to +\infty,$$

  so $y = 0$ is a horizontal asymptote of $f(x)$ as $x \to +\infty$

  Similarly, since

  $$x/(2x^2 - x + 3) \to 0 \qquad \text{as } x \to -\infty,$$

  so $y = 0$ is a horizontal asymptote of $f(x)$ as $x \to -\infty$.

  Observe that the f approaches the $y = 0$ in different manners

An example on p. 96

- **Example** $\lim_{x\to\infty} \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

$$\frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = \frac{x^3\left(10 - \frac{3x^2}{x^3} + \frac{8}{x^3}\right)}{|x^3|\sqrt{25 + \frac{1}{x^2} + \frac{2}{x^6}}} \to \frac{10}{\sqrt{25}} = 2$$

as $x \to +\infty$ and since $x^3/|x^3| = 1$ as $x > 0$. So
$\lim_{x\to\infty} \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = 2$

- We have

$$\lim_{x\to-\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = -2$$

since $x^3/|x^3| = -1$ when $x < 0$.

Other asymptotes

- **Definition** If $f(x) \to g(x)$ or $f(x) - g(x) \to 0$ as $x \to \infty$ (resp. $-\infty$), then we say $y = g(x)$ is an asymptote of $f(x)$ as $x \to \infty$ (resp. $-\infty$).
- **Remark** Usually the asymptote function $y = g(x)$ is simpler and more familiar to us.
- **Example** (p. 98) $\lim_{x\to\infty} \frac{x^2-1}{x+2}$

  Since
  $$\frac{x^2-1}{x+2} - x = \frac{-2x-1}{x+1} \to -2$$
  as $x \to \infty$, so
  $$\frac{x^2-1}{x+2} - (x-2) = \frac{3}{x+1} \to 0$$
  as $x \to \infty$. Hence $y = x - 2$ is an asymptote of the function.
- What happens if $x \to -\infty$?

Examples

- **Example** Find asymptotes of $f(x) = x^3 - 100,000\,x^2$.

$$f(x) = x^3\left(1 - \frac{100,000}{x^3}\right)$$

so that

$$\frac{f(x) - x^3}{x^3} = 1 - \frac{100,000}{x^3} \to 0 \qquad \text{as } x \to \infty.$$

We deduce that $g(x) = 1$ is an asymptote of $\frac{f(x)}{x^3}$ as $x \to \infty$.

- What happens if $x \to -\infty$?
- **Definiton** If $f(x)$ becomes arbitrarily large and positive when $x \to \infty$ or $x \to -\infty$, then we write

$$\lim_{x\to\infty} f(x) = \infty, \qquad \text{or} \quad \lim_{x\to-\infty} f(x) = \infty$$

respectively. If $f(x)$ becomes arbitrarily large and negative

$$\lim_{x\to\infty} f(x) = -\infty, \qquad \text{or} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Example with no asymptote

- **Example** Let

$$f(x) = \begin{cases} n, & \text{if } n < x \leq n+1 \quad (n = 0, 2, 4, \cdots), \\ -n, & \text{if } n < x \leq n+1 \quad (n = 1, 3, 5, \cdots). \end{cases}$$

This function has no limit at both $+\infty$ and $-\infty$. This is because f is oscillating between n and $-n$. It will never "tend" to any fixed value either finite or infinite.

Continuity

- **Definition** Let f be a function and that $\lim_{x\to a} f(x) = \ell$ exists. Then f is continuous at a if $f(a)$ exists and $f(a) = \ell$. We say that f is continuous on an interval I if it is continuous at every point of I.
- Generally speaking a function is continuous at $x = a$, say, if the curve of f at a has no *jump*, or that one does not need to lift a pen when drawing that part of curve containing the point a.
- **Example** The function $f(x) = \begin{cases} 2 & \text{if } x \neq 1/2; \\ 1 & \text{if } x = 1/2. \end{cases}$ is not continuous at $x = 1/2$. Otherwise, it is continuous everywhere.

