MATH1013 Calculus I

Derivatives IV (§4.3-4.5)[1]

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[1] Based on Stewart, James, "Single Variable Calculus, Early Transcendentals", 7th edition, Brooks/Coles, 2012
Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Signs of derivatives

1st order test

Concavity

Graphing

L'Hôpital's Rule

Consequences of MVT

- We recall as a consequence of the Mean Value Theorem that **Theorem** Suppose $f(x)$ is continuous on an interval I and differentiable at all interior points (i.e., not the end points) of I, then
  - if $f'(x) > 0$ at all interior points of I, then f is increasing on I;
  - if $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

- We show this result can help us to
  - sketch graph of functions,
  - to determine the nature of nature of critical points (first order derivative test
  - A further investigation of the rate of change of f' (i.e., f'') can help in the graph sketching, but not necessarily on determining the nature of the critical points.
- We start on observing the f'.

Example: Stewart §4.3

- Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing/decreasing.
- Since $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$ which is positive (increasing) on $(-1, 0),\ (2, \infty)$ and negative on $(-\infty, -1),\ (0, 2)$ (decreasing).

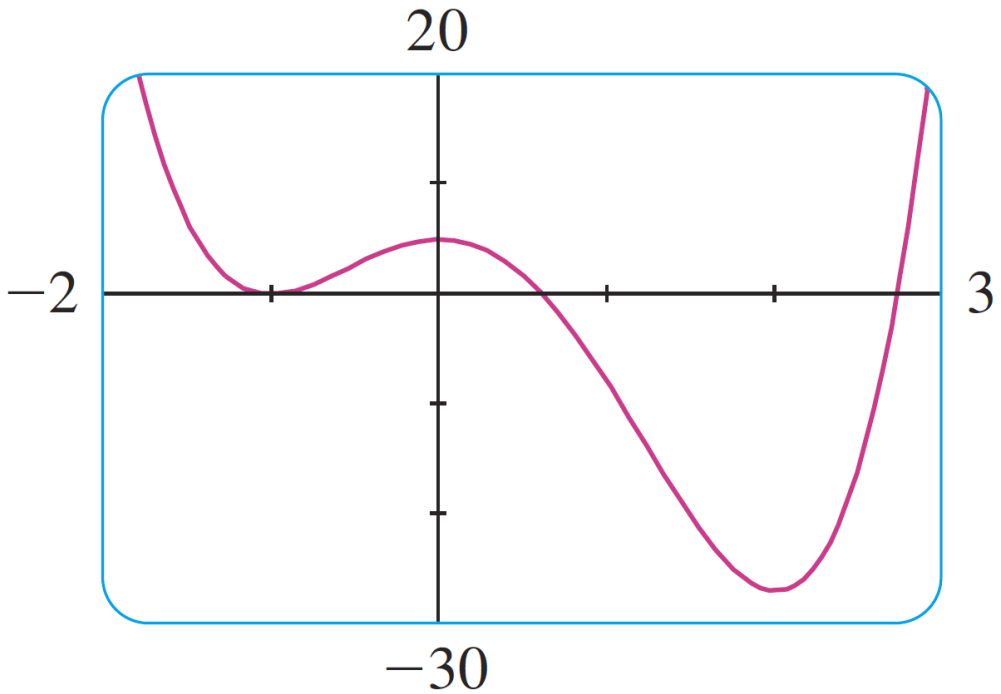


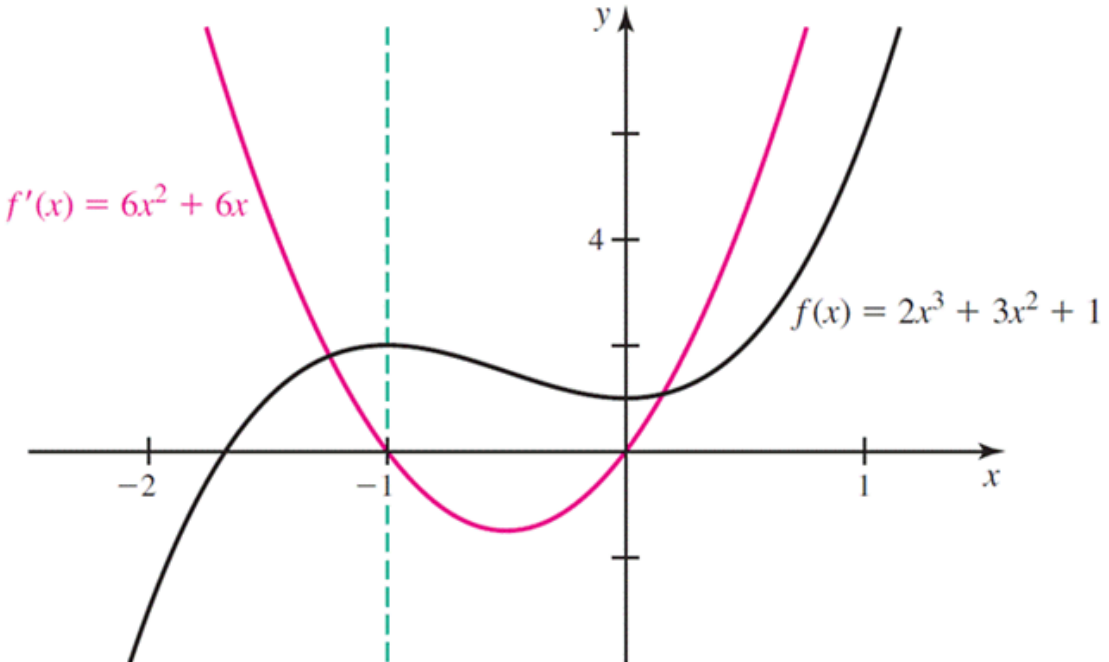
Figure: (4.3.2)

Example (Briggs, et al: p. 269)

Find the intervals of increase or decrease of $f(x) = 2x^3 + 3x^2 + 1$.
Since

$$f'(x) = 6x^2 + 6x = 6x(x+1) = \begin{cases} > 0, & \text{if } x < -1; \\ < 0, & \text{if } -1 < x < 0; \\ > 0. & \text{if } x > 0 \end{cases}$$

so that $f(x)$ is increasing on $(-\infty, -1)$ and $(0, +\infty)$, and decreasing on $(-1, 0)$.

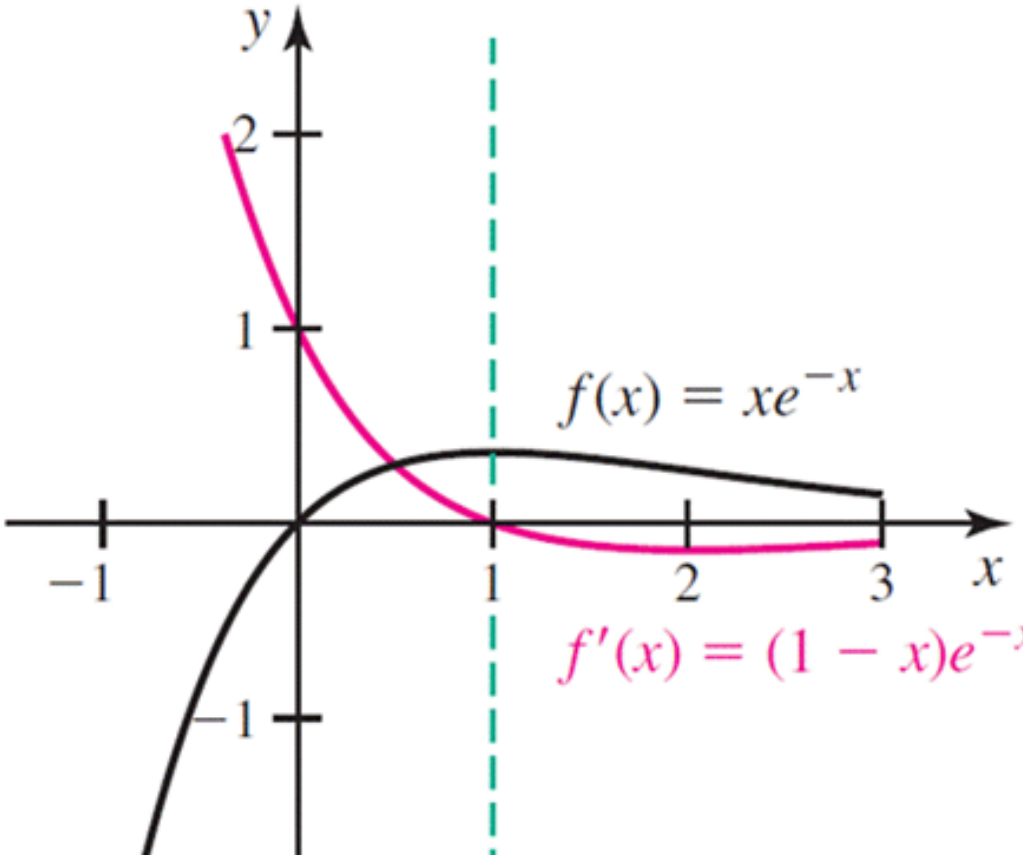


Example (Briggs, et al: p. 270)

Find the intervals of increase or decrease of $f(x) = xe^{-x}$

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} = \begin{cases} > 0, & \text{if } x < 1; \\ < 0, & \text{if } x > 1 \end{cases}$$

so that $f(x)$ is increasing on $(-\infty, 1)$ and decreasing on $(1, +\infty)$.



Exercises

Find the intervals of increase and decrease for the following functions

- $f(x) = x^2 - 4x + 5$, (f is increasing for $x > 2$; f is decreasing for $x < 2$)
- $f(x) = x^3 - 3x - 4$. (f is increasing for $x < -1, x > 1$; f is decreasing for $-1 < x < 1$)
- $f(x) = x^5 - 5x^4 + 100$ (f is increasing for $x < 0, x > 4$; $f(x)$ is decreasing for $0 < x < 4$)
- $f(x) = \frac{1}{x^3}$. (f is increasing for $x > 0$; f is decreasing for $x < 0$)

Behaviour at extrema

- **Example** Let $y = f(x) = x^2 - 4x + 4 = (x-2)^2$.
  Sketch the graphs of f and f' on the **same axis** and discuss any findings.
  The curve of the quadratic $f(x) = (x-2)^2$ has a minimum at $x = 2$. But
  $$f'(x) = 2(x-2) = 2x - 4.$$
  is a straight line with gradient 2. It equals to zero when $x = 2$.
- Suppose we don't know in advance that $x = 2$ is a minimum of f, then how do we find out this from f' what happens to f at $x = 2$?

