MATH1013 Calculus I

Derivatives III (§4.1-4.3) [1]

Edmund Y. M. Chiang

Department of Mathematics
Hong Kong University of Science & Technology

November 5, 2014

---

[1] Based on Stewart, James, "Single Variable Calculus, Early Transcendentals", 7th edition, Brooks/Coles, 2012
Briggs, Cochran and Gillett, Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Maximum/Minimum

- **Definitions** Let c be in the domain D of f. Then $f(x)$ is the
  1. absolute maximum value of f on D if $f(c) \geq f(x)$ for all $x \in D$;
  2. absolute minimum value of f on D if $f(c) \leq f(x)$ for all $x \in D$
- **Definitions** Let c be in the domain D of f. Then $f(c)$ is a
  1. local maximum value of f if $f(c) \geq f(x)$ when x is near c;
  2. local minimum value of f if $f(c) \leq f(x)$ when x is near c.

Figure of Maximum/Minimum

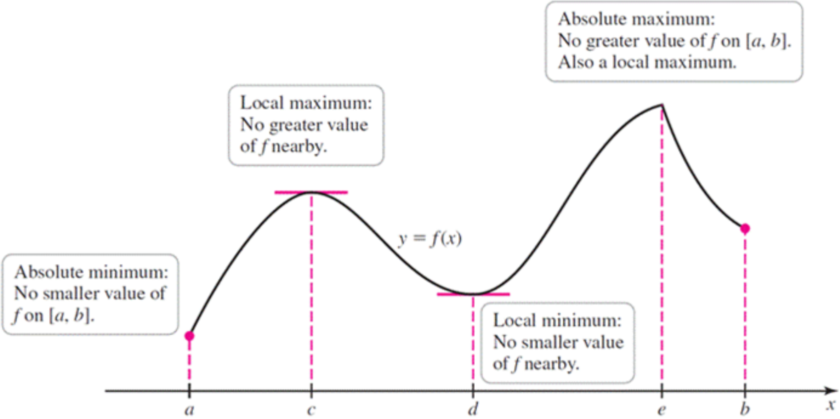
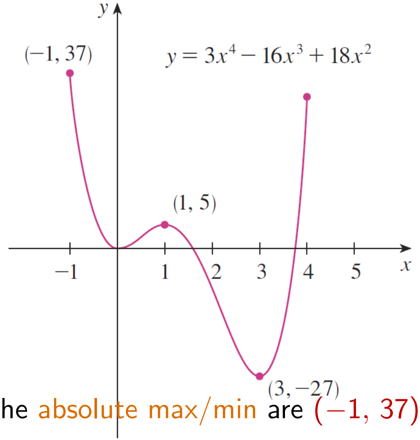


Figure: (Briggs, et al, Figure 4.5 (p. 259))

Stewart: Example 4

- The graph of the function $f(x) = 3x^4 - 16c^3 + 18x^2$ $(-1 \leq x \leq 4)$ is shown below



- We see that the absolute max/min are $(-1, 37)$ and $(3, -27)$ respectively.
- The local max/min are $(1, 5)$ and $(3, -27)$ respectively.

Extreme Value theorem

- **Theorem** A function $f(x)$ continuous on a closed interval $[a, b]$ attains its absolute maximum/minimum on $[a, b]$. That is, there exist c, d in $[a, b]$ such that

$$f(x) \geq f(c), \qquad f(x) \leq f(d) \qquad \text{for all } x \text{ in } [a, b].$$

- This result looks very trivial is in fact a deep result in elementary mathematical analysis. It is proved vigorously in chapter 5 (Theorem 5.3) of my supplementary notes on Mathematical Analysis course found in my web site of this course.
- What we will do in the following sides is to show the Extreme Value theorem does not hold when any one of the hypotheses fails to hold.

Stewart: Example 8

- Find the absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1, \qquad -\frac{1}{2} \leq x \leq 4.$$

We locate all the critical points:

$$f'(x) = 3x^3 - 6x = 3x(x-2).$$

Thus, the critical points $x = 0,\ 2$. Their coordinates are $(0,\ 1)$ and $(2,\ -3)$.