Continuity examples

- **Example** Show that $f(x) = 2x^2 + 3x$ is continuous at $x = 1$. Since $\lim_{x \to 1} 2x^2 = 2 \lim_{x \to 1} x^2 = 2 = 2(1)^2$, and $\lim_{x \to 1} 3x = 3$. Thus both $2x^2$ and $3x$ are continuous at $x = 1$. Obviously, $\lim_{x \to 1}(2x^2 + 3x) = 5 = f(1) = 2(1)^2 + 3(1)$. Thus f is continuous at 1. The above argument clearly applies to any x other than 1. So f is continuous not only at 1 but on $\mathbb{R}$.
- **Example** Polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous at every point in $\mathbb{R}$. This follows from the fact that the sum of two continuous functions is still a continuous function.

Continuity example I

- **Example** Determine the region of continuity of

$$f(x) = \frac{x^2 - 3}{x^2 + 2x - 8}.$$

Since both $x^2 - 3$ and $x^2 + 2x - 8$ are continuous functions (being polynomials), their quotient is also continuous whenever $x^2 + 2x - 8 \neq 0$. But $x^2 + 2x - 8 = (x + 2)(x - 4)$ equals zero only when $x = -2$ and 4. Thus f is continuous except when $x = -2$ or 4, i.e., the region of continuity is $\mathbb{R} \backslash \{-2, 4\}$, that is the whole real line except the points -2 and 4.

Continuity example II

- **Example** Find the region of discontinuity of

$$f(x) = \begin{cases} x^2, & \text{if } x < 3, \\ x + 6, & \text{if } x \geq 3. \end{cases}$$

Since both x^2 and $x + 6$ are continuous on the real axis, we conclude from the definition of f that it must be continuous except perhaps when $x = 3$. The left limit is

$$\lim_{x \to 3+} (x + 6) = 3 + 6 = 9 = f(x),$$

whereas the right limit is

$$\lim_{x \to 3-} f(x) = \lim_{x \to 3-} x^2 = 9 = f(3) = 3 + 6.$$

Since the left and right limits are equal, it follows from the definition that $\lim_{x \to 3} f(x) = f(3)$ i.e., $f(3)$ exists and f is continuous at 3. Hence the region of discontinuity is an empty set.

Continuity example III

- **Example** Find the region of discontinuity of

$$g(x) = \begin{cases} x^2, & \text{if } x < 3, \\ x+6, & \text{if } x > 3. \end{cases}$$

Define a new function F so that it is continuous on $\mathbb{R}$.

- Since $g(x)$ is almost identical to f in the last example, we conclude that (from the definition of g) g is continuous on $\mathbb{R}$ except when $x = 3$ at which g is undefined. But

$$\lim_{x\to 3-} g(x) = \lim_{x\to 3-} x^2 = 9 = \lim_{x\to 3+} (x+6) = \lim_{x\to 3+} g(x).$$

and this shows that g actually converges to the right value 9 as x approaches 3. Thus the following function

$$F(x) = \begin{cases} g(x), & \text{if } x \neq 3; \\ 9, & \text{if } x = 3. \end{cases}$$

is continuous on $\mathbb{R}$ and $F(x)$ is thus identical to the function f in the last slide.

Continuity example IV

- **Example** The function $f(x) = \frac{x^3-8}{x-2}$ is not continuous at $x = 2$.

  Although we have

  $$\frac{x^3 - 8}{x - 2} = \frac{\cancel{(x - 2)}(x^2 + 2x + 4)}{\cancel{x - 2}} = x^2 + 2x + 4$$

  but the above cancellation is only valid when $x - 2 \neq 0$ or $x \neq 2$. So the function $f(x)$ only equals to $x^2 + 2x + 4$ when $x \neq 2$. So the f is still undefined at $x = 2$. So the function $f(x)$ must be discontinuous at $x = 2$.
- But we do have the limit $\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} x^2 + 2x + 4 = 12$ although $f(x)$ is undefined there.
- We define $F(x) = \begin{cases} f(x), & \text{if } x \neq 2; \\ 12, & \text{if } x = 2. \end{cases}$ Then we have $\lim_{x\to 2} F(x) = 12 = F(2)$ so that $F(x)$ is continuous at $x = 2$.