Comparing two graphs

To answer this question, let us plot f and f' against x on the same coordinate axis in the following way:

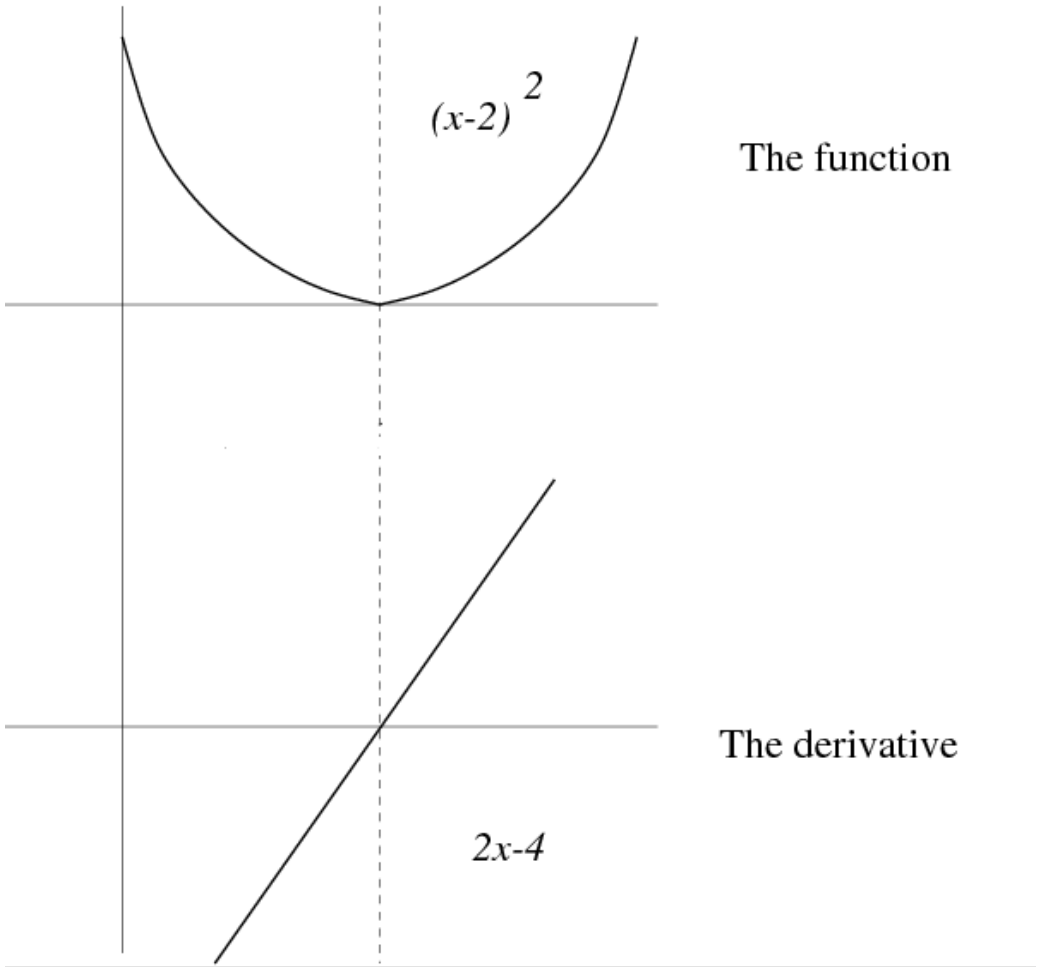


Figure: (Horizontal tangents)

Close up analysis of signs

- We note from the above figure that when f is decreasing in the region $x < 2$, $f'(x) = 2x - 4 < 0$ for $x < 2$. When f is increasing for the region $x > 2$, since $f'(x) = 2x - 4 > 0$ for $x > 2$. And at the critical point, i.e., $x = 2$, f reaches its minimum.
- We summarize the fundings below. Around the local minimum[2] of $f(x)$ at $x = 2$, the behaviour of $f'(x)$ is

$$f'(x) = \begin{cases} = 2x - 4 < 0, & \text{if } x < 2 \\ = 0, & \text{if } x = 2 \\ = 2x - 4 > 0, & \text{if } x > 2. \end{cases}$$

for x close to $x = 2$.

[2] It's a local minimum since the above analysis holds good only around the critical point in general even though it works globally for this particular example under discussion.

Close up analysis of signs: general quadratic

For $y = f(x) = ax^2 + bx + c$ $(a < 0)$, we have the critical point at $f'(x) = 2ax + b = 0$, i.e., $x = -b/(2a)$.
The behaviour of $f'(x)$ is

$$y = f'(x) = \begin{cases} 2ax + b > 0, & \text{if } x < -b/(2a) \\ 0, & \text{if } x = -b/(2a) \\ 2ax + b < 0, & \text{if } x > -b/(2a). \end{cases}$$

for x close to $x = -b/(2a)$.

Observation When $f(x)$ is quadratic, then around a critical point:

- f' increases from negative to positive around the critical point being minimum,
- f' decreases from positive to negative around the critical point being maximum.

Comparing two graphs: general quadratic ($a < 0$)

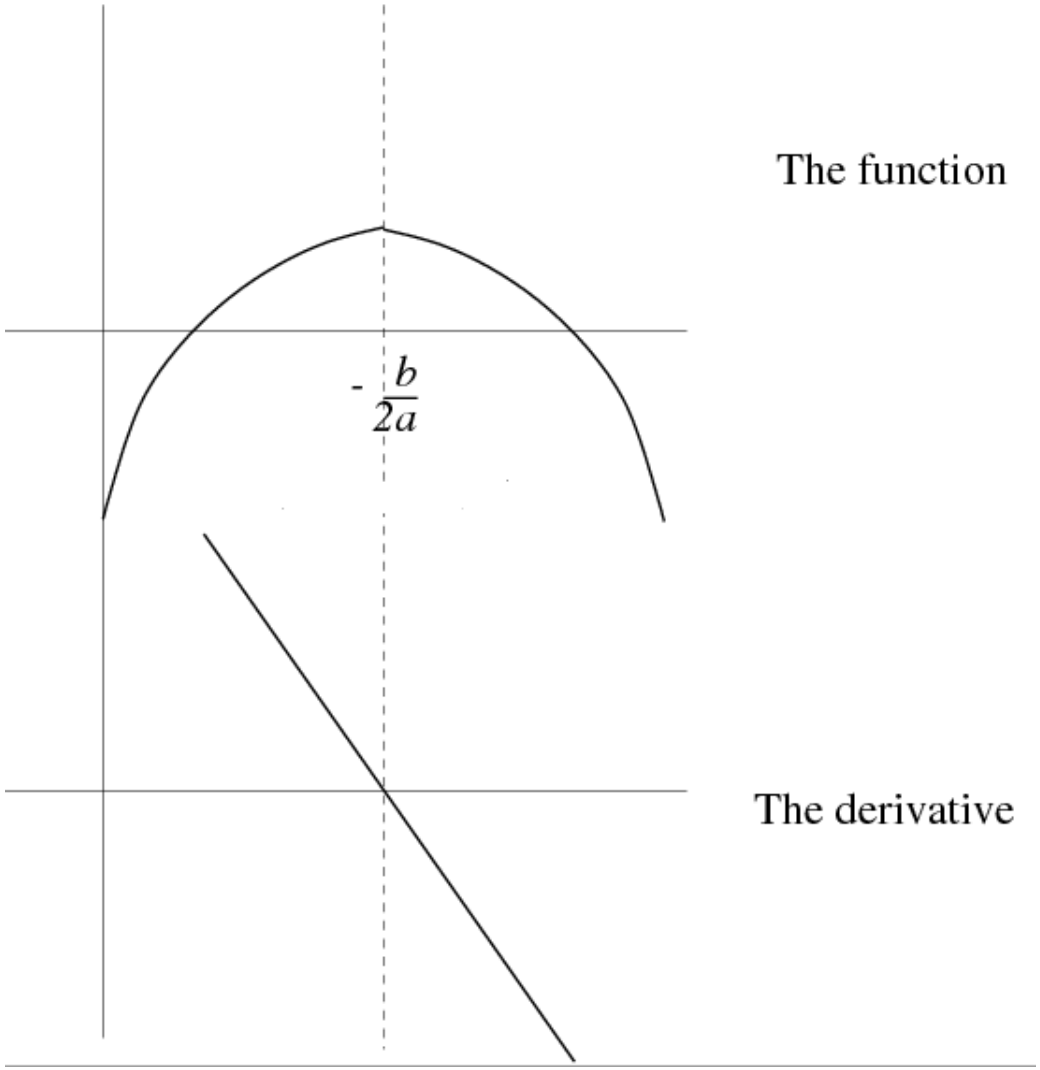


Figure: (Horizontal tangents)

Critical points is necessary but not sufficient

- **Definition** We recall that $x = a$ a *critical point* of f if $f'(a) = 0$ or if $f'(a)$ is undefined.
- If *f has a maximum or a minimum at a*, then $f'(a) = 0$ is a critical point.

- The converse is **not necessarily true**.
  - That is, at a *critical point a* ($f'(a) = 0$) may **not** represent $f(a)$ has either a maximum or minimum there.
  - **Example** $f(x) = (x-2)^3 + 2$ has $f'(2) = 0$ but $f(2)$ is **neither** a maximum **nor** a minimum.
  - **Example** $f(x) = (x-1)^4$ has $f'(1) = 0$ and $f(1)$ is a maximum

- That is, knowing $f'(a) = 0$ is insufficient to decide if $f(a)$ is an extrema.

Derivatives and behaviour

- We have seen that when a function $y = f(x)$ reaches a local maximum or local minimum, then $f'(x) = 0$.
- We have seen that if we only know that f has a critical point at a, then the natural of $f(a)$ being a max/min is inconclusive.
- What we want to show next is that a more detailed investigation on the behaviour of $f(x)$ around the critical point with our knowledge of how the $f'(x)$ affect the behaviour of $f(x)$. This would allow us to decide the nature of the point.
- This is done via the so-called first order derivative test. However, a completely vigorous argument will only be given later.