- The values of f at the end points:

$$f(-\frac{1}{2}) = \frac{1}{8}, \qquad f(4) = 17.$$

- We see that the absolute maximum locates at $f(4) = 17$ which is at an end point, while the absolute minimum occurs at the critical point $f(2) = -3$.

Absolute extrema example

- **Example** (Briggs, et al, p. 262) Find the maximum/minimum of $f(x) = x^4 - 2x^3$ on $[-2, 2]$.
  We note that since this is a smooth function, so f' exists at all points in $[-2, 2]$.

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$$

- so that the critical points are at $\{0, \frac{3}{2}\}$.
- But $f(0) = 0$, $f(\frac{3}{2}) = -\frac{27}{16}$, $f(-2) = 32$, $f(2) = 0$ so that
  - $f(\frac{3}{2}) = -\frac{27}{16}$ is both a local and global minimum,
  - while $f(0) = 0$ is neither a max nor a min, and that
  - $f(-2) = 32$ is a global maximum on $[-2, 2]$.
- So f can attend an absolute maximum/minimum at end points of a finite interval rather than at the critical points.

Absolute extrema example (cont.)

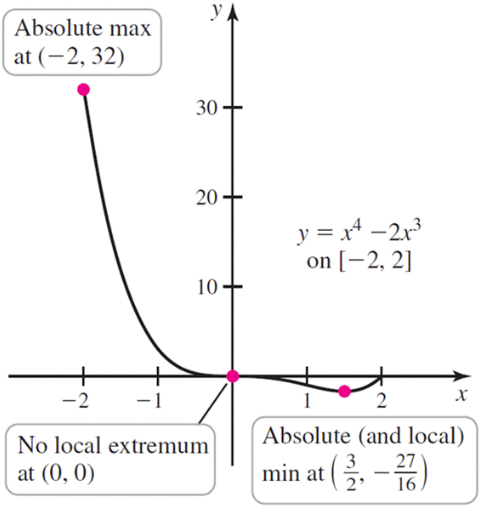


Figure: (Figure 4.11 (p. 236))

Extreme value theorem example I

$f(x) = x^2$. It requires boundedness and closedness of the interval $[a, b]$ assumption.

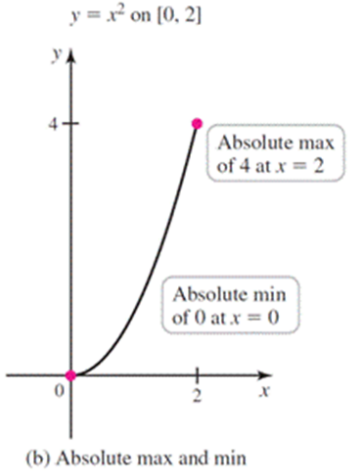


Figure: (Briggs, et al, Figure 4.2 (b))

Extreme value theorem example II

It requires continuity $f(x)$ assumption.

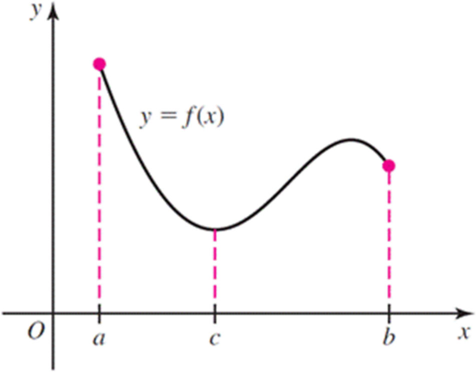


Figure: (Briggs, et al, Figure 4.4 (a))

How extreme value theorem can fail I

$f(x) = x^2$. Dropping boundedness of interval $[a, b]$ assumption.