First order derivative test

We have seen that when a critical point a is a maximum/minimum, the derivative $f'(x)$ changes signs around a. That is,

- when $f(a)$ is a local maximum,

$$f'(x)\begin{cases} > 0, & \text{if } x < a; \\ = 0, & \text{if } x = a; \\ < 0, & \text{if } x > a. \end{cases}$$

$f' \downarrow$ that is $\nearrow \longrightarrow \searrow$

- when $f(a)$ is a local minimum,

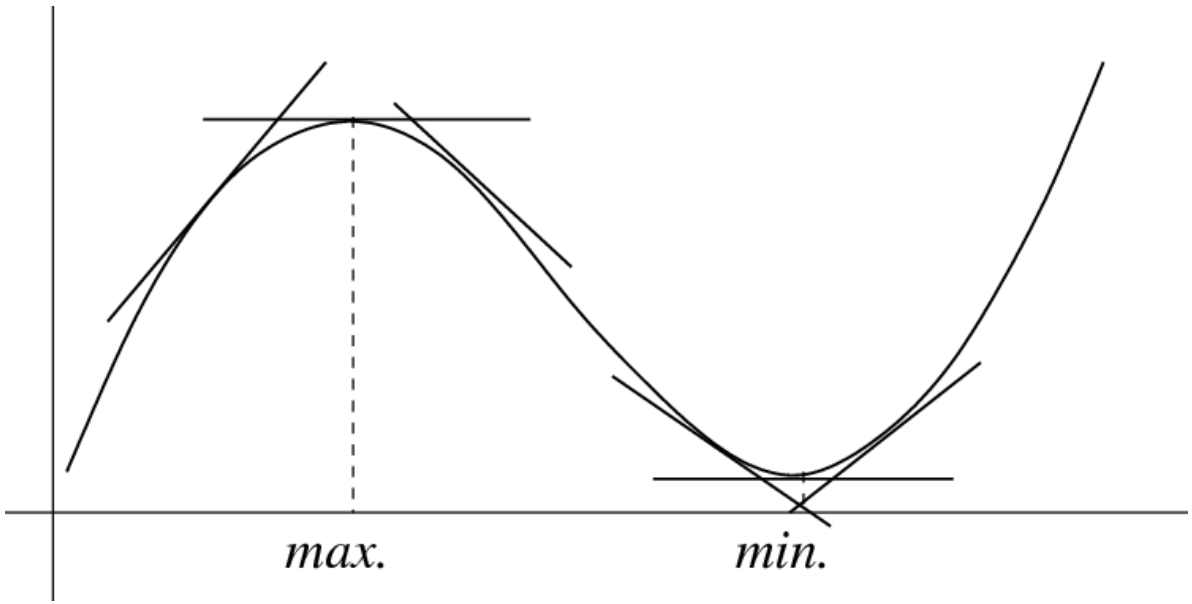
$$f'(x)\begin{cases} < 0, & \text{if } x < a; \\ = 0, & \text{if } x = a; \\ > 0, & \text{if } x > a. \end{cases}$$

$f' \uparrow$ that is $\searrow \longrightarrow \nearrow$

First order derivative test: Converse statements

It is not difficult to see that the converses also hold if $x = a$ is a critical point: $f'(a) = 0$. That is,

- if $f'(x)$ decreases from being positive to being negative, then $f(a)$ is a local maximum;
- if $f'(x)$ increases from being negative to being positive, then $f(a)$ is a local minimum
- if neither of the above happen, then $f(a)$ is neither a local maximum nor a local minimum.



Example

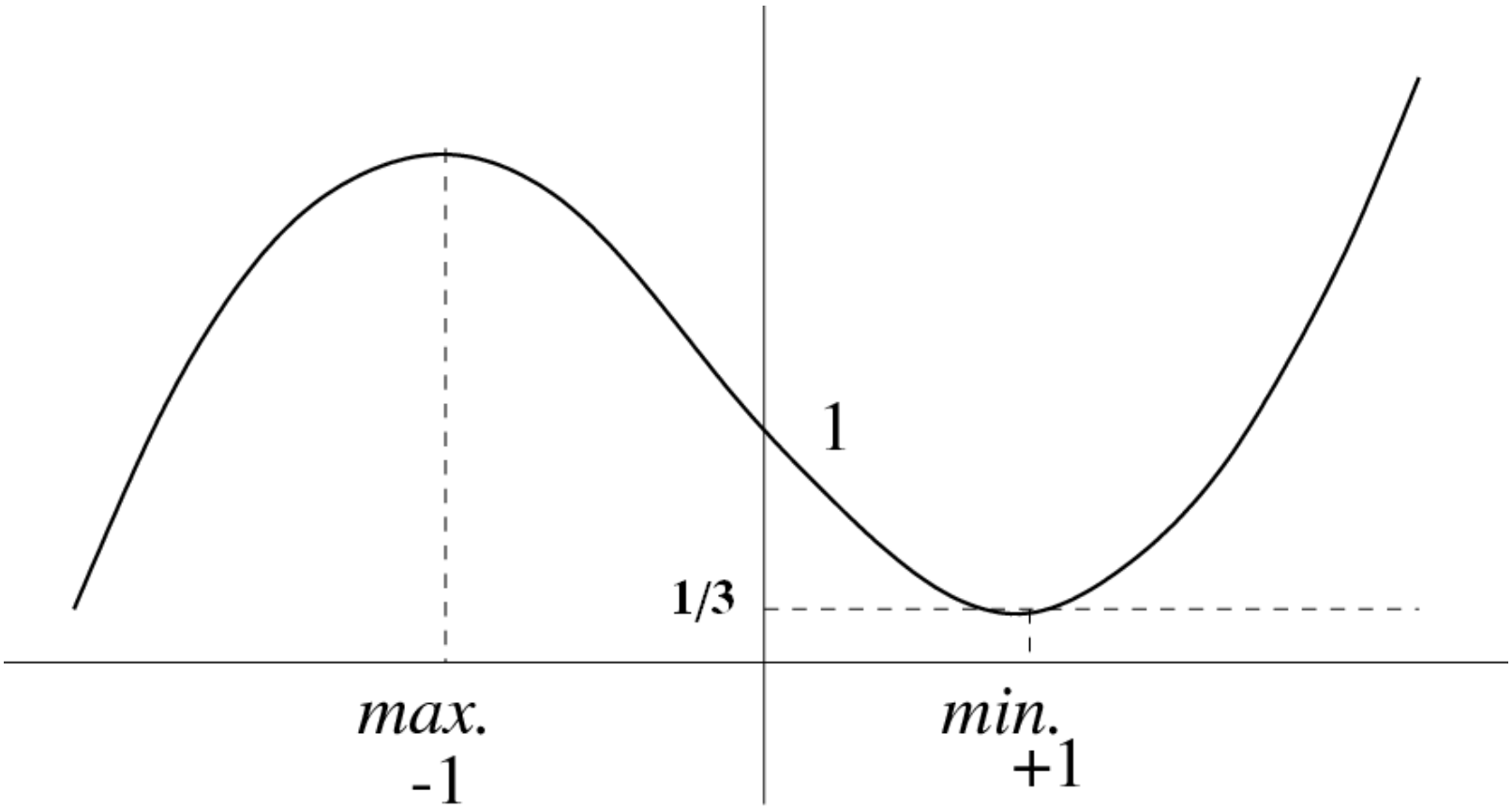
Question Find the intervals of increase/decrease and investigate the nature of the critical points $f(x) = \frac{1}{3}x^3 - x + 1$.
The critical points of f occurs when
$0 = f'(x) = x^2 - 1 = (x+1)(x-1)$. That is, when $x = \pm 1$. We have

$$f'(x) = (x+1)(x-1) = \begin{cases} > 0, & \text{if } x < -1; \\ = 0, & \text{if } x = -1; \\ < 0, & \text{if } -1 < x < 1; \\ = 0, & \text{if } x = 1; \\ > 0, & \text{if } x > 1. \end{cases}$$

Hence intervals of increase are $(-\infty, -1)$ and $(1, \infty)$, and interval of decrease is $(-1, 1)$.

Example (cont.)

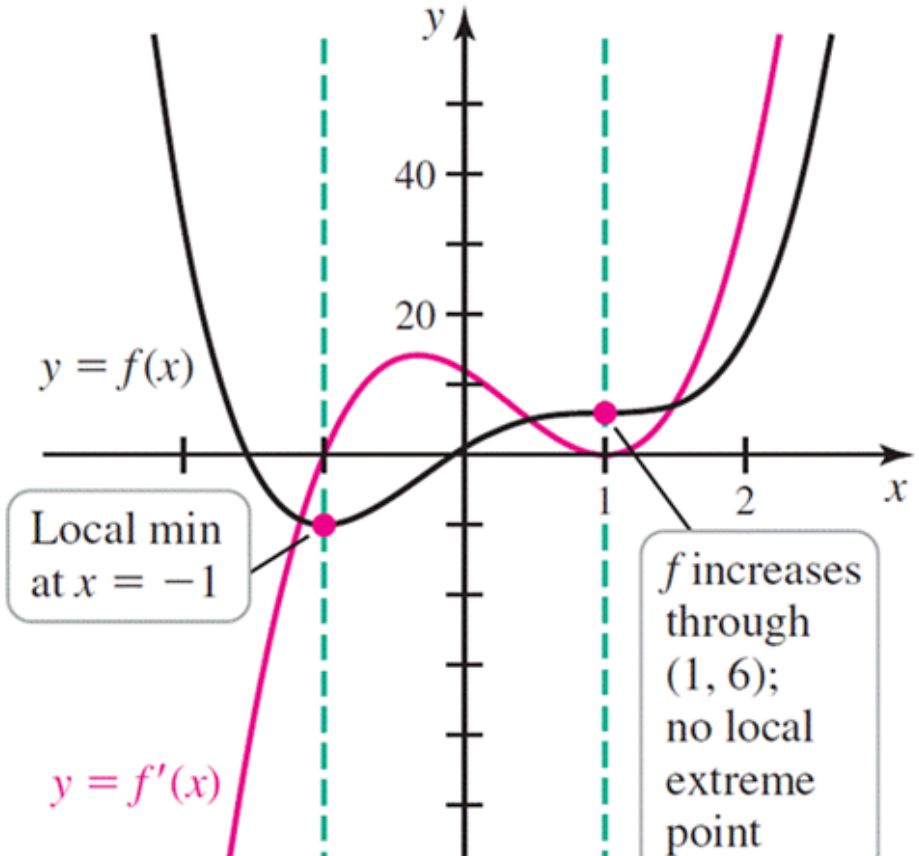
We can also apply the first order test to conclude that the critical point $x = -1$ is a local maximum and the other critical point $x = 1$ is a local minimum.



Example (Briggs, et al: p. 270 publisher)

Let $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. Find the intervals of increase/decrease and any local extrema of f.

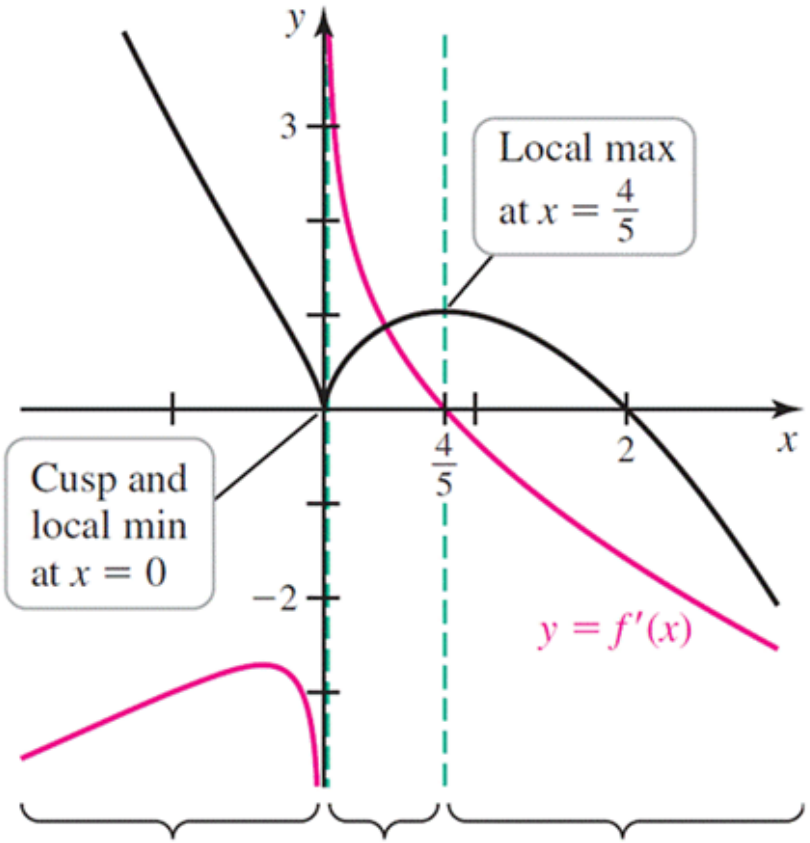
$$f'(x) = 12(x+1)(x-1)^2 = \begin{cases} < 0, & \text{if } x < -1; \\ > 0, & \text{if } -1 < x < 1; \\ > 0, & \text{if } x > 1. \end{cases}$$



Example (Briggs, et al: p. 271, publisher)

Let $f(x) = x^{2/3}(2-x)$. Find the intervals of in increase/decrease and any local extrema of f.

$$f'(x) = \frac{4}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{4-5x}{3x^{1/3}} = \begin{cases} < 0, & \text{if } x < 0; \\ > 0, & \text{if } 0 < x < 4/5; \\ < 0, & \text{if } x > 4/5. \end{cases}$$



Example

Find the maximum(s) and minimum(s) of $f(x) = x^3 - 2x^2 - 4x + 6$.

- critical points: $0 = f'(x) = 3x^2 - 4x - 4 = (3x + 2)(x - 2)$, which gives the *possible* $x = \{-\frac{3}{2}, 2\}$ for local maximums/minimums or neither.
- Determine critical points'nature:

$$f'(x) = (3x + 2)(x - 2) = \begin{cases} > 0, & \text{if } x < -2/3, \ x > 2; \\ = 0, & \text{if } x = -2/3 \text{ or } 2; \\ < 0, & \text{if } x = -2/3 < x < 2. \end{cases}$$

  So f is increasing on $(-\infty, -2/3)$ and $(2, \infty)$, and f is decreasing on $(-2/3, 2)$.
- We conclude that $f(-2/3) \approx 7.48$ is a local maximum and $f(2) = -2$ is a local minimum. Note also that $f(0) = 6$.

Nature of critical point exercises

Determine the nature of the critical points of the following functions:

- $f(x) = 3x^4 - 8x^3 + 6x^2 + 2$, $(0, 2)$ relative minimum; $(1, 3)$ neither
- $f(t) = 2t^3 + 6t^2 + 6t + 5$, $(-1, 3)$ neither
- $f(x) = (x-1)^5$, $(1, 0)$ neither,
- $f(x) = (x^2 - 1)^5$, $(-1, 0)$ neither; $(0, -1)$ relative minimum; $(1, 0)$ neither
- $f(x) = (x^3 - 1)^4$, $(0, 1)$ neither, $(1, 0)$ relative minimum.

Concavity I (Stewart §4.3)

- **Definition** A differentiable function f is concave up over an interval I if the graph of f lies above all its tangents over I.
- **Definition** A differentiable function f is concave down over an interval I if the graph of f lies below all its tangents over I.

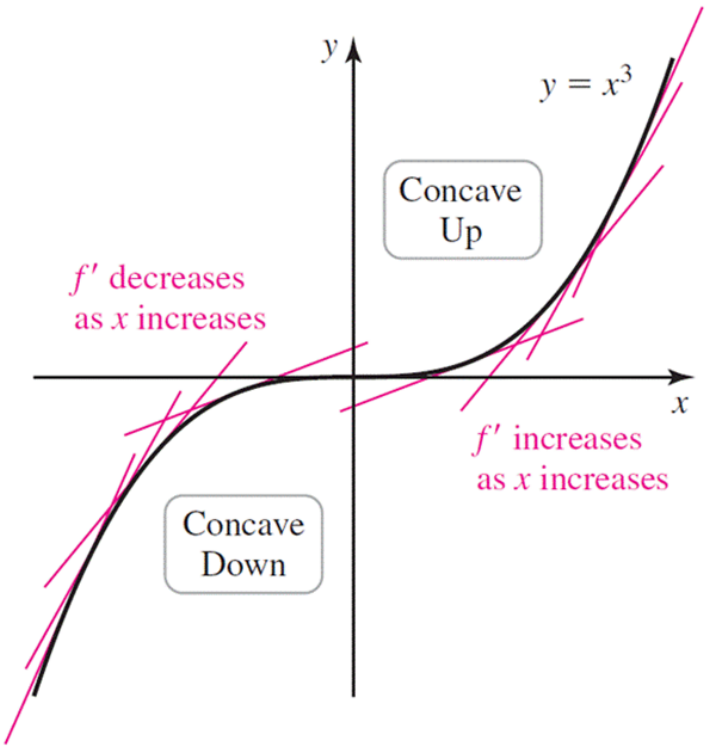


FIGURE 4.27

Figure: (Briggs, et al)

Concavity (test) II

- **Theorem** Suppose that $f''(x)$ exists over an interval I.
  1. If $f''(x) > 0$ at each point of I, then f is concave up over I;
  2. If $f''(x) < 0$ at each point of I, then f is concave down over I.

We need to show

$$f(x) > f(a) + (x - a)f'(a)$$

whenever $x > a$. But the MVT asserts that there is a c with$a < c < x$ such that

$$f(x) - f(a) = f'(c)(x - a)$$

But $f''(x) > 0$ implies that $f'(a) < f'(c)$ and hence

$$f(a) + f'(a)(x - a) < f(a) + f'(c)(x - a) = f(x)$$

as required. We omit the case $f''(x) < 0$.

Example

- **Example** $f(x) = \frac{1}{x}$ is concave down over $(-\infty, 0)$ and concave up over $(0, \infty,)$.

Inflection point I

- **Definition** A point c is called a point of inflection for a function $f(x)$ if $f(x)$ is continuous at a *and* if there is a change of concavity:
- Thus the concavity test suggests that, if

$$f''(x) > 0, \qquad x < c;$$
$$f''(x) < 0, \qquad x > c,$$

or

$$f''(x) < 0, \qquad x < c;$$
$$f''(x) > 0, \qquad x > c.$$

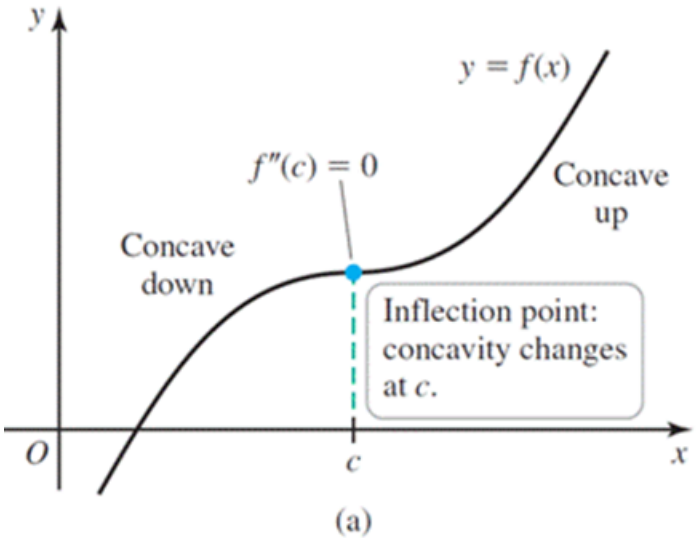
then $x = c$ is an inflection point.

- Depending on the behaviour of $f(x)$ at the inflection point, its second derivative could be zero there, i.e., $f''(c) = 0$ (necessary but not sufficient) or $f''(c)$ is simply undefined.