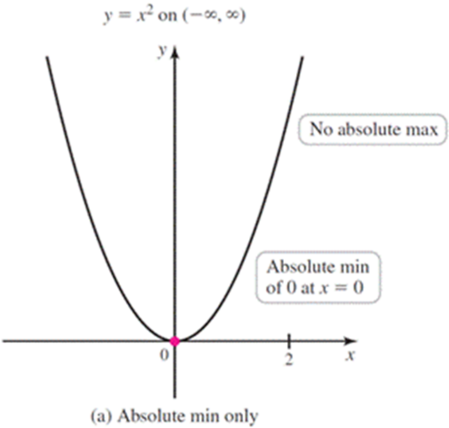


Figure: (Briggs, et al, Figure 4.2 (a))

How extreme value theorem can fail II

$f(x) = x^2$. Dropping closed interval $[a, b]$ assumption.

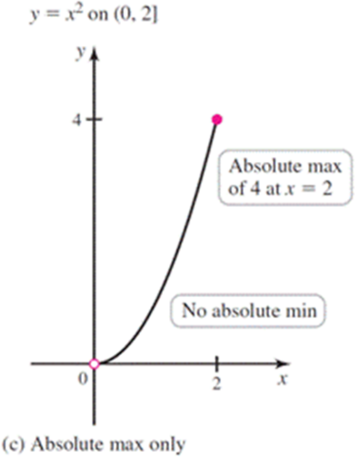


Figure: (Briggs, et al, Figure 4.2 (c))

How extreme value theorem can fail III

$f(x) = x^2$. Dropping closed interval $[a, b]$ assumption.

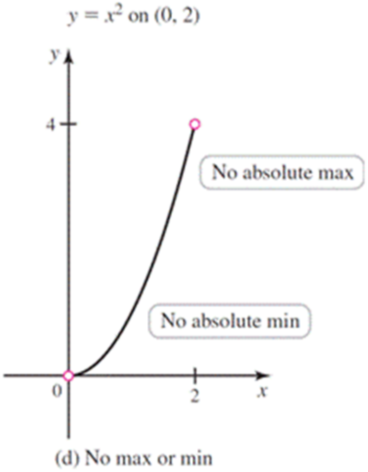


Figure: (Briggs, et al, Figure 4.2 (d))

How extreme value theorem can fail IV

Dropping continuity $f(x)$ assumption.

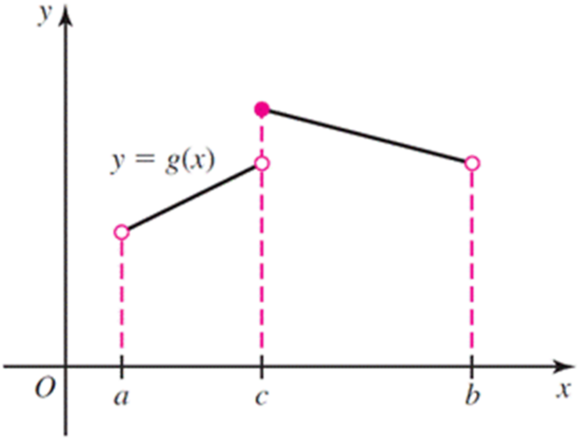


Figure: (Briggs, et al, Figure 4.4 (b))

Fermat's theorem and critical points

- **Theorem** If f has a local maximum/minimum at c, then $f'(c) = 0$.
- **Definition** A number c that lies in the domain D of a function f is called a critical point/number, if either $f'(c) = 0$ or $f'(c)$ does not exist.
- We note that if $f(a)$ is a local maximum/minimum, then it implies $f'(a) = 0$
- However, if $f'(a) = 0$, then $f(a)$ may or may not be a maximum/minimum or neither of $f(x)$.

Critical point example

- (Stewart p. 278) Find the critical point(s) of $f(x) = x^{3/5}(4 - x)$.
- Then it is easy to check that

$$f'(x) = x^{3/5}(-1) + (4 - x)(\frac{3}{5}x^{-2/5}) = \frac{12 - 8x}{5x^{2/5}}$$

so that there is only one critical point at $x = 3/2$.

Rolle's theorem

- **Theorem** Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.