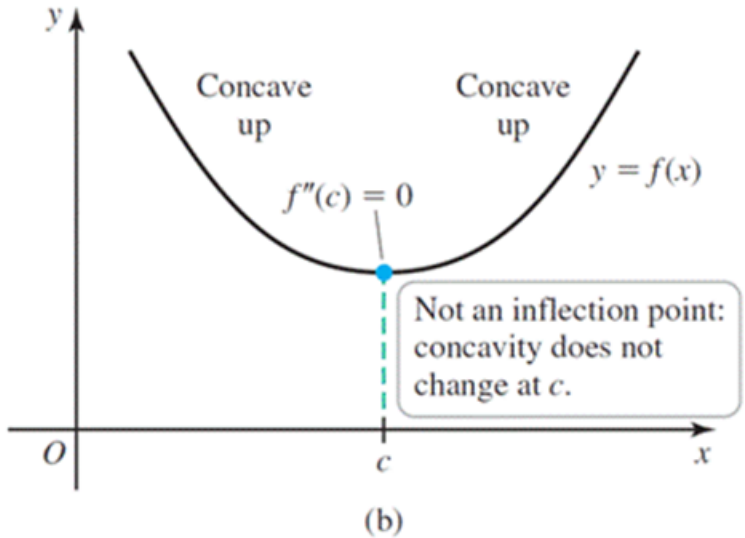
Inflection point II

- If $f(x)$ has an inflection point at $x = c$ and that $f''(x)$ is continuous over an interval containing $x = c$, then $f''(c) = 0$.
- **Example** $f(x) = x^3$ has an inflection point at $x = 0$ since there is a change of concavity and $f''(0) = 0$.
- **Example** $f(x) = -x^3$ has an inflection point at $x = 0$ since there is a change of concavity and $f''(0) = 0$.
- **Remark** Even if $f''(x)$ is continuous around $x = c$, and $f''(0) = 0$, it does not imply necessarily that $f(x)$ has an inflection point at $x = c$.
- **Example** $f(x) = x^4$ is concave up over $(-\infty, \infty)$ and although $f''(x) > 0$ whenever $x \neq 0$, yet it has $f''(0) = 0$.
- $f''(0) = 0$ it is a necessary condition for concavity, it is , however, **not** sufficient.
- The next slide shows that f'' is undefined at a point of inflection.

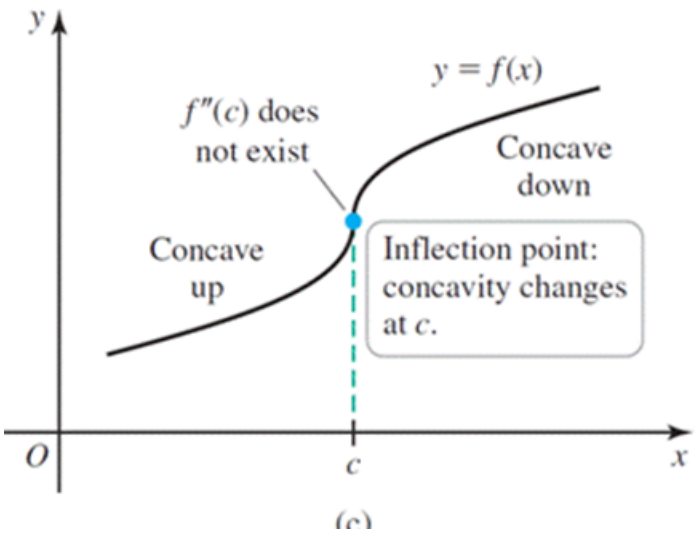
Inflection point III (publisher)



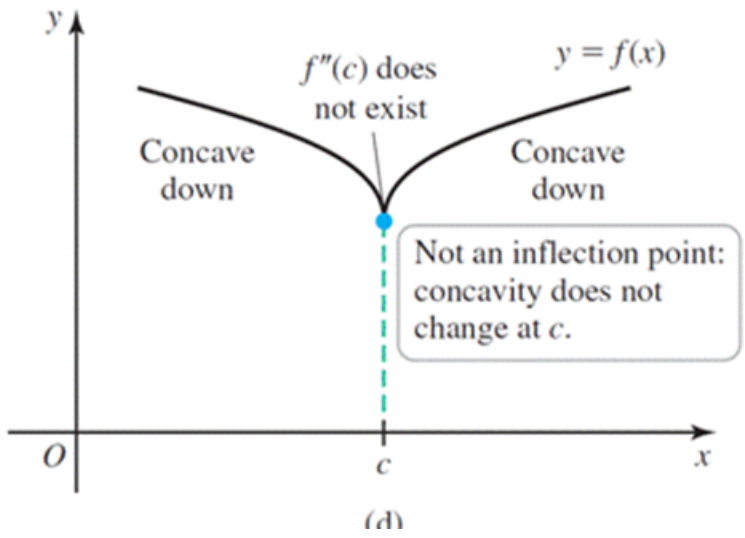
(a)



(b)



(c)



(d)

Example (Briggs, et al: p. 274) I

Identify the intervals of concave up/down of $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$.
We have already computed

$$f'(x) = 12(x+1)(x-1)^2 = \begin{cases} < 0, & \text{if } x < -1; \\ > 0, & \text{if } -1 < x < 1; \\ > 0, & \text{if } x > 1. \end{cases}$$

$$f''(x) = 12(x-1)(3x+1) \begin{cases} > 0, & \text{if } x < -1/3 \text{ or } x > 1; \\ = 0, & \text{if } x = -1/3 \text{ or } x = 1; \\ < 0, & \text{if } -1/3 < x < 1; \end{cases}$$

We deduce that the critical points are $\{-1, 1\}$ and the inflection points are $\{-1/3, 1\}$.

Example (Briggs, et al:p. 275) II

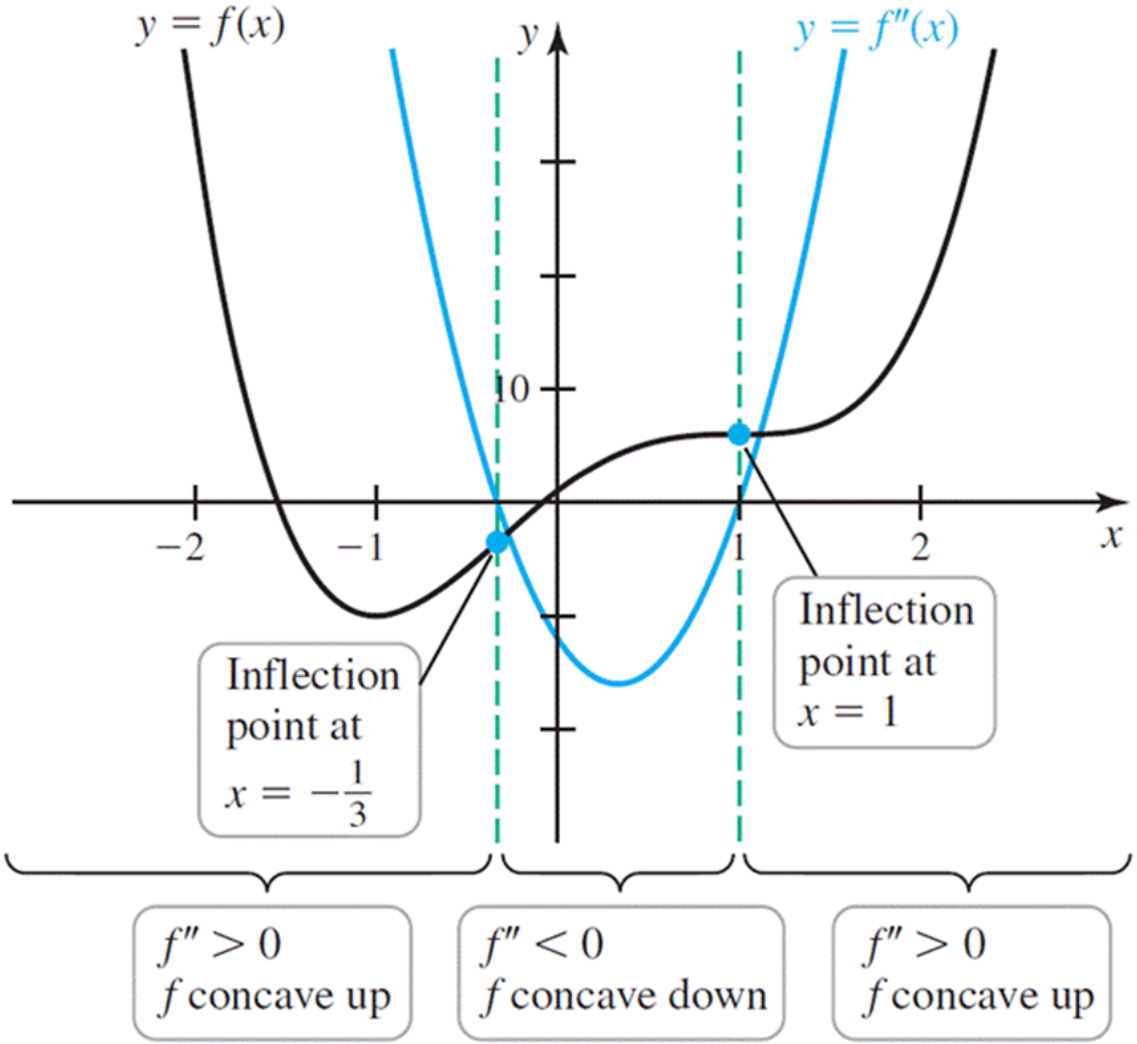


Figure: (Figure 4.31 (publisher))

Example (Briggs, et al: p. 275) III

Identify the intervals of concave up/down of $f(x) = \arcsin(x)$.

$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad f''(x) = \frac{x}{\sqrt{(1-x^2)^3}}.$$

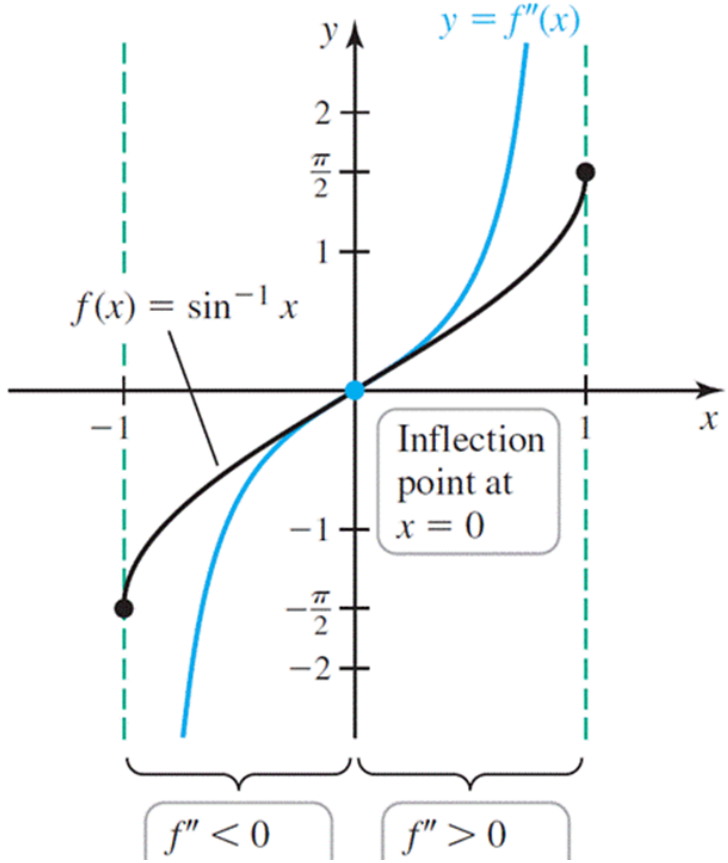


Figure: (Figure 4.31 (publisher))

Example I

Sketch the graph of $f(x) = 3x^4 - 4x^3 - 12x^2 + 17$.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x+1)(x-2)$$

$$= \begin{cases} < 0, & \text{if } x < -1,\ 0 < x < 2; \\ = 0, & \text{if } x = -1,\ 0,\ 2 \\ > 0, & \text{if } -1 < x < 0,\ x > 2. \end{cases}$$

$$f''(x) = 12(3x^2 - 2x - 2)$$

$$= 36\left(x - \frac{1+\sqrt{7}}{3}\right)\left(x - \frac{1-\sqrt{7}}{3}\right)$$

$$= \begin{cases} > 0, & \text{if } x < (1-\sqrt{7})/3,\ \text{or } x > (1+\sqrt{7})/3 \\ = 0, & \text{if } x = (1-\sqrt{7})/3 \text{ or } (1+\sqrt{7})/3 \\ < 0, & \text{if } (1-\sqrt{7})/3 < x < (1+\sqrt{7})/3 \end{cases}$$

Inflection points $(1-\sqrt{7})/3 \approx -0.55$ and $(1+\sqrt{7})/3 \approx 1.27$.

Example II

Sketch the graph of $y = f(x) = x + \dfrac{4}{x+1}$.

- The easiest is to find the where f intersects with with the axes. Suppose $f(x) = 0$, i.e., $0 = x + \frac{4}{x+1}$ or $x^2 + x + 4 = 0$

$$x = \frac{-1 \pm \sqrt{1^4 - 4 \cdot 1 \cdot 4}}{2},$$

  which has no solution since $1^2 - 16 < 0$. So f will never be zero, and so f will never intersect the $x-$axis. Besides, $f(0) = 4$.
- The next step is to consider $x \to +\infty$ and $x \to -\infty$. When x is large and positive $f(x) - x$ is approaching zero. i.e.,

$$\lim_{x \to +\infty} \left(f(x) - x \right) = \lim_{x \to +\infty} \left(\frac{4}{x+1} \right) = 0.$$

  Similarly,

$$\lim_{x \to -\infty} \left(f(x) - x \right) = \lim_{x \to -\infty} \left(\frac{4}{x+1} \right) = 0.$$

Example II (cont.)

- That is f is "essentially" like x when $x \to \pm\infty$.
  - In fact, since $\frac{4}{x+1} > 0$ as $x \to +\infty$, f approaches $y = x$ from above,
  - $\frac{4}{x+1} < 0$ when $x \to -\infty$, so f tends to $y = x$ from below.
- The third step is to note that $\dfrac{4}{x+1}$ is meaningless when $x = -1$. We have

$$\lim_{x \to (-1)+} f(x) = \lim_{x \to (-1)+} \left(x + \frac{4}{x+1} \right) = +\infty,$$

and

$$\lim_{x \to (-1)-} f(x) = \lim_{x \to (-1)-} \left(x + \frac{4}{x+1} \right) = -\infty,$$

Example II (cont.)

- The fourth step is to identify the critical points.

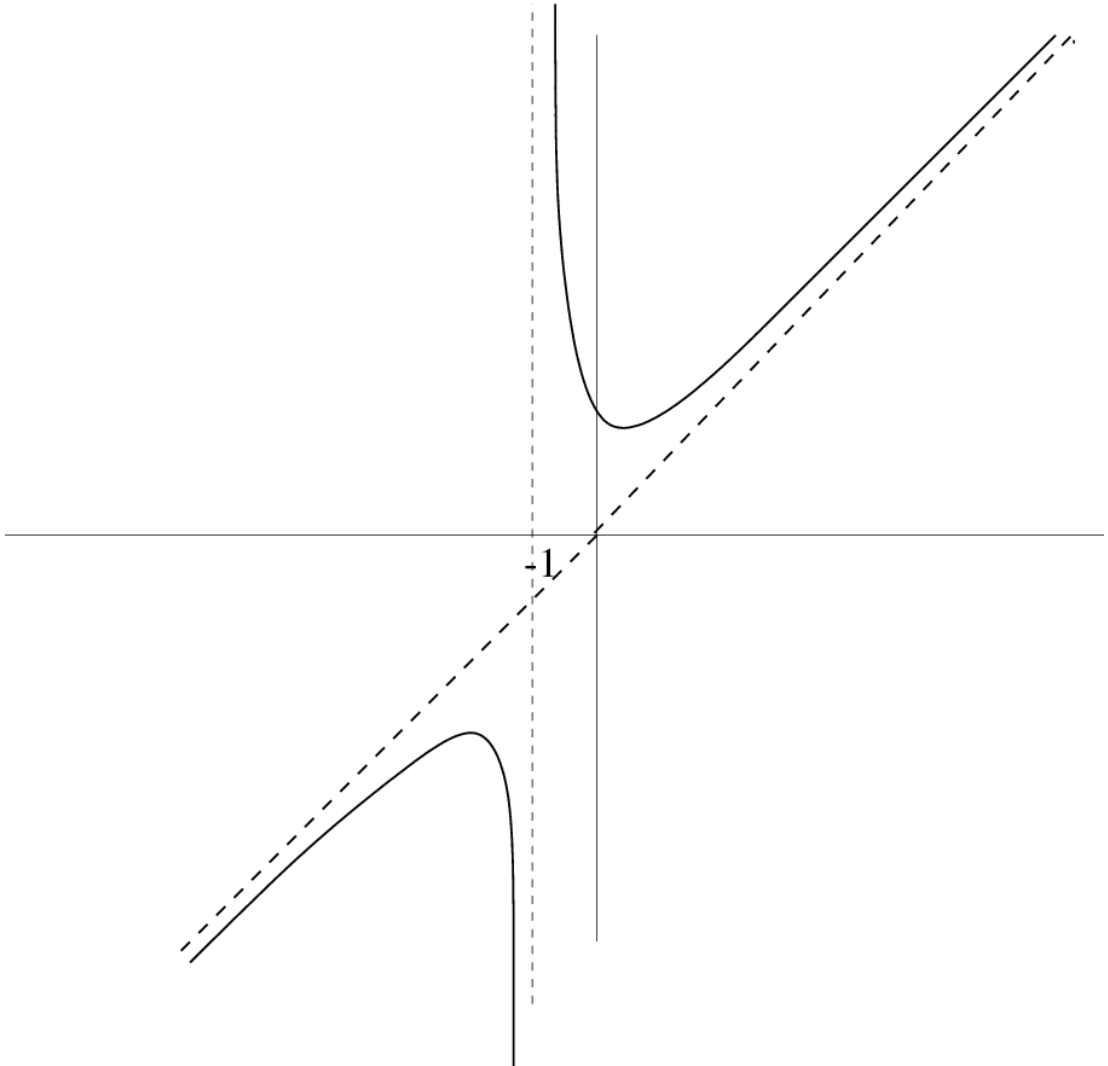
$$f'(x) = 1 - \frac{4}{(x+1)^2} = \frac{(x-1)(x+3)}{(x+1)^2}$$

$$= \begin{cases} > 0, & \text{if } x < -3; \\ < 0, & \text{if } -3 < x < -1; \\ < 0, & \text{if } -1 < x < 1 \\ > 0, & \text{if } x > 1. \end{cases}$$

We deduce f has a maximum at $x = -3$ and a minimum at $x = 1$. In fact, f is increasing on the intervals $x < -3$ and $x > 1$, and decreasing on the intervals $-3 < x < -1$ and $-1 < x < 1$.

Example II (cont.)

- Now the concavity $f''(x) = \dfrac{8}{(x+1)^3} = \begin{cases} > 0, & \text{if } x > -1; \\ < 0 & \text{if } x < -1. \end{cases}$



Example III (Briggs, et al: p. 286)

Sketch the curve of $f(x) = \dfrac{10x^3}{x^2-1}$

- Clearly $f(0) = 0$ and f is undefined on $x = \pm 1$.
- Asymptotes: $f(x) = 10x + \dfrac{10x}{x^2-1}$
- Derivatives:

$$f'(x) = \frac{10x^2(x^2-3)}{(x^2-1)^2} = \begin{cases} > 0, & \text{if } x > \sqrt{3} \text{ or } x < -\sqrt{3}; \\ = 0, & \text{if } x = \pm\sqrt{3}; \\ < 0, & \text{if } -\sqrt{3} < x < \sqrt{3}. \end{cases}$$

and

$$f''(x) = \frac{20x(x^2+3)}{(x^2-1)^3} = \begin{cases} < 0, & \text{if } x < -1, \text{ or } 0 < x < 1; \\ = 0, & \text{if } x = 0; \\ > 0, & \text{if } -1 < x < 0, \text{ or } x > 1. \end{cases}$$

So we see that $x = -\sqrt{3}$ is a local maximum and $x = \sqrt{3}$ is a local minimum. Moreover, there is a change of concavity, so $x = 0$ is a point of inflection.