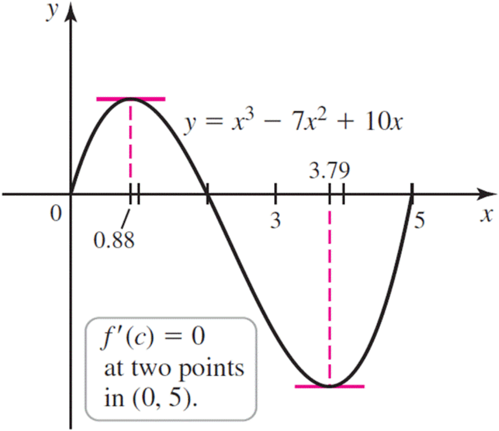
- **Proof** Case I: $f(x) =$constant. The $f'(x) = 0$ at all point in (a, b).
- Case II: If $f(x)$ attains its maximum and minimum at the end points of $[a, b]$, then $f(a) = f(b)$. That is, $f(x)$ is a constant on (a, b). So $f'(c) = 0$ for all c in (a, b).

- Case II: If at least one of the extreme points doesn't coincide with the end points a, b, then this extreme point, c say must lie in (a, b). But then we know $f'(c) = 0$.

Rolle's theorem example

$f(x) = x^3 - 7x^2 + 10x$ has critical points that satisfy the Rolle's theorem at

$$x = \frac{7 \pm \sqrt{19}}{3}, \text{ or } x \approx 0.88 \text{ and } x \approx 3.79.$$



Rolle's theorem example

Show $f(x) = x^3 + x - 1$ has exactly one real root.

- Notice that $f(0) < 0$ while $f(1) > 0$ so that the Intermediate value theorem asserts there is a point d between 0 and 1 such that $f(c) = 0$. This show that there is at least one root between 0 and 1.
- Suppose there are more than two roots a, b between 0 and 1. Then Rolle's theorem asserts that there must be a c between 0 and 1 such that $f'(c) = 0$.
- However,

  $$f'(x) = 3x^2 + 1 > 0$$

  for all x lying between 0 and 1. This contradicts the above argument that there must be a c between 0 and 1 such that $f'(c) = 0$.
- This contradiction shows that there cannot be two roots a, b.

How Rolle's theorem can fail I

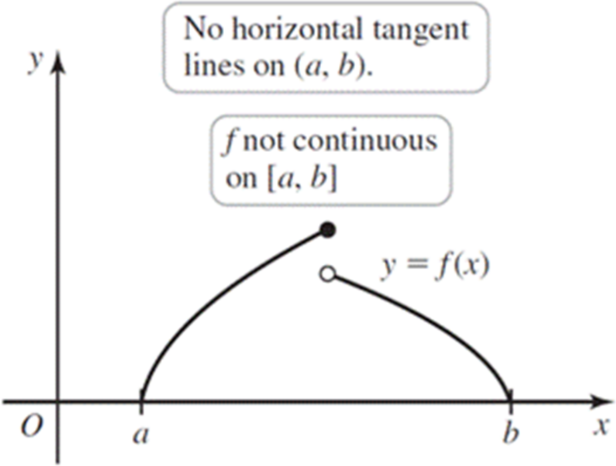


Figure: (Publisher Figure 4.66 (a))

How Rolle's theorem can fail II

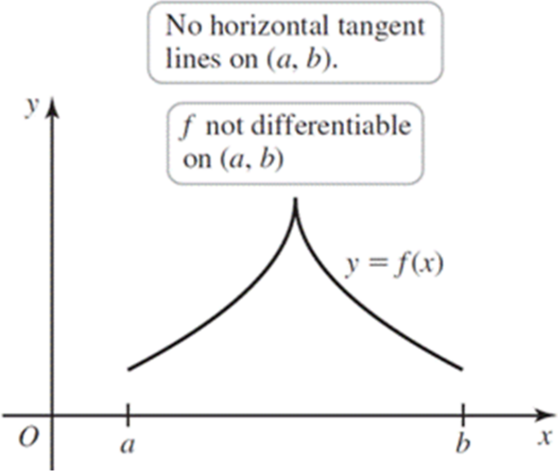


Figure: (Publisher Figure 4.66 (b))

Mean Value theorem

Theorem If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is a c in $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