Example (Briggs, et al: p. 286) (cont.)

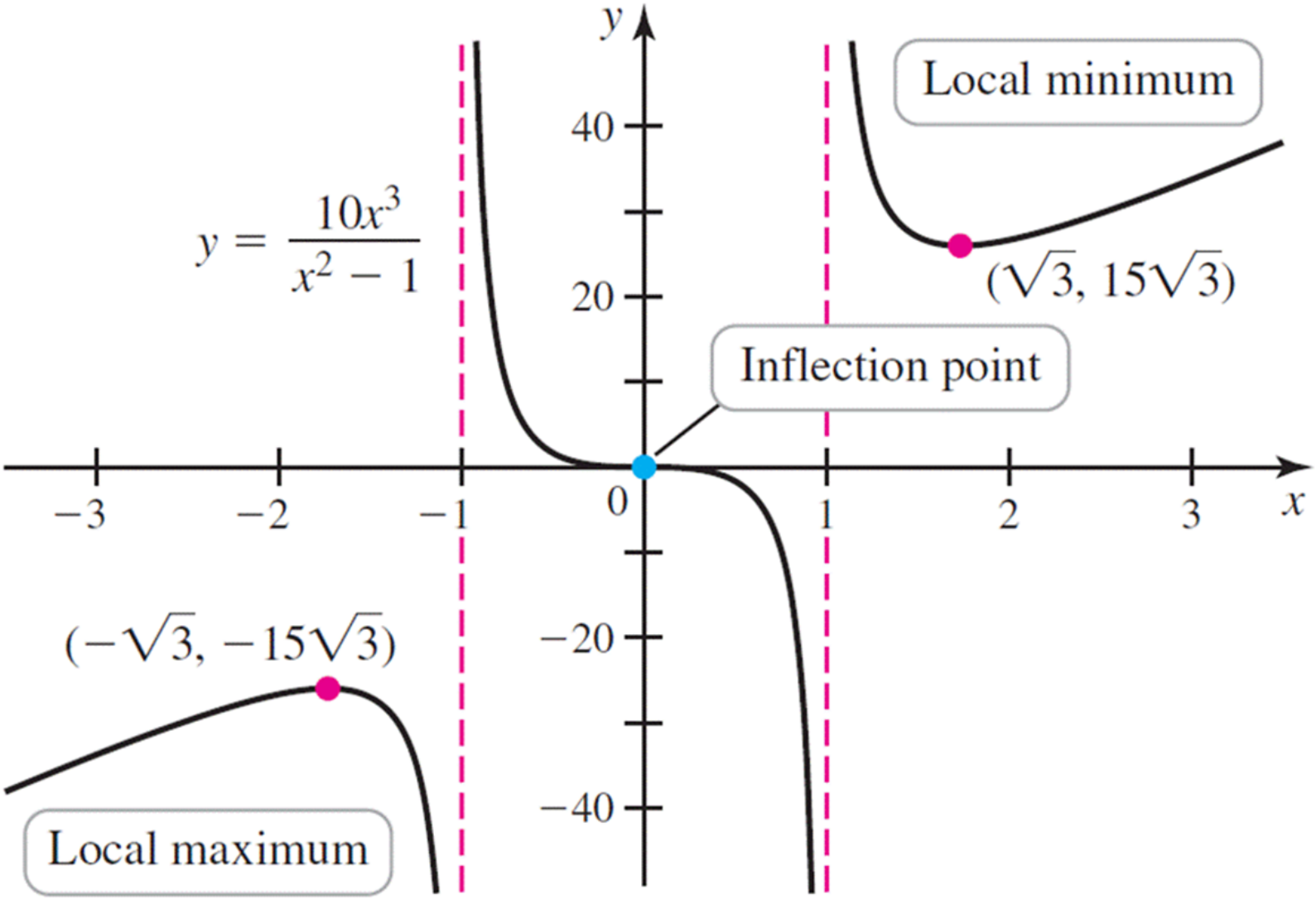


Figure: (Figure 4.45 (publisher))

Normal distribution (Briggs, et al: p. 287)

Sketch the curve of $f(x) = e^{-x^2}$.

- $f(0) = 1$, and $f(x) > 0$ for all x.
- $\lim_{x \to \pm\infty} f(x) = 0$ from above the $x-$axis.
- $$f'(x) = -2xe^{-x^2} = \begin{cases} < 0, & \text{if } x > 0; \\ = 0, & \text{if } x = 0 \\ > 0, & \text{if } x < 0, \end{cases}$$

  and that $x = 0$ is a local maximum by the first order derivative test.
- $$f''(x) = 2(2x^2-1)e^{-x^2} = \begin{cases} < 0, & \text{if } x < -1/\sqrt{2} \text{ or } x > 1/\sqrt{2}; \\ = 0, & \text{if } x = \pm 1/\sqrt{2} \\ > 0, & \text{if } -1/\sqrt{2} < x < 1/\sqrt{2}, \end{cases}$$

  so that $x = \pm 1/\sqrt{2}$ are points of inflection.

Normal distribution (Briggs, et al: p. 286) II

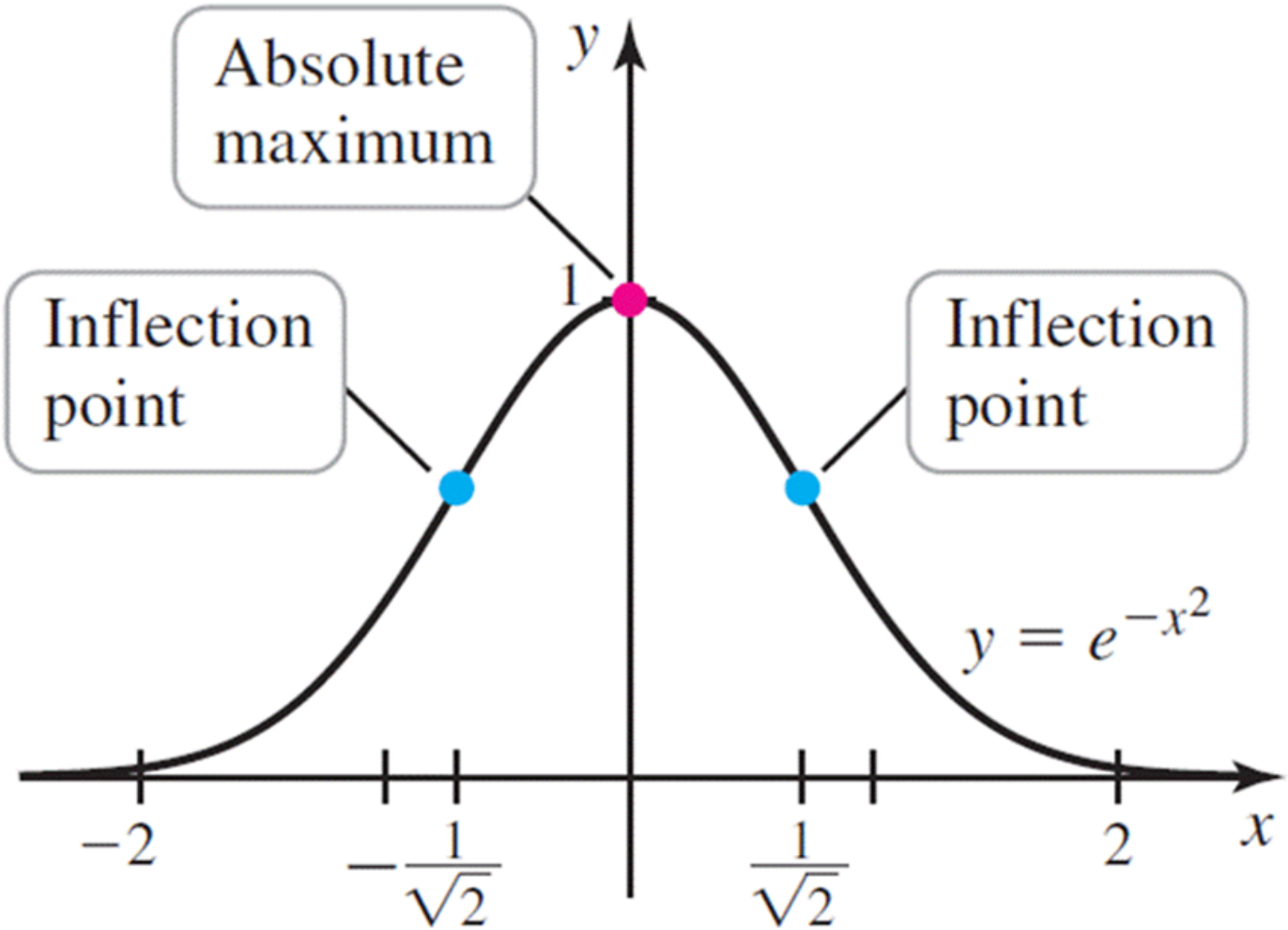


Figure: (Figure 4.47 (publisher))

Graphing procedure (Stewart)

Here is a useful procedure for reference:

- Domain
- Intercepts
- Symmetry
- Asymptotes
- Intervals of increase and/or decrease
- Local maximum and minimum
- Concavity and inflection points (if any)
- Sketch the curve

L'Hôptial's rule 0/0 form

- **Theorem (p.316)** *Suppose f and g are differentiable on (a, b) and c lies in (a, b) such that $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ and $g'(x) \neq 0$ $(x \neq c)$. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the last limit exists.

- **Remarks** The above also holds for $x \to \pm\infty$ or $x \to c\pm$.
- **Example**

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} 1} = 1.$$

Proof of "0/0" form

We assume in addition that f' and g' are also continuous over (a, b) and $g'(c) \neq 0$

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)} = \frac{\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}}{\lim\limits_{x\to c} \dfrac{g(x)-g(c)}{x-c}}$$

$$= \lim_{x\to c} \frac{\dfrac{f(x)-f(c)}{x-c}}{\dfrac{g(x)-g(c)}{x-c}} = \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x\to c} \frac{f(x)}{g(x)}$$

The basic rationale behind is the use of first order approximation formula: $f(x) \approx f(a) + f'(a)(x-a) = f'(a)(x-a)$.

Examples (p. 317)

- $\lim\limits_{x\to 1} \dfrac{x^3+x^2-2x}{x-1}$
- $\lim\limits_{x\to 0} \dfrac{\sqrt{9+3x}-3}{x}$

More examples (p. 318)

The following examples require repeated applications of L'Hôpital's rule:

- $\lim_{x\to 0} \dfrac{e^x - x - 1}{x^2}$
- $\lim_{x\to 2} \dfrac{x^3 - 3x^2 + 4}{x^4 - 4x^3 + 7x^2 - 12x + 12}$

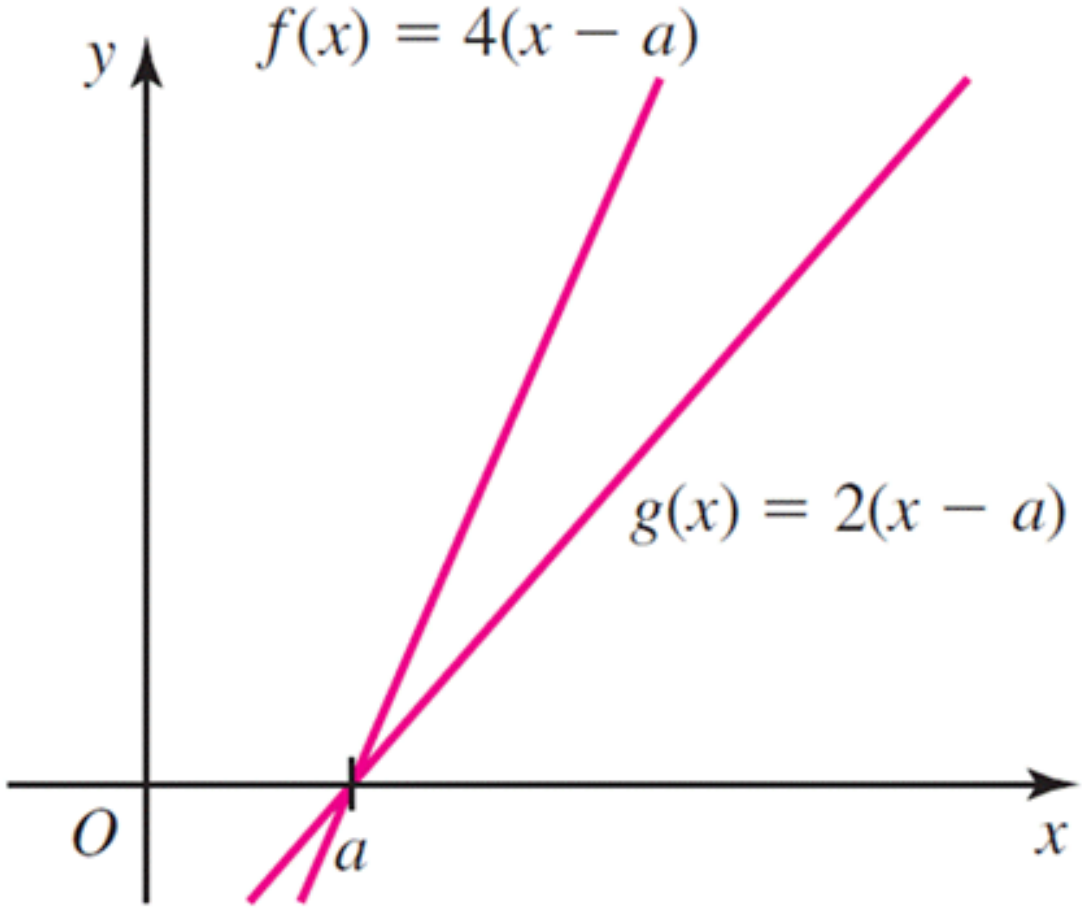
Double check in the indeterminate forms

$$\lim_{x\to 0} \frac{3x - \sin x + 1}{x + 2} = \lim_{x\to 0} \frac{3 - \cos x}{1} = \frac{3 - \cos 0}{1} = 2.$$

But in fact, the above limit is NOT in the indeterminate form

$$\lim_{x\to 0} \frac{3x - \sin x + 1}{x + 2} = \frac{0 - 0 + 1}{0 + 2} = \frac{1}{2}.$$

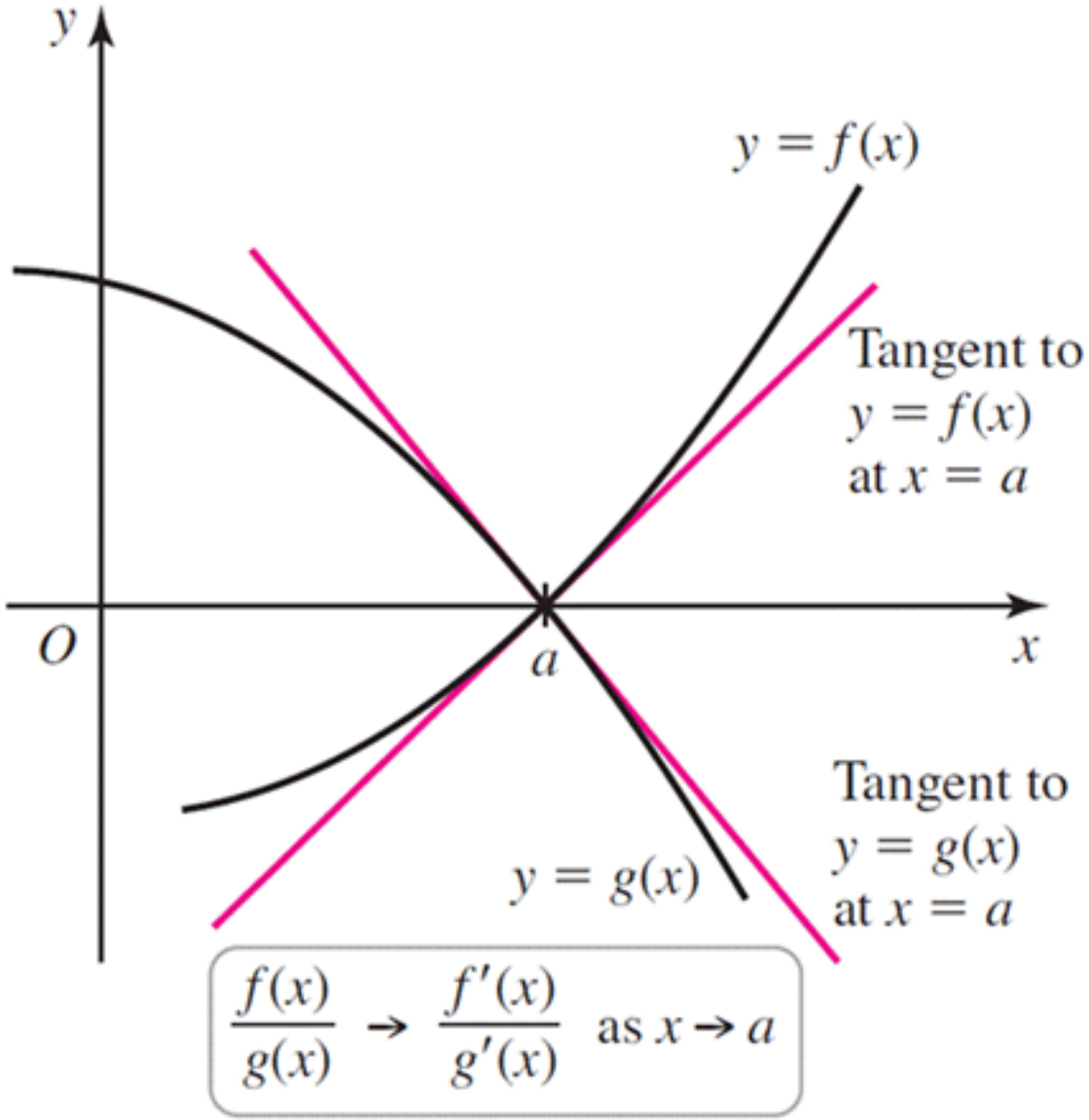
Geometric reason I (p. 317)



$$\frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)} = \frac{4}{2} = 2$$

Figure: (Publisher Figure 4.72)

Geometric reason II (p. 317)



"∞/∞" form

Theorem (p.319) *Suppose f and g are differentiable on (a, b) and c lies in (a, b) such that $\lim_{x\to c} f(x) = \pm\infty = \lim_{x\to c} g(x)$ and $g'(x) \neq 0$ $(x \neq c)$. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided that the last limit exists.

Remark The above also holds for $x \to \pm\infty$ or $x \to c\pm$.

- **Example** $\lim_{x\to\infty} \dfrac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$
- **Example** $\lim_{x\to\pi/2-} \dfrac{1 + \tan x}{\sec x}$

Double check "∞/∞" form

Example

$$\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x-5}}=\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x+2}}}{\frac{1}{2\sqrt{x-5}}}=\lim_{x\to\infty}\frac{\sqrt{x-5}}{\sqrt{x+2}}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x-5}}}{\frac{1}{2\sqrt{x+2}}}=\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x-5}}.$$

But in fact,

$$\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x-5}}=\lim_{x\to\infty}\frac{\sqrt{1+2/x}}{\sqrt{1-5/x}}=\frac{\sqrt{1+0}}{\sqrt{1-0}}=1.$$

"$0 \cdot \infty$", "$\infty - \infty$", "1^∞", "0^∞", "∞^0"

They can be changed to $0/0$ or ∞/∞.

- "$0 \cdot \infty$": $\lim_{x\to\infty} x \sin\frac{1}{x} = 1$
- "$\infty - \infty$": $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$
- "0^0": $\lim_{x\to 0+} x^x = 1$
- 1^∞: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
- Cannot apply L'Hôpital's rule to 1^∞, 0^∞, ∞^0.
- Rewrite $\lim_{x\to a} f(x)^{g(x)}$ as

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}.$$

Thus it remains to compute

$$\lim_{x\to a} g(x)\ln f(x)$$

perhaps again using L'Hôpital's rule.