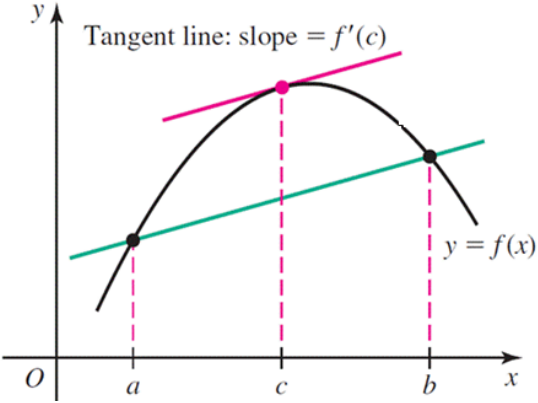


Figure: (Publisher Figure 4.68)

Proof of Mean Value theorem

The idea is to transform the problem into the setting of the Rolle theorem.

The gradient of the straight line equation $\ell(x)$ that connects the pair of points $(a, f(a))$ and $(b, f(b))$ is given by:

$$y = \ell(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

where $\ell(a) = f(a)$ and $\ell(b) = f(b)$. So the function $F(x) = f(x) - \ell(x)$ satisfies $F(a) = F(b) = 0$. Since $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b), so the Rolle theorem implies that there is a c in (a, b) so that $0 = F'(c) = f'(c) - \ell'(c)$. That is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof of Mean Value theorem II

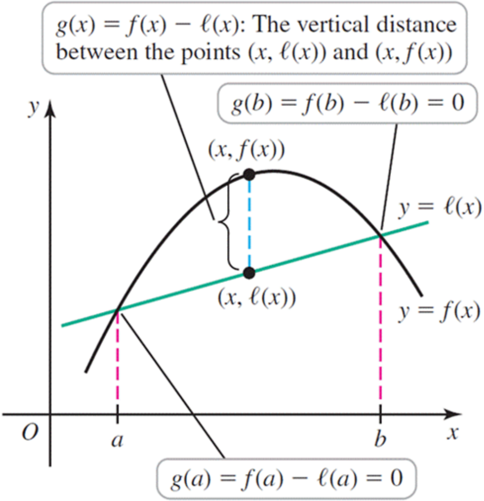


Figure: (Publisher Figure 4.69)

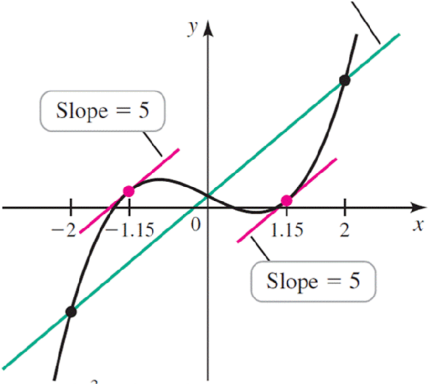
Example of Mean Value theorem

Determine the points in $[-2, 2]$ for $f(x) = 2x^3 - 3x + 1$ that are guaranteed to exist by the Mean Value theorem.

There are point(s) c in $[-2, 2]$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = 5.$$

But $f'(x) = 6x^2 - 3 = 5$ so that $x^2 = 4/3$ or $x = \pm 2/\sqrt{3}$.



Stewart: Example 5

- Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ be?
- Since we are given f is differentiable everywhere, so we can apply the Mean Value theorem on $[0, 2]$ that there is a c lying in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

or

$$f(2) = f(0) + 2f'(c) = -3 + 2f'(c) \leq -3 + 2(5) = 7,$$

- Thus the largest possible value for $f(2)$ is 7.

Consequences of Mean Value theorem

- **Theorem** If $f'(x) = 0$ at all points of an interval I, then f is a constant on I.

- **Theorem** If two functions have the property that $f'(x) = g'(x)$ on I, then $f(x) = g(x) + k$ holds for all x in I for some constant k.

- **Theorem** Suppose $f(x)$ is continuous on an interval I and differentiable at all interior points of I, then
  - if $f'(x) > 0$ at all interior points of I, then f is increasing on I;
  - if $f'(x) < 0$ at all interior points of I, then f is decreasing on